that $\mathfrak{F}_{f\pi\pi}(q^2) \approx \mathfrak{F}_{fK\bar{K}}(q^2) \approx F_{fNN}(q^2)$ will be a good approximation.

[18]We write the Reggeized f contribution to the $\pi\pi$ amplitude, for instance, as

$$F_{\pi\pi}(s,t) = S(t)(\alpha+\tfrac{1}{2})\beta(t)E_{00}^{\alpha,+}(\cos\theta_T)$$
$$\xrightarrow[s\to\infty]{} S(t)\gamma_{\pi\pi}(t)\Gamma(\alpha+\tfrac{3}{2})\pi^{-1/2}[\Gamma(\alpha+1)]^{-1}s^{\alpha},$$

where $\alpha=\alpha(t)$ is the f trajectory, $S(t)=\frac{1}{2}(1+e^{-i\pi\alpha})/\sin\pi\alpha$, $\beta(t)=\gamma_{\pi\pi}(t)\alpha(t)(p_tq_t)^{\alpha}$, and $p_t=q_t=\frac{1}{2}(t-4\mu_\pi^2)^{1/2}$. Note that on taking into account the correct threshold behavior of $\beta(t)$, the dependence on p_t and q_t cancels out, and there is no explicit mass dependence of the residue functions, contrary to the results of Ref. 2. For convenience, we have written the amplitude here with the Gell-Mann ghost-eliminating mechanism. However, the results in the text are not sensitive to the mechanism assumed.

[19]E.g., see the analysis of forward scattering by Barger *et al.*; see V. Barger, review talk in *Proceedings of the Topical Conference on High-Energy Collisions of Hadrons, CERN, 1968* (CERN, Geneva, 1968), and references contained therein. In the analysis of Barger *et al.*, the f lies on the P'.

[20]Recently, the possibility that the f may be on the Pomeranchuk trajectory has been revived; e.g., see Ref. 12 and references quoted therein.

[21]G. Dass and S. Papageorgiou, Nuovo Cimento 64A, 36 (1970). Our definition of $g(A_1\epsilon\pi)$ is half that of these authors, and is the same as that in Ref. 15.

[22]Alternatively, one may postulate that the universality (31) holds in the infinite-momentum frame.

[23]The subtraction terms would obey $a_N^T=a_\Lambda^T=\cdots$ and $a_\pi=a_K=a_\eta$, if they arise from a unitary singlet piece of the stress tensor.

[24]I is the isospin and J is the angular momentum. Note that ρ dominance of the pion form factor gives a reasonable picture at small q^2.

[25]E.g., see H. Munczek *et al.*, Phys. Rev. 145, 1154 (1968).

[26]See M. Gell-Mann, Ref. 6; P. Carruthers, Phys. Rev. D 2, 2265 (1970); L. N. Chang and P. G. O. Freund, Ann. Phys. (N.Y.) 61, 182 (1970).

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Phenomenological Analysis for the Electromagnetic Form Factor of the Pion*

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(Received 22 January 1971)

Using the results from electron-positron colliding-beam experiments, a phenomenological analysis has been made by means of the continuous-dispersion sum rules for the pion electromagnetic form factor. Results such as $\Gamma_\rho=0.110$ GeV, $a_1(\pi\pi)=0.028\mu^{-2}$, $|F_\pi(m_\rho^2)|^2=48.8$, $r_\pi=0.62$ F, $\delta\mu$ (the pion mass difference) $=4.3$ MeV have been obtained, and sum rules involving amplitudes accessible to $e^+e^-\to\pi^+\pi^-$ and $\pi e\to\pi e$ processes tested.

In this note, we present a phenomenological analysis for the electromagnetic form factor of the pion,[1] $F_\pi(s)$, by means of continuous-dispersion sum rules.[2]

By definition, a phenomenological analysis makes use of available experimental data only, without attempting to understand the underlying dynamics. For the dispersion approach, phenomenological parametrizations in the experimentally unfeasible regions, e.g., near threshold or at an unattainable high energy, are also necessary.

