$$-\tfrac{1}{3}+\tfrac{4}{3}\langle 1/N\rangle \le \int_1^\infty [F^p(\omega)-F^n(\omega)]\frac{d\omega}{\omega^2} \le \tfrac{1}{3}-\tfrac{2}{3}\langle 1/N\rangle \qquad \text{(b)}$$

and

$$\tfrac{4}{3}-\tfrac{1}{3}\langle N\rangle \le \int_1^\infty [F^p(\omega)-F^n(\omega)]\frac{d\omega}{\omega} \le -\tfrac{2}{3}+\tfrac{1}{3}\langle N\rangle\,. \qquad \text{(c)}$$

The λ and $\bar{\lambda}$ quarks have I^3Y value zero and so these expressions give us an overestimation, thus making the bounds too weak.

[8]There are several other inequalities that can be derived from Eq. (3.20) but they do not give us new information.

[9]Note that Eq. (2.4) has not been used in Sec. III C. Only Eqs. (3.1), (3.4), (3.5), and (3.10) make use of the assumption that the momentum distribution of the N partons are symmetric.

[10]We have found that this relation can also be derived with the help of Eq. (3.7) and the following inequality:

$$\tfrac{1}{9}\langle N\rangle+\tfrac{1}{3} \le \int_1^\infty F^n(\omega)\frac{d\omega}{\omega} \le \tfrac{4}{9}\langle N\rangle-\tfrac{2}{3}\ .$$

[11]C. H. Llewellyn Smith, Nucl. Phys. 17B, 277 (1970).
[12]M. Gourdin, CERN Report No. CERN-TH-1266 (unpublished).

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General Treatment of the Breaking of Chiral Symmetry and Scale Invariance in the $SU(3)$ σ Model*

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We study the model of nine scalar and nine pseudoscalar fields interacting by means of the most general nonderivative chiral $SU(3)\times SU(3)$ invariant and any particular symmetry-breaking term. Two basic "generating" equations which express the complete content of chiral symmetry are derived. The masses and coupling constants of arbitrary order for this model are simply found by differentiating the "generating" equations an arbitrary number of times and using an equation which expresses the stability of the "ground" state. In this way, previous results on this model can be easily recaptured and a systematic framework for the investigation of different symmetry-breaking terms is provided. Numerical estimates are made for scalar-meson widths, $\pi\pi$ and πK scattering lengths, and $\eta'\to\eta 2\pi$ decays. The consequences of imposing scale invariance on the invariant part of the interaction are also investigated by writing down a scale invariance "generating equation." Finally, we discuss the relation between our approach and the method of using the divergences of currents and trace of the energy-momentum tensor.

I. INTRODUCTION

The subject of chiral $SU(3)\times SU(3)$ symmetry breaking[1] has recently been one of the most actively pursued branches of strong-interaction theory. There is great interest in this field not only because it searches for a way to estimate corrections to the interesting "current-algebra" results but also because it is hoped that the answer to the symmetry-breaking problem will elucidate some deep mysteries of elementary-particle structure.

Now, once we depart from the exact symmetry limit of any theory, a large number of alternatives usually present themselves. Therefore, in order not to get lost in a maze of complications, it is normally desirable to study a relatively simple model which contains (it is hoped) the key features of the problem. For the case of $SU(3)\times SU(3)$ breaking, the model which is generally taken as a prototype is the so-called "$SU(3)$ σ model"[2] which contains nine pseudoscalar and nine scalar fields transforming *linearly* under the chiral $SU(3)\times SU(3)$ group of transformations.

The advantage of the $SU(3)$ σ model over the quark-model approach to symmetry breaking (as exemplified by the recent work of Gell-Mann, Oakes, and Renner[3] and its descendants) is that everything is explicit in the case of the σ model so that results can be obtained relatively easily. Otherwise the structure of the two models, as we shall illustrate, is very similar. One example of the practical advantage of the σ model lies in the specification of the "ground state" or "vacuum state" of the system. In the σ-model approach the symmetry breaking of the "vacuum" is correlated to the choice of symmetry-breaking interaction by means of a "stability" or "extremum" equation. On the other hand, in the quark-model approach this physical condition is more difficult to enforce and often it is just assumed that the "vacuum" has a certain symmetry property [e.g., $SU(3)$ invariance] which may or may not be consistent.

One variation of the $SU(3)$ σ model suppresses the scalar particles by essentially allowing their masses to become infinite. The transformation properties of the pseudoscalars then become complicated *nonlinear*[4,5] ones. Since this procedure was introduced precisely to suppress reference to symmetry-breaking efforts, it can be appreciated that (while better for deriving the current-algebra "theorems") it is less interesting for studying symmetry breaking.

Since, as pointed out above, the $SU(3)$ σ model is so well worth study, it has been investigated by many authors.[6] However, the structure of this model, though simple, is far from trivial and it would seem that (except for the discussion of the mass spectrum in paper I) a general approach has not yet been given. In the present paper we propose to do this, if we are permitted the freedom of calling a treatment in the semiclassical framework a general one.

Here we shall investigate the $SU(3)$ σ model with the most general nonderivative chiral $SU(3)\times SU(3)$-invariant interaction and some particular symmetry-breaking terms. It is not necessary for us to specify the detailed form of the invariant part of the interaction. All our results are obtained from the requirements of chiral invariance, together with an extremum condition which implies the stability of the "ground state."[7] We shall derive two basic equations which are "generating" relations in the sense that the formulas for masses and coupling constants of any order can be derived essentially by just differentiating these basic equations any number of times.

As detailed applications of this procedure we discuss the mass spectrum of the theory (which is found by differentiating the basic equations once), the trilinear coupling constants (which come from differentiating twice), and the quadrilinear coupling constants (differentiating three times). For the simplest choice of a symmetry-breaking term, numerical estimates are made of scalar-meson widths, $\pi\pi$ and πK scattering lengths, and $\eta'\to\eta 2\pi$ decay.

We stress that the present formalism provides a systematic way of investigating different choices of symmetry-breaking terms. The addition of other particles than spin-0 mesons into the theory is clearly a desirable future step; the present method may also be extended in this direction.

It turns out that chiral symmetry alone is not sufficient to relate all the masses and coupling constants to each other. The masses of most scalar mesons and the coupling constants for vertices involving only isoscalar particles can be chosen freely. On the other hand, it is possible to try to get some information on these objects by imposing additional symmetries on the chiral-invariant part of the interaction. One promising choice which can be tested is scale invariance.[8] This subject is also taken up here and we derive a generating relation to express the consequences of scale invariance. In this case our generating relation is nothing but Euler's formula for homogeneous functions. The consequences of scale invariance on the scalar-meson masses and trilinear coupling constants are investigated in detail. Generally the predictions seem to be reasonable, but they involve quantities which are not well known experimentally.

In all of our treatment it is unnecessary to introduce vector currents, axial-vector currents, or the energy-momentum tensor of the system. However, alternative approaches to the subject of symmetry breaking deal with the current divergences[9] and the trace of the energy-momentum tensor.[8] For our model the present approach is far simpler. Nevertheless, for the purposes of comparison and going beyond the framework of the present model, we also show how our relations can be derived from these considerations.

The basic equations expressing the chiral symmetry of the problem are derived in Sec. II. Application to the mass spectrum, trilinear couplings, quadrilinear couplings, and arbitrary couplings are given in Secs. III, IV, V, and Appendix B, respectively. Scale invariance is treated in Secs. VI and VII. Finally, the currents and energy-momentum tensor are discussed in Appendices A and C.

II. BASIC EQUATIONS

We will consider a theory constructed out of nine scalar fields, S_a^b $(a,b=1,2,3)$, and nine pseudoscalar fields, ϕ_a^b $(a,b=1,2,3)$. These fields will be allowed to interact by means of the most general nonderivative chiral $SU(3)\times SU(3)$ terms as well as some particular (but initially unspecified) symmetry-breaking (SB) term.

The appropriate Lagrangian density is then, with matrix notation for the fields,

$$\mathfrak{L} = -\tfrac{1}{2}\mathrm{Tr}(\partial_\mu\phi\partial_\mu\phi) - \tfrac{1}{2}\mathrm{Tr}(\partial_\mu S\partial_\mu S) - V_0 - V_{\mathrm{SB}}\,. \tag{2.1}$$

(We are using the Dirac-Pauli metric.) In (2.1) V_0 stands for the most general chiral invariant while V_{SB} is the symmetry-breaking term. The quantity V_0 may be considered to be an arbitrary function of the following independent invariants:

$$I_1 = \mathrm{Tr}(MM^\dagger),$$

$$I_2 = \mathrm{Tr}(MM^\dagger MM^\dagger),$$

$$I_3 = \mathrm{Tr}(MM^\dagger MM^\dagger MM^\dagger),$$
$$I_4 = 6(\det M + \det M^\dagger), \tag{2.2}$$

where the 3×3 matrices M and $M^\dagger$ are taken to transform according to the (3,3*) and (3*,3) representations of $SU(3)\times SU(3)$, respectively, and are related to S and ϕ by

$$M = S + i\phi,$$
$$M^\dagger = S - i\phi. \tag{2.3}$$

A discussion of the invariants in (2.2) was given in paper I, which treats the mass spectrum of this model in great detail. Although we shall constantly refer to paper I, an attempt is being made to keep the present paper as self-contained as possible.

To find the experimental consequences of the expression (2.1), we shall adopt a semiclassical approach. That is to say, we first consider $\mathcal{L}$ as representing a classical system of coupled fields. As in any "small-oscillation" theory, we must first find the "equilibrium point" (or "ground state") of the system by imposing the extremum conditions

$$\left\langle \frac{\partial V_0}{\partial \phi} \right\rangle_0 + \left\langle \frac{\partial V_{SB}}{\partial \phi} \right\rangle_0 = 0, \tag{2.4a}$$

$$\left\langle \frac{\partial V_0}{\partial S} \right\rangle_0 + \left\langle \frac{\partial V_{SB}}{\partial S} \right\rangle_0 = 0, \tag{2.4b}$$

where the notation $\langle\ \rangle_0$ means that the enclosed expression is evaluated at the equilibrium point. Next, the fields must be expanded about their equilibrium values, so we introduce the "physical" objects

$$\tilde{\phi} = \phi - \langle \phi \rangle_0, \tag{2.5a}$$

$$\tilde{S} = S - \langle S \rangle_0, \tag{2.5b}$$

and expand the Lagrangian density as

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}(\partial_\mu \tilde{\phi}\partial_\mu \tilde{\phi}) - \tfrac{1}{2}\mathrm{Tr}(\partial_\mu \tilde{S}\partial_\mu \tilde{S}) - \tfrac{1}{2}\sum_{a,b,c,d}\left(\left\langle \frac{\partial^2 V}{\partial \phi_a^b \partial \phi_c^d}\right\rangle_0 \tilde{\phi}_a^b \tilde{\phi}_c^d + \left\langle \frac{\partial^2 V}{\partial S_a^b \partial S_c^d}\right\rangle_0 \tilde{S}_a^b \tilde{S}_c^d\right)$$
$$-\tfrac{1}{2}\sum_{a,b,c,d,e,f}\left\langle \frac{\partial^3 V}{\partial S_a^b \partial \phi_c^d \partial \phi_e^f}\right\rangle_0 \tilde{S}_a^b \tilde{\phi}_c^d \tilde{\phi}_e^f - \frac{1}{4!}\sum_{a,b,c,d,e,f,g,h}\left\langle \frac{\partial^4 V}{\partial \phi_a^b \partial \phi_c^d \partial \phi_e^f \partial \phi_g^h}\right\rangle_0 \tilde{\phi}_a^b \tilde{\phi}_c^d \tilde{\phi}_e^f \tilde{\phi}_g^h + \cdots, \tag{2.6}$$

with

$$V = V_0 + V_{SB}. \tag{2.7}$$

Equation (2.6) is the one that will be quantized and whose consequences will be investigated in the well-known tree approximation. We see from (2.6) that the set of coefficients

$$\left\langle \frac{\partial^2 V}{\partial \phi_a^b \partial \phi_c^d}\right\rangle_0,$$

for example, represents the matrix of pseudoscalar-meson squared masses. The terms involving more derivatives of V are the coefficients of the corresponding multilinear interaction vertices. We have explicitly shown two of the interaction terms which will be calculated here.