We begin with the following sum rules for $F_\pi(s)$, and its derivative with respect to s, at $s=0$:

$$F_\pi(0)=\frac{s_0^{\beta}}{\pi}\int_{s_0}^{\infty}\frac{ds}{s}\,\frac{\cos\pi\beta\,\mathrm{Im}F_\pi(s)+\sin\pi\beta\,\mathrm{Re}F_\pi(s)}{(s-s_0)^{\beta}},\tag{1}$$

$$F_\pi'(0)=-\frac{\beta}{s_0}F_\pi(0)+\frac{s_0^{\beta}}{\pi}\int_{s_0}^{\infty}\frac{ds}{s^2}\times\frac{\cos\pi\beta\,\mathrm{Im}F_\pi(s)+\sin\pi\beta\,\mathrm{Re}F_\pi(s)}{(s-s_0)^{\beta}},\tag{2}$$

where $0\le\beta<1$, $s_0=4\mu^2$ ($\mu=$pion mass). The sensitive threshold factor $1/(s-s_0)^{\beta}$ in Eq. (2) can be avoided, if desired, by subtracting off the representation for $F_\pi(0)$ in Eq. (1), yielding (for $0\le\beta\le1$)

$$F'(0)=\frac{1-\beta}{s_0}F_\pi(0)-\frac{s_0^{\beta-1}}{\pi}\int_{s_0}^{\infty}ds\frac{(s-s_0)^{1-\beta}}{s^2}\times[\cos\pi\beta\,\mathrm{Im}F_\pi(s)+\sin\pi\beta\,\mathrm{Re}F_\pi(s)].\tag{2'}$$

Both Eqs. (1) and (2) are valid under the assumption that $F_\pi(s)\xrightarrow[|s|\to\infty]{}0$. If a definite asymptotic behavior like $1/s$ for $F_\pi(s)$ is used, as in Eq. (5′) below, we will be able to derive two more useful dispersion sum rules[3] (for $0<\beta\le1$):

$$\int_{s_0}^{\infty}ds\frac{\cos\pi\beta\,\mathrm{Im}F_\pi(s)+\sin\pi\beta\,\mathrm{Re}F_\pi(s)}{(s-s_0)^{\beta}}=0,\tag{3}$$

$$\int_{-\infty}^{0}ds\frac{\sin\pi\beta\,\mathrm{Re}F_\pi(s)}{[s(s-s_0)]^{\beta}}=\int_{s_0}^{\infty}ds\frac{\cos\pi\beta\,\mathrm{Im}F_\pi(s)+\sin\pi\beta\,\mathrm{Re}F_\pi(s)}{[s(s-s_0)]^{\beta}}.\tag{4}$$

Numerical values for $F_\pi(s)$ in the *timelike* region ($s>0$) can be measured either by the process[4] $\pi N \to e^+e^- N$ or by $e^+e^- \to \pi^+\pi^-$.[5] At the moment, the colliding-beam experiment[5] has been performed and has yielded informative results. We therefore concentrate on the latter process.

For $s_0 \le s \le 16\mu^2$, it is well known that $F_\pi(s) = |F_\pi(s)|e^{i\delta_1}$, where δ_1 is the p-wave phase shift of elastic $\pi\pi$ scattering. Since the p-wave $\pi\pi$ scattering is dominated by the ρ-meson resonance, we write, even beyond $16\mu^2$,

$$F_\pi(s) = \frac{f(s)}{s_r - s - im_r\Gamma_r(q/q_r)^3(m_r/\sqrt{s})}, \qquad (5)$$

where m_r (Γ_r) is the mass (width) of the ρ meson, $s_r = m_r^2$, $s = 4(q^2+\mu^2)$, and $f(s)$ is an unknown real function. To date, many different forms for $F_\pi(s)$ have been proposed[6]; we find that Eq. (5) is simple but satisfactory, for the following reasons:

(1) As far as the process $e^+e^- \to \pi^+\pi^-$ is concerned, the point $s=0$ lies outside the physical region; hence, the factor $1/\sqrt{s}$ does not generate a singularity, nor is the normalization $F_\pi(0)=1$ in Eq. (5) necessary.