It will be seen that by proceeding along the line suggested above, the usual "current-algebra" or "phenomenological-Lagrangian" results can be quickly recovered. The advantage of this approach is that it permits us to easily and systematically investigate different kinds of symmetry-breaking terms as well as to obtain some insight into the structure of the theory.

Parity invariance for this model requires that the "equilibrium" value of pseudoscalar objects vanish, i.e.,

$$\langle \phi \rangle_0 = \left\langle \frac{\partial V_0}{\partial \phi} \right\rangle_0 = \left\langle \frac{\partial V_{SB}}{\partial \phi} \right\rangle_0 = 0. \tag{2.8}$$

Thus $\tilde{\phi} = \phi$, and (2.4a) is identically satisfied.

The "equilibrium" values of the scalar fields need not vanish, of course, and it is just this fact which gives theories of this type their unique aspect. A brief study of the extremum condition (2.4b) (given in paper I) shows that if the 3×3 matrix $\langle \partial V_{SB}/\partial S_a^b\rangle_0$ can be simultaneously diagonalized with $\langle S_a^b\rangle_0$, we may choose

$$\langle S_a^b\rangle_0 = \delta_a^b \alpha_a \quad \text{(no sum)}, \tag{2.9}$$

where δ_a^b is the Kronecker δ and the α_a are three real constants characterizing the "ground state" of this model. Isotopic-spin invariance requires $\alpha_1 = \alpha_2 \equiv \alpha$, and in this limit it is convenient to define a quantity[5]

$$w = \alpha_3/\alpha \tag{2.10}$$

whose deviation from unity expresses the amount of $SU(3)$ noninvariance of the "ground state."

The basic equations which give nontrivial experimental information for this model follow directly from the chiral $SU(3)\times SU(3)$ invariance of V_0. From a mathematical standpoint we consider transformations in two separate $SU(3)$ groups – a "left-handed" $SU(3)$ and a "right-handed" $SU(3)$. For our present purposes it is somewhat easier to consider infinitesimal transformations corresponding to the left-plus-right (or vector) and left-minus-right (or axial-vector) transformations. Explicitly, under a "vector" transformation, the change in the fields is given by

$$\delta\phi = [E_V, \phi], \quad \delta S = [E_V, S], \tag{2.11}$$

where E_V is an arbitrary 3×3 infinitesimal matrix satisfying

$$E_V^\dagger = -E_V \,. \tag{2.12}$$

[The requirement (2.12) comes from demanding that $1+E_V$ be unitary to first order in E_V.] The change in the fields under an "axial-vector" transformation is given by

$$\delta\phi = -i[E_A, S]_+ , \quad \delta S = i[E_A, \phi]_+ , \tag{2.13}$$

where E_A again satisfies (2.12). We must point out that the transformations (2.11) and (2.13) actually correspond to the $U(3)\times U(3)$ group rather than $SU(3)\times SU(3)$. For a unitary *unimodular* transformation the requirement that $\det(1+E)=1$ gives

$$\mathrm{Tr}(E)=0 \tag{2.14}$$

as a condition that must be satisfied in addition to (2.12). All four quantities listed in (2.2) are invariant under the "vector" transformation of (2.11) with no restriction on $\mathrm{Tr}(E)$. However, in the case of the "axial-vector" transformations of (2.13), the quantity I_4 is invariant only when (2.14) is imposed. In other words, I_1, I_2, and I_3 are invariant under the full $U(3)\times U(3)$ group while I_4 is only invariant under its $SU(3)\times SU(3)$ subgroup. We may easily demonstrate this with the aid of the following matrix identity:

$$\delta(\det M) = (\det M)\mathrm{Tr}(M^{-1}\delta M), \tag{2.15}$$

where M is some matrix. Thus under the transformation (2.13),

$$\begin{aligned}\delta I_4 &= 6\{\det M\,\mathrm{Tr}(M^{-1}[E_A, M]_+) \\ &\quad - \det M^\dagger\,\mathrm{Tr}(M^{\dagger -1}[E_A, M^\dagger]_+)\} \\ &= 12(\mathrm{Tr}E_A)(\det M - \det M^\dagger),\end{aligned} \tag{2.16}$$

which is clearly vanishing only when $\mathrm{Tr}(E_A)=0$. In deriving (2.16), we of course used (2.3).

Now the invariance of V_0 under the "vector" transformation (2.11) is expressed by the equation

$$\begin{aligned}\delta V_0 &= \mathrm{Tr}\left(\frac{\partial V_0}{\partial\phi}\,\delta\phi + \frac{\partial V_0}{\partial S}\,\delta S\right) \\ &= \mathrm{Tr}\left(E_V\left(\left[\phi, \frac{\partial V_0}{\partial\phi}\right] + \left[S, \frac{\partial V_0}{\partial S}\right]\right)\right) = 0,\end{aligned} \tag{2.17}$$

where E_V is restricted by (2.12) but not (2.14). Since E_V is a 3×3 matrix of complex numbers which is restricted by nine conditions in (2.12), it requires nine real numbers for its specification. These nine real numbers may be chosen arbitrarily so that (2.17) leads to the following nine equations:

$$[\phi, \partial V_0/\partial\phi] + [S, \partial V_0/\partial S] = 0\,. \tag{2.18}$$

For the case of the "axial-vector" transformations (2.13), V_0 will only be invariant when E_A is restricted by (2.14) in addition to (2.12). However, for technical purposes, it is more convenient to relax the restriction (2.14) so that V_0 will not be invariant; its change will be determined by

$$\delta V_0 = \frac{\partial V_0}{\partial I_4}\,\delta I_4, \tag{2.19}$$

since I_4 is the only quantity in (2.2) which is not invariant when (2.14) is relaxed. Substituting (2.16) into (2.19) gives, as before,

$$\begin{aligned}\delta V_0 &= i\,\mathrm{Tr}\left(E_A\left(\left[\frac{\partial V_0}{\partial S}, \phi\right]_+ - \left[\frac{\partial V_0}{\partial\phi}, S\right]_+\right)\right) \\ &= 12\frac{\partial V_0}{\partial I_4}(\det M - \det M^\dagger)\mathrm{Tr}(E_A)\,.\end{aligned} \tag{2.20}$$

Because E_A in (2.20) is restricted by nine rather than ten conditions, we have the nine equations

$$\left[\frac{\partial V_0}{\partial S}, \phi\right]_+ - \left[\frac{\partial V_0}{\partial\phi}, S\right]_+ = -12i\frac{\partial V_0}{\partial I_4}(\det M - \det M^\dagger)\times \underline{1}, \tag{2.21}$$

where $\underline{1}$ is the 3×3 unit matrix.

Equations (2.18) and (2.21) are the basic ones for our purposes. By differentiating them any number of times with respect to the fields and evaluating the resulting expressions at the "equilibrium" point with the help of (2.4b), we will obtain a large number of relations between the particle masses and interaction vertices of the theory. This will be illustrated in detail in the following sections. It will be seen that the relations are of the same form as the "current-algebra" ones in the sense that n-point vertices become related to $(n-1)$-point vertices. However, in the present approach it is straightforward to handle the effects of symmetry breaking.

We stress that V_0 may be a completely general chiral invariant. The relations we obtain will be the ones that follow from chiral invariance together with the extremum condition. It is not necessary for us to specify the explicit form of V_0 in terms of I_1, I_2, I_3, and I_4.

Until now the choice of V_{SB} has not been discussed. Clearly, the most interesting choices are simple ones, since the whole model is constructed on an $SU(3)\times SU(3)$-invariant framework. The *simplest* nontrivial one is

$$V_{SB} = -2(A_1 S_1^1 + A_2 S_2^2 + A_3 S_3^3), \tag{2.22}$$

where A_1, A_2, A_3 are three real constants. In the isotopic-spin limit, it is convenient to define[1]

$$A_1 = A_2 = g_0,$$
$$A_3 = g_0 + g. \tag{2.23}$$

The situation $g=0$, $g_0 \neq 0$ corresponds to the symmetry-breaking term *retaining* ordinary $SU(3)$ while the situation $g_0 = 0$, $g \neq 0$ corresponds to the symmetry-breaking term *retaining* chiral $SU(2)\times SU(2)$.

The Lagrangian density (2.1) with V_{SB} given by (2.22) has the same group-theoretical structure as the following quark model of Gell-Mann[3,10]:

$$\mathcal{L}(\text{quark}) = -\sum_{a=1}^{3} \bar{q}_a \gamma_\mu \partial_\mu q_a + [\text{nonderivative } SU(3)\times SU(3)\text{-invariant interaction}]$$
$$-m_1 \bar{q}_1 q_1 - m_2 \bar{q}_2 q_2 - m_3 \bar{q}_3 q_3 . \tag{2.24}$$

In (2.24) q_1, q_2, and q_3 stand for the three quark fields. The explicit correspondence between the two theories is expressed by requiring

$$A_a = K m_a ,$$
$$\langle \bar{q}^a q_a \rangle_0 = K' \alpha_a , \tag{2.25}$$

where the m_a are the quark "masses" and K and K' are constants of proportionality. The advantage of this approach over the quark-model one is that it is more explicit and we do not have to employ roundabout means to suppress reference to the fermionic quark fields.

It may be worthwhile to point out that when V_{SB} is given by (2.22), the extremum condition (2.4b) cannot be satisfied for $w=1$ [see (2.10)] and $g \neq 0$. In other words, the "vacuum" can *not* be exactly $SU(3)$-invariant when the symmetry-breaking term (2.22) violates $SU(3)$. This result follows immediately upon inspection of Eq. (22) in paper I. However, if more complicated choices of V_{SB} are made, it is possible[11] to have an $SU(3)$-invariant vacuum.

III. MASS SPECTRUM

Although the pseudoscalar and scalar-meson mass spectra were discussed in paper I, we shall derive the same results here with a quicker method and point out some connection with other work.

The constraints on the scalar-meson mass spectrum which follow from chiral symmetry are obtained immediately by differentiating the basic equation (2.18) with respect to the scalar field. [For this purpose it is convenient to write (2.18) with the $SU(3)$ tensor indices on the fields explicitly displayed.] Thus we find

$$\sum_c \left(\phi_a^c \frac{\partial^2 V_0}{\partial \phi_b^c \, \partial S_f^e} - \phi_c^b \frac{\partial^2 V_0}{\partial \phi_c^a \, \partial S_f^e} + S_a^c \frac{\partial^2 V_0}{\partial S_b^c \, \partial S_f^e} - S_c^b \frac{\partial^2 V_0}{\partial S_c^a \, \partial S_f^e} \right) + \delta_a^f \frac{\partial V_0}{\partial S_b^e} - \delta_e^b \frac{\partial V_0}{\partial S_f^a} = 0 . \tag{3.1}$$

Evaluating (3.1) at the "equilibrium" point with the aid of (2.4b), (2.8), (2.9), and the assumed relation

$$\left\langle \frac{\partial V_{SB}}{\partial S_a^b} \right\rangle_0 = \delta_b^a \left\langle \frac{\partial V_{SB}}{\partial S_a^a} \right\rangle_0 \tag{3.2}$$

gives the final result for $\langle \partial^2 V_0 / \partial S_b^a \, \partial S_f^e \rangle_0$:

$$(\alpha_a - \alpha_b) \left\langle \frac{\partial^2 V_0}{\partial S_b^a \, \partial S_f^e} \right\rangle_0 = \delta_a^f \delta_e^b \left(\left\langle \frac{\partial V_{SB}}{\partial S_b^b} \right\rangle_0 - \left\langle \frac{\partial V_{SB}}{\partial S_a^a} \right\rangle_0 \right). \tag{3.3}$$

The scalar-mass (squared) matrix which is to be compared with experiment is just

$$\left\langle \frac{\partial^2 V}{\partial S_b^a \, \partial S_f^e} \right\rangle_0 = \left\langle \frac{\partial^2 V_0}{\partial S_b^a \, \partial S_f^e} \right\rangle_0 + \left\langle \frac{\partial^2 V_{SB}}{\partial S_b^a \, \partial S_f^e} \right\rangle_0 , \tag{3.4}$$

so that once V_{SB} is specified a certain number of the squared masses of the scalar mesons can be read off immediately from (3.3) and (3.4).