(2) The correct q^3 threshold behavior yields an expression for the p-wave $\pi\pi$ scattering length[7] $a_1 = s_r\Gamma_r/8q_r^5$, independent of the value of $f(s_0)$.

(3) Even with the usual assumption that $f(s)$ = constant = C, we can extrapolate $F_\pi(s)$ to $s=\infty$ without difficulty, with both $\mathrm{Im}F_\pi(s)$ and $\mathrm{Re}F_\pi(s)$ following a $1/s$-type behavior [the constant C can be determined from the dispersion sum rules (1) and (2), as will be done below]. The problem that $F_\pi(s)$ has a distant singularity [for constant $f(s)$] at negative s is also not present.

Thus our phenomenological parametrization for $F_\pi(s)$ in the range $s_0 \le s \le \infty$ is given by

$$F_\pi(s) = \frac{C}{(s_r - s) - i\gamma(s-s_0)(1-s_0/s)^{1/2}}, \qquad (5')$$

with $\gamma = s_r\Gamma_r/(s_r - s_0)^{3/2}$, a dimensionless constant.

In the *spacelike* region ($s<0$), the experiment $\pi e \to \pi e$ is just under way.[8] The pion electroproduction experiment[9] did give a (model-dependent) pion form factor of the form

$$F_\pi(s) = 1/(1 - s/s_V) \qquad (6)$$

for $-0.4\ (\mathrm{GeV}/c)^2 \le s \le 0$ and $s_V = 0.56^2\ (\mathrm{GeV}/c)^2$. We shall adopt a similar form factor, namely, $F_\pi(k^2) = 1/(1 + k^2/m_V^2)$, where $k^2 \equiv -s$, but take into account the connection between $F_\pi(k^2)$ and the pion mass difference formula of Riazuddin,[10]

$$\delta\mu = -\frac{i\alpha}{8\pi^3\mu}\left[\int \frac{d^4k}{k^2}\frac{k^2+4\mu^2}{k^2-2p\cdot k}F_\pi{}^2(k^2) + 2\int\frac{d^4k}{k^2}F_\pi{}^2(k^2)\right],$$

which, after making a Wick rotation, reads[11]

$$\delta\mu = \frac{\alpha}{16\pi\mu^3}\int_0^\infty dk^2\left[\frac{(k^2+4\mu^2)^{3/2}}{k} - k^2\right]F_\pi{}^2(k^2), \qquad (7)$$

or, following the Feynman method of symmetric integration, becomes[12]

$$\delta\mu = \frac{\alpha}{8\pi\mu}m_V{}^2\left[3 + \tfrac{3}{2}x - \tfrac{8}{3}x^2 - 3x(1 + \tfrac{4}{3}x + 3x^2)\ln x\right], \qquad (8)$$

where $\alpha = \frac{1}{137}$ and $x = \mu^2/m_V{}^2$.

We proceed with the phenomenological analysis as follows. (We use the natural units $\hbar = c = \mu = 0.140$ GeV = 1.) First the ρ mass is fixed at 0.765 GeV.[13] This permits us to plot several graphs of $|F_\pi(s)|^2$ from Eq. (5′), for several Γ_r (ranging from 125 to 100 MeV) and C (from $29.0\mu^2$ to $32.5\mu^2$). A few graphs are obtained, compatible with the data points of Orsay[5] and Novosibirsk,[5] but differing in Γ_r and C among themselves.

To allow for further selection, we employ the dispersion sum rules (1)–(3).

The superconvergent sum rule (3) is independent of C. Numerical studies with Eq. (5′) reveal that in general *the smaller the width, the better the sum rule*. In particular, for each Γ_r, although the left-hand side always remains around zero as $\beta \to 1$, it is not negligible at $\beta = \frac{1}{2}$ (where it becomes a Gilbert-type sum rule). This may be an indication that resonances other than the ρ meson are necessary for the description of $F_\pi(s)$.

Ideally, the right-hand integral of Eq. (1) should, for given Γ_r, be stationary as β is varied continuously; the normalization $F_\pi(0)=1$ then determines a value of C corresponding to that Γ_r. In practice, C varies slightly with β. Thus when Γ_r is varied over the allowed widths (from 125 to 100 MeV), Eq. (1) leads to a band of adoptable C, lying between $29.0\mu^2$ and $32.5\mu^2$. This result has already been used to plot those graphs of $|F_\pi(s)|^2$ mentioned earlier.