The matrix of pseudoscalar-meson squared masses is

$$\left\langle \frac{\partial^2 V}{\partial \phi_b^a \, \partial \phi_f^e} \right\rangle_0 = \left\langle \frac{\partial^2 V_0}{\partial \phi_b^a \, \partial \phi_f^e} \right\rangle_0 + \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_b^a \, \partial \phi_f^e} \right\rangle_0 , \tag{3.5}$$

and the quantity

$$\left\langle \frac{\partial^2 V_0}{\partial \phi_b^a \, \partial \phi_f^e} \right\rangle_0$$

is evaluated as in the scalar case by differentiating (2.21) with respect to the pseudoscalar field. This gives the result[12]

$$(\alpha_a+\alpha_b)\left\langle\frac{\partial^2 V_0}{\partial\phi_b^a\,\partial\phi_f^e}\right\rangle_0=-\delta_a^f\,\delta_e^b\left(\left\langle\frac{\partial V_{SB}}{\partial S_a^a}\right\rangle_0+\left\langle\frac{\partial V_{SB}}{\partial S_b^b}\right\rangle_0\right)-24V_4\delta_a^b\,\delta_e^f\,\alpha_1\alpha_2\alpha_3/\alpha_e\,, \tag{3.6}$$

where

$$V_4=\left\langle\frac{\partial V_0}{\partial I_4}\right\rangle_0. \tag{3.7}$$

Again, once V_{SB} is specified, the pseudoscalar-meson squared masses can be read off immediately from (3.5) and (3.6).

Now we are in a position where it is easy to see what the characteristic mass spectrum of this model looks like. An amusing situation is the so-called "spontaneous breakdown" case where $V_{SB}=0$. Equations (3.3) and (3.6) then show that, depending on the choice of the α_a's there will be some pseudoscalar or scalar mesons with zero mass (Goldstone bosons). Hence this case is clearly unrealistic. We shall not give details here since a complete chart of the spontaneous break-down spectrum was presented in paper I. One point worth emphasizing, though, is that in this type of model the symmetry is in general *broken* even though $V_{SB}=0$. Thus in this case it is not in general correct to make "symmetry" transformations on the theory and expect to reach a physically equivalent situation.[13]

As previously indicated, one very interesting choice of V_{SB} is that of (2.22). Here

$$\left\langle\frac{\partial V_{SB}}{\partial S_a^a}\right\rangle_0=-2A_a$$

and

$$\left\langle\frac{\partial^2 V_{SB}}{\partial S_b^a\,\partial S_f^e}\right\rangle_0=\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_b^a\,\partial\phi_f^e}\right\rangle_0=0,$$

so the formulas become rather simple. We designate the nonet of pseudoscalar mesons as usual by the symbols (π, K, η, η') and the *corresponding* nonet of scalar mesons by the symbols $(\epsilon, \kappa, \sigma, \sigma')$. (Note that we are using the symbol ϵ to stand for the *isovector* scalar particle.) For convenience we shall also consider the particle symbol to stand for its mass. Then the squared masses of the pseudoscalar particles with nonzero internal quantum numbers are found from (3.6) to be

$$\pi_+^{\,2}=2\frac{A_1+A_2}{\alpha_1+\alpha_2},\qquad K_+^{\,2}=2\frac{A_1+A_3}{\alpha_1+\alpha_3},\qquad K_0^{\,2}=2\frac{A_2+A_3}{\alpha_2+\alpha_3}, \tag{3.8}$$

while the squared masses of the corresponding scalar particles are found from (3.3) to be

$$\epsilon_+^{\,2}=2\frac{A_1-A_2}{\alpha_1-\alpha_2},\qquad \kappa_+^{\,2}=2\frac{A_1-A_3}{\alpha_1-\alpha_3},\qquad \kappa_0^{\,2}=2\frac{A_2-A_3}{\alpha_2-\alpha_3}. \tag{3.9}$$

Finally (3.6) shows that the squared masses of the π^0, η, and η' particles correspond to the roots of the secular equation for the following matrix whose (ab) element is $\langle\partial^2V_0/\partial\phi_a^a\,\partial\phi_b^b\rangle_0$:

$$\begin{pmatrix} 2A_1/\alpha_1-12V_4\alpha_2\alpha_3/\alpha_1 & -12V_4\alpha_3 & -12V_4\alpha_2\\ -12V_4\alpha_3 & 2A_2/\alpha_2-12V_4\alpha_1\alpha_3/\alpha_2 & -12V_4\alpha_1\\ -12V_4\alpha_2 & -12V_4\alpha_1 & 2A_3/\alpha_3-12V_4\alpha_1\alpha_2/\alpha_3 \end{pmatrix} \tag{3.10}$$

On the other hand, we see from (3.3) that no information is given on

$$\left\langle\frac{\partial^2 V_0}{\partial S_a^a\,\partial S_b^b}\right\rangle_0,$$

so that the ϵ^0, σ, and σ' masses are not determined by the chiral invariance of V_0.

The structure of the preceding mass formulas are such that all masses are related to each other except for the ϵ^0, σ, and σ'. If we impose isotopic-spin invariance on the theory, $\alpha_1=\alpha_2\equiv\alpha$ and $A_1=A_2$ so the expression for $\epsilon_+^{\,2}$ in (3.9) becomes undefined. In this (usual) case the masses of the pseudoscalar nonet and the κ are related to each other while the masses of σ, σ', and ϵ are undetermined. Thus, information about some of the poorly observed scalar particles tends to be suppressed in this model. We recall that one reason[5] for introducing nonlinear transformations of the pseudoscalar mesons in models of this type is just to suppress this information. Evidently the introduction of nonlinear transformations is not absolutely required on this account.

If the interaction term V_0 is restricted with additional symmetries, then some of the presently unrelated scalar masses get related. This will be illustrated in a later section, where scale invariance will be imposed on V_0.

Recently, Mathur and Okubo[14] have studied the spectral functions of the scalar and pseudoscalar densities in the quark model of (2.24). From general positivity requirements they determined the allowed ranges of the quark "mass" ratio m_3/m_1 and a quantity which is analogous to w defined in (2.10). We note that exactly the same results emerge[15] in a transparent way from our model when we make the identification (2.25) and require that the squared masses as computed above be positive. This would seem to strengthen the similarity between the quark model and the present one with V_{SB} given by (2.22).

The question also arises as to how well the Eqs. (3.8)–(3.10) agree with experiment in the isotopic-spin invariant limit. It was shown in paper I that there is surprisingly reasonable agreement for such a simple model. Analysis of these equations shows that η'^2 can be predicted in terms of π^2, K^2, η^2, and w. (w is not a completely free parameter since it is expected by the usual weak-interaction theory[16] to be around 1.56.) Exact agreement for η'^2 was found for $w=1.7$, which is not bad. For the purpose of making further numerical estimates, we shall adopt the following set of values which satisfy the above equations:

$$\pi^2=1,\quad K^2=13.6,\quad \eta^2=16.5,\quad \eta'^2=50.4,\\ \kappa^2=49.6,\quad w=1.7\,. \tag{3.11}$$

The value of κ^2 above is also a theoretical prediction. We are using a system of units where the π^0 mass is unity.

Now it is convenient to record for future use our conventions about η-η' mixing. We define the mixing angle θ_p in such a way that

$$\begin{pmatrix}\eta\\ \eta'\end{pmatrix}=\begin{pmatrix}\cos\theta_p & -\sin\theta_p\\ \sin\theta_p & \cos\theta_p\end{pmatrix}\begin{pmatrix}\eta_8\\ \eta_0\end{pmatrix}, \tag{3.12}$$

where η and η' stand here for the corresponding physical fields and η_8 and η_0 are the mathematical objects transforming as an $SU(3)$ octet and singlet, respectively. In the tensor notation we are using,

$$\begin{pmatrix}\pi^0\\ \eta\\ \eta'\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0\\ \frac{\cos\theta_p-\sqrt2\,\sin\theta_p}{\sqrt6} & \frac{\cos\theta_p-\sqrt2\,\sin\theta_p}{\sqrt6} & \frac{-(\sin\theta_p+\sqrt2\,\cos\theta_p)}{\sqrt3}\\ \frac{\sin\theta_p+\sqrt2\,\cos\theta_p}{\sqrt6} & \frac{\sin\theta_p+\sqrt2\,\cos\theta_p}{\sqrt6} & \frac{\cos\theta_p-\sqrt2\,\sin\theta_p}{\sqrt3}\end{pmatrix}\begin{pmatrix}\phi_1^1\\ \phi_2^2\\ \phi_3^3\end{pmatrix}. \tag{3.13}$$

In terms of the mixing angle, the relations between the matrix elements $\langle\partial^2V/\partial\phi_a^a\,\partial\phi_b^b\rangle_0$ and the physical masses are given by (in the isotopic-spin limit)

$$\begin{aligned}
\left\langle\frac{\partial^2V}{\partial\phi_1^1\partial\phi_1^1}\right\rangle_0&=\left\langle\frac{\partial^2V}{\partial\phi_2^2\partial\phi_2^2}\right\rangle_0=\tfrac12\pi_0{}^2+\tfrac16(\cos\theta_p-\sqrt2\,\sin\theta_p)^2\eta^2+\tfrac16(\sin\theta_p+\sqrt2\,\cos\theta_p)^2\eta'^2,\\
\left\langle\frac{\partial^2V}{\partial\phi_1^1\partial\phi_2^2}\right\rangle_0&=-\tfrac12\pi_0{}^2+\tfrac16(\cos\theta_p-\sqrt2\,\sin\theta_p)^2\eta^2+\tfrac16(\sin\theta_p+\sqrt2\,\cos\theta_p)^2\eta'^2,\\
\left\langle\frac{\partial^2V}{\partial\phi_1^1\partial\phi_3^3}\right\rangle_0&=\left\langle\frac{\partial^2V}{\partial\phi_2^2\partial\phi_3^3}\right\rangle_0=-\frac{1}{3\sqrt2}(\cos\theta_p-\sqrt2\,\sin\theta_p)(\sin\theta_p+\sqrt2\,\cos\theta_p)(\eta^2-\eta'^2),\\
\left\langle\frac{\partial^2V}{\partial\phi_3^3\partial\phi_3^3}\right\rangle_0&=\tfrac13(\sin\theta_p+\sqrt2\,\cos\theta_p)^2\eta^2+\tfrac13(\cos\theta_p-\sqrt2\,\sin\theta_p)^2\eta'^2\,.
\end{aligned} \tag{3.14}$$

Exactly analogous formulas hold for mixing in the σ-σ' system when we use θ_s instead of θ_p and replace all the pseudoscalar fields by their scalar analogs.

The mixing angle θ_p corresponding to the situation described in (3.11) is, of course, also predicted.[17] It turns out to be extremely small: $\theta_p\simeq 0.3°$. Thus the η and η' seem to be essentially unmixed in the model where V_{SB} is given by (2.22).

To end this section we note that mass formulas of the type we have obtained can also be derived by relating the divergences of the currents for this model to V_{SB} and V_4. This approach is less direct than the present one since it requires us to first introduce currents and then get rid of them. However, a discussion of this method is given in Appendix A.