The inhomogeneous equations (2) and (2′), finally, serve to determine C and $F'(0)$ easily, because it happens that these two sum rules are nearly independent of Γ_r [except for low β in Eq. (2′)]. This is due to the intrinsic property of Eq. (5′) and to the fact that the two integrals are highly convergent. The right-hand sides of Eqs. (2) and (2′) would be the same if Eq. (5′) were the true pion electromagnetic form factor, which is not the case here. Indeed, requiring the constancy of $F_\pi'(0)$ as a function of β, our best result from Eq. (2) is $C = 29.0\mu^2$, $F'(0) = 0.031\mu^{-2}$, while from Eq. (2′) we obtain $C = 31.0\mu^2$, $F'(0) = 0.033\mu^{-2}$. The over-all analysis therefore yields $C = 30.0\mu^2$ and $F'(0) = 0.032\mu^{-2}$ as our best estimate.

With $C = 30.0\mu^2$, we now go back to the graphs

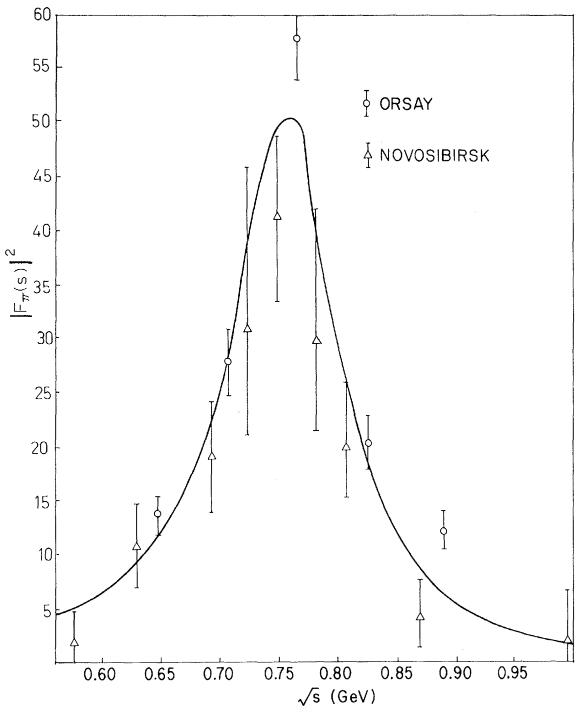

FIG. 1. A plot of $|F_\pi(s)|^2$ for Eq. (5′), with $C=30.0\mu^2$, $\Gamma_r=0.110$ GeV. Experimental data points (Ref. 5) are compared.

we had plotted earlier and obtain $\Gamma_r=110$ MeV.[14] This width gives $a_1=0.028\mu^{-2}$. The diagram for $|F_\pi(s)|^2$ is shown in Fig. 1, and is compared with the data points of Ref. 5. Other relevant quantities are $|F_\pi(m_\rho)|^2=(C/m_r\Gamma_r)^2=48.8$, $F_\pi(s_0)=C/(s_r-s_0)=1.16$.

The slope $F'_\pi(0)=0.032\mu^{-2}$, on the other hand, yields a mass $m_V=0.780$ GeV in Eq. (6), and gives $r_\pi=0.62$ F, $\delta\mu=4.3$ MeV from both Eqs. (7) and (8). The experimental pion mass difference $\delta\mu=4.6$ MeV required $r_\pi=0.60$ F, $m_V=0.810$ GeV, $F'_\pi(0)=0.030\mu^{-2}$.

Combining both timelike parameters $C=30.0\ \mu^2$,

TABLE I. Test of the sum rule (4). On the right-hand side (RHS), Eq. (5′) with $C=30.0\mu^2$, $\Gamma_r=0.110$ GeV is used. On the left-hand side (LHS), use is made of Eq. (6), together with $s_V=m_V^2$, $m_V=0.780$ GeV (present analysis), $m_V=765$ GeV (ρ-meson dominance), and $m_V=0.560$ GeV (pion electroproduction experiment).