IV. $S\phi\phi$ COUPLING CONSTANTS

The computation of the $S\phi\phi$ coupling constants will enable us to estimate the scalar-meson widths as well as to calculate processes like $\pi\pi\to\pi\pi$, $\pi K\to\pi K$, and $\eta'\to\eta 2\pi$, all of which involve $S\phi\phi$ vertices. These coupling constants in our approach are to be determined from the expression

$$\left\langle \frac{\partial^3 V}{\partial S_h^g\,\partial\phi_f^e\,\partial\phi_b^a}\right\rangle_0 = \left\langle \frac{\partial^3 V_0}{\partial S_h^g\,\partial\phi_f^e\,\partial\phi_b^a}\right\rangle_0 + \left\langle \frac{\partial^3 V_{\rm SB}}{\partial S_h^g\,\partial\phi_f^e\,\partial\phi_b^a}\right\rangle_0 . \tag{4.1}$$

The restrictions from chiral invariance of V_0 on the first term on the right-hand side above may be found by differentiating the basic equation (2.21) once with respect to the pseudoscalar field and once more with respect to the scalar field. This gives when evaluated at the "equilibrium" point as before:

$$(\alpha_a+\alpha_b)\left\langle \frac{\partial^3 V_0}{\partial S_h^g\,\partial\phi_f^e\,\partial\phi_b^a}\right\rangle_0 = \delta_e^b\left\langle \frac{\partial^2 V_0}{\partial S_h^g\,\partial S_f^a}\right\rangle_0 + \delta_a^f\left\langle \frac{\partial^2 V_0}{\partial S_h^g\,\partial S_b^e}\right\rangle_0 - \delta_g^b\left\langle \frac{\partial^2 V_0}{\partial\phi_f^e\,\partial\phi_h^a}\right\rangle_0 - \delta_a^h\left\langle \frac{\partial^2 V_0}{\partial\phi_f^e\,\partial\phi_b^g}\right\rangle_0$$
$$+12i\delta_a^b\left\langle \frac{\partial^2}{\partial S_h^g\,\partial\phi_f^e}\left(\frac{\partial V_0}{\partial I_4}(\det M-\det M^\dagger)\right)\right\rangle_0 . \tag{4.2}$$

We see from (4.2) that if the indices a and b are unequal,[18] the trilinear coupling becomes related to the scalar and pseudoscalar masses. These relations are of the same form as the so-called generalized Goldberger-Treiman relations (see Appendix A), but in the present approach we can easily see how these equations get modified for different choices of $V_{\rm SB}$.

If the indices a and b are equal in (4.2), additional *a priori* unknown quantities

$$V_{4j} = \left\langle \frac{\partial^2 V_0}{\partial I_4\,\partial I_j}\right\rangle_0$$

enter into the picture. A little thought shows that these additional quantities will arise only when we are calculating the coupling constants for vertices involving three *isoscalar* objects. Thus, as in the case of the scalar masses, only a certain number of coupling constants get related by the chiral invariance of V_0. The others can be chosen freely or alternatively can get related when V_0 is restricted with another symmetry like scale invariance. This situation continues to vertices of higher order.

Now let us give the explicit relations. The $S\phi\phi$ part of $\mathcal{L}$ may be written as

$$\mathcal{L}(S\phi\phi)=\mathcal{L}_1(S\phi\phi)+\mathcal{L}_2(S\phi\phi), \tag{4.3}$$

where

$$-\mathcal{L}_1(S\phi\phi)=\left[(\tfrac{1}{2})^{1/2}g_{\kappa K\pi}\overline{K}\vec{\tau}\cdot\vec{\pi}\kappa+g_{\kappa K\eta}\overline{K}\kappa\eta+g_{\kappa K\eta'}\overline{K}\kappa\eta'+\text{H.c.}\right]+(\tfrac{1}{2})^{1/2}g_{\epsilon K\overline{K}}\overline{K}\vec{\tau}\cdot\vec{\epsilon}K+g_{\epsilon\pi\eta}\vec{\epsilon}\cdot\vec{\pi}\eta+g_{\epsilon\pi\eta'}\vec{\epsilon}\cdot\vec{\pi}\eta'$$
$$+\tfrac{1}{2}g_{\sigma\pi\pi}\vec{\pi}\cdot\vec{\pi}\sigma+\tfrac{1}{2}g_{\sigma'\pi\pi}\vec{\pi}\cdot\vec{\pi}\sigma'+g_{\sigma K\overline{K}}\overline{K}K\sigma+g_{\sigma' K\overline{K}}\sigma'\overline{K}K \tag{4.4}$$

and

$$-\mathcal{L}_2(S\phi\phi)=\tfrac{1}{2}g_{\sigma\eta\eta}\sigma\eta\eta+\tfrac{1}{2}g_{\sigma'\eta\eta}\sigma'\eta\eta+\tfrac{1}{2}g_{\sigma\eta'\eta'}\sigma\eta'\eta'+\tfrac{1}{2}g_{\sigma'\eta'\eta'}\sigma'\eta'\eta'+g_{\sigma\eta\eta'}\sigma\eta\eta'+g_{\sigma'\eta\eta'}\sigma'\eta\eta' . \tag{4.5}$$

In the above we have used the usual isospin notation and each particle symbol stands for the corresponding field. The g's are the coupling constants. $\mathcal{L}_1(S\phi\phi)$ contains those vertices which do get related by chiral symmetry of V_0 and $\mathcal{L}_2(S\phi\phi)$ contains those vertices which do not.

As an example of using (4.1) and (4.2), consider the $\kappa K\pi$ coupling constant $g_{\kappa K\pi}$. By identifying this with the appropriate term in (2.6), we find

$$g_{\kappa K\pi}\equiv\left\langle \frac{\partial^3 V}{\partial S_1^3\partial\phi_2^1\partial\phi_3^2}\right\rangle_0 = \left\langle \frac{\partial^3 V_{\rm SB}}{\partial S_1^3\partial\phi_2^1\partial\phi_3^2}\right\rangle_0 + \frac{1}{\alpha_2+\alpha_3}\left[(\kappa^2-\pi^2)-\left(\left\langle \frac{\partial^2 V_{\rm SB}}{\partial S_1^3\partial S_3^1}\right\rangle_0 - \left\langle \frac{\partial^2 V_{\rm SB}}{\partial\phi_1^2\partial\phi_2^1}\right\rangle_0\right)\right].$$

If $V_{\rm SB}$ is linear in the fields as is the case for (2.22), this expression simplifies to

$$g_{\kappa K\pi}=(\alpha_2+\alpha_3)^{-1}(\kappa^2-\pi^2) . \tag{4.6}$$

Equation (4.6) can also be derived by using the partial conservation of either the π-type axial-vector current, the K-type axial-vector current, or the κ-type vector current (see Appendix A).

Proceeding as above, we may find the following list of the coupling constants in $\mathcal{L}_1(S\phi\phi)$, when we adopt the symmetry-breaking term of (2.22):

$$
\begin{aligned}
&g_{\kappa K\pi}=\frac{1}{F_K}(\kappa^2-\pi^2),\quad g_{\kappa K\eta}=\frac{1}{\sqrt{6}\,F_K}(\cos\theta_p+2\sqrt{2}\,\sin\theta_p)(\eta^2-\kappa^2),\\
&g_{\kappa K\eta'}=\frac{1}{\sqrt{6}\,F_K}(2\sqrt{2}\,\cos\theta_p-\sin\theta_p)(\kappa^2-\eta^2),\\
&g_{\epsilon K\bar{K}}=\frac{1}{F_K}(\epsilon^2-K^2),\quad g_{\epsilon\pi\eta}=\frac{2}{\sqrt{6}\,F_\pi}(\cos\theta_p-\sqrt{2}\,\sin\theta_p)(\epsilon^2-\eta^2),\\
&g_{\epsilon\pi\eta'}=\frac{2}{\sqrt{6}\,F_\pi}(\sin\theta_p+\sqrt{2}\,\cos\theta_p)(\epsilon^2-\eta'^2),\\
&g_{\sigma\pi\pi}=\frac{2}{\sqrt{6}\,F_\pi}(\cos\theta_s-\sqrt{2}\,\sin\theta_s)(\sigma^2-\pi^2),\quad g_{\sigma'\pi\pi}=\frac{2}{\sqrt{6}\,F_\pi}(\sin\theta_s+\sqrt{2}\,\cos\theta_s)(\sigma'^2-\pi^2),\\
&g_{\sigma K\bar{K}}=\frac{1}{\sqrt{6}\,F_K}(\cos\theta_s+2\sqrt{2}\,\sin\theta_s)(K^2-\sigma^2),\quad g_{\sigma' K\bar{K}}=\frac{1}{\sqrt{6}\,F_K}(2\sqrt{2}\,\cos\theta_s-\sin\theta_s)(\sigma'^2-K^2),
\end{aligned}
\tag{4.7}
$$

where we set[16]

$$F_\pi=2\alpha\,,\quad F_K=\alpha+\alpha_3\,. \tag{4.8}$$

[To see how the mixing angles enter into (4.7) we note that, for example,

$$g_{\kappa K\eta}=\sum_a\left\langle\frac{\partial^3 V}{\partial S_1^3\partial\phi_a^a\,\partial\phi_3^1}\right\rangle\frac{\partial\phi_a^a}{\partial\eta}\,,$$

where $\partial\phi_a^a/\partial\eta$ is computed from the transposed matrix in (3.13). The use of (3.14) and its scalar analog is also required for some of (4.7).]

With (4.7), some of the scalar widths may be estimated. Unfortunately the experimental situation is not well established for these mesons. Using the prediction for the κ mass in (3.11) allows us to calculate $\Gamma(\kappa\to K\pi)\simeq 500$ MeV. The calculation of the σ and σ' widths is seen from (4.7) to require knowledge of their masses and the scalar mixing angle θ_s. At present we do not have enough information to do this, but in a later section we will see that the imposition of scale invariance on V_0 gives us this needed information. With this extra assumption it again turns out that σ and σ' are extremely broad. Finally for the case of the ϵ meson, only its mass is needed to calculate the widths from (4.7). There are two conceivable experimental candidates for the ϵ: one is the $\delta(962)$ and the other is the $\pi_N(1016)$. For the former case we predict from (4.7), $\Gamma(\epsilon\to\eta\pi)\simeq 180$ MeV, while for the latter case we predict $\Gamma(\epsilon\to\eta\pi)\simeq 250$ MeV and $\Gamma(\epsilon\to K\bar{K})\simeq 80$ MeV. These widths are all much larger than the experimental widths.[19]

The general picture that emerges (on the theoretical side, at least) is that all the scalar mesons in our model should be rather broad. There is some experimental support of this for the σ meson, from phase-shift analyses[20] of deduced π-π scattering data. However, real experimental confirmation seems to be essentially an open question at present.

V. ϕ^4 COUPLING CONSTANTS

By the ϕ^4 coupling constants we mean the following set of quantities which appear in (2.6):

$$\left\langle\frac{\partial^4 V}{\partial\phi_n^m\partial\phi_h^g\,\partial\phi_f^e\,\partial\phi_b^a}\right\rangle_0=\left\langle\frac{\partial^4 V_0}{\partial\phi_n^m\partial\phi_h^g\,\partial\phi_f^e\,\partial\phi_b^a}\right\rangle_0+\left\langle\frac{\partial^4 V_{\rm SB}}{\partial\phi_n^m\partial\phi_h^g\,\partial\phi_f^e\,\partial\phi_b^a}\right\rangle_0\,. \tag{5.1}$$

The first term on the right-hand side may be calculated by differentiating the basic equation (2.21) three times with respect to the pseudoscalar field and evaluating the resulting expression at the equilibrium point. This gives

$$
\begin{aligned}
(\alpha_a+\alpha_b)\left\langle\frac{\partial^4 V_0}{\partial\phi_n^m\partial\phi_h^g\,\partial\phi_f^e\,\partial\phi_b^a}\right\rangle_0=\;&\delta_e^b\left\langle\frac{\partial^3 V_0}{\partial\phi_n^m\partial\phi_h^g\,\partial S_f^a}\right\rangle_0+\delta_g^b\left\langle\frac{\partial^3 V_0}{\partial\phi_n^m\partial\phi_f^e\,\partial S_h^a}\right\rangle_0+\delta_m^b\left\langle\frac{\partial^3 V_0}{\partial\phi_h^g\,\partial\phi_f^e\,\partial S_n^a}\right\rangle_0+\delta_a^f\left\langle\frac{\partial^3 V_0}{\partial\phi_n^m\partial\phi