	RHS		LHS	
β		$m_V=0.780$	$m_V=0.765$	$m_V=0.560$
1.0	0.79	0.79	0.79	0.79
0.9	1.18	1.15	1.15	1.13
0.8	1.50	1.48	1.47	1.42
0.7	1.93	1.91	1.90	1.78
0.6	2.60	2.57	2.56	2.30
0.5	3.70	3.65	3.61	3.08

$\Gamma_r=0.110$ GeV [Eq. (5′)], and the spacelike $s_V=m_V^2=(0.810\text{ GeV})^2$ [Eq. (6)], we are ready to test the sum rule (4). The result is satisfactory, as shown in Table I. Also shown in Table I are the results from using the ρ-meson dominance ($m_V=0.765$ GeV) and the pion electroproduction experiment ($m_V=0.580$ GeV) in the spacelike region. It is worth remarking that no knowledge of $F_\pi(s)$ in the range $0<s<s_0$ is involved. This energy region is unphysical for both $e^+e^-\to\pi^+\pi^-$ and $\pi e\to\pi e$, but it can, at least in principle, be explored by the reaction $\pi N\to e^+e^-N$.[4]

In conclusion, we have studied the pion electromagnetic form factor by means of experimental data and dispersion sum rules.[15] Our analysis leads to the well-known result[14] $\Gamma_r=0.110$ GeV and $C\simeq m_r^2$, although the machinery used is completely different. We look forward to the high-energy colliding-beam experiment[16] and direct πe scattering[8] now under way for a better understanding of $F_\pi(s)$, in view of its connection with many interesting physical quantities.

We thank S. Okubo for a communication on the $F_\pi(s)$ threshold behavior, and D. Clysdale for computational assistance.

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*Work supported in part by the National Research Council of Canada.

[1]See, for example, D. V. Shirkov *et al.*, *Dispersion Theory of Strong Interactions at Low Energy* (North-Holland, Amsterdam, 1969), Chap. 4.

[2]Y. C. Liu and S. Okubo, Phys. Rev. Letters 19, 190 (1967); Y. C. Liu and I. J. McGee, Phys. Rev. D 2, 166 (1970).

[3]The particular case at $\beta=\frac{1}{2}$ has been obtained earlier by L. A. Khalfin, Yadern. Fiz. 7, 876 (1968)[Soviet J. Nucl. Phys. 7, 529 (1968)].

[4]D. A. Geffen, Phys. Rev. 125, 1745 (1962); M. P. Rekalo, Yadern. Fiz. 1, 1066 (1965) [Soviet J. Nucl. Phys. 1, 760 (1965)].

[5]J. E. Augustin *et al.*, Phys. Letters 28B, 508 (1969); V. L. Auslander *et al.*, Yadern. Fiz. 9, 114 (1969) [Soviet J. Nucl. Phys. 9, 69 (1968)]. We refer to them as Orsay and Novosibirsk, respectively, hereafter.

[6]See the discussion in M. Roos and J. Pisut, Nucl. Phys. B10, 563 (1969); G. J. Gounaris and J. J. Sakurai, Phys. Rev. Letters 21, 244 (1968).

[7]M. G. Olsson, Phys. Rev. 162, 1338 (1967); L. S. Brown and R. L. Goble, Phys. Rev. Letters 20, 346 (1968). We define a_1 as in $(2q^3/\sqrt{s})\cot\delta_1=1/a_1+\cdots$.

[8]G. B. Lubkin, Phys. Today 23 (No. 9), 18 (1970).

[9]C. Mistretta *et al.*, Phys. Rev. 184, 1487 (1969); C. W. Akerlof *et al.*, *ibid.* 163, 1482 (1967); F. A. Berends, Phys. Rev. D 1, 2590 (1970).

[10]Riazuddin, Phys. Rev. 114, 1184 (1959); Fayyazuddin and Riazuddin, Nuovo Cimento 54A, 520 (1968).

[11]Y. Taguchi and K. Yamamoto, Progr. Theoret. Phys. (Kyoto) 38, 1152 (1967).

[12]D. J. Hall, Phys. Rev. Letters 6, 31 (1961). Notice that Hall's result is correct only to the order x. We also have used Eq. (6) for both spacelike and timelike k^2 to obtain Eq. (8), in contradiction with the spirit of the present approach toward $F_\pi(s)$.

[13]The ρ mass could have been treated as a parameter to be determined from the data in the same way as the ρ width. Its value is less controversial, however, so it was not varied throughout the calculation.

[14]Unfortunately, both results are not new. $C=30\mu^2 \simeq m_\rho{}^2$ corresponds to the ρ-meson dominance together with an ordinary-type Breit-Wigner formula for $F_\pi(s)$. $\Gamma_r = 110$ MeV is the best fit from both Orsay and Novosibirsk, Ref. 5.

[15]Our treatment of the timelike and spacelike regions is different for two reasons: (1) Both the right-hand and left-hand (whether present or not) cuts are independent in the dispersion approach, and (2) an analytic formula like the Veneziano one does not permit a satisfactory continuation from the forward to backward direction, say.

[16]G. B. Lubkin, Phys. Today 23 (No. 12), 17 (1970).

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Exact Sum Rule for Transition Amplitudes and $K_L - K_S$ Decay

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(Received 11 January 1971)

It is pointed out that there is just one exact form of the sum rule holding for specific transition amplitudes. The assumptions and approximations implicit in the various usual formulations are analyzed on the basis of this result. The adequate description of the time dependence is discussed.

The "unitarity sum rule" of Bell and Steinberger[1] plays an important role in the analysis of the K^0 system. It is related to the Weisskopf-Wigner treatment[2] of this decay problem. Starting from a phenomenological S matrix, further sum rules have been obtained stating that the result is different[3] or that the phase information agrees.[4] Subsequently, several authors[5-8] have come to the conclusion that the rules should agree completely. The recent effort spent on getting a satisfactory description of the K^0 system also includes additional scattering treatments[9-11] and an investigation[12] of the deviations from the Weisskopf-Wigner equations caused by postulating the Bell-Steinberger sum rule.

In the following, it is shown that the difficulties indicated above disappear by using recent results of the present author on specific time-dependent probabilities[13] and on the detailed structure of related amplitudes.[14] In particular, the exact form of the sum rule can be written down. Since the various amplitudes of interest are properly specified and given explicitly, the assumptions implicit in the usual scattering treatments as well as the nature of the conventional Weisskopf-Wigner approach can be analyzed. The adequate description of the time dependence is derived from first principles.

First we review some general relations.[14] Defining

$$P=\sum_\nu |\chi_\nu\rangle\langle\chi_\nu| ,$$

with $\langle\chi_\mu|\chi_\nu\rangle=\delta_{\mu\nu}$, $Q=1-P$, and $g(z)=(QHQ-z)^{-1}$, a biorthogonal basis with

$$\langle v_m(z)|u_n(z)\rangle=\delta_{mn}$$

and

$$P=\sum_n |u_n(z)\rangle\langle v_n(z)|$$

is introduced which diagonalizes

$$P(H-HQg(z)H)P=\sum_n \lambda_n(z)|u_n(z)\rangle\langle v_n(z)| .$$

If the χ_ν are (proper) eigenvectors of an "unperturbed Hamiltonian" H_r, writing the total Hamiltonian $H=H_r+V_r$, one has

$$P(H-HQgH)P=PH_r+P(V_r-V_rQgV_r)P .$$

We now put $z=E+i0$ and introduce

$$E_n(E)=\mathrm{Re}\lambda_n(E+i0), \quad \Gamma_n(E)=-2\,\mathrm{Im}\,\lambda_n(E+i0) .$$

Then the amplitude which turns out to be essential for the description of decays, involving an outgoing scattering state $\psi^-_{E,\beta}$ (improper eigenvector of H), gets the form

$$\langle\psi^-_{E,\beta}|u_n\rangle=A_{\beta,n}(E-E_n+\tfrac{1}{2}i\Gamma_n)^{-1} , \qquad (1)$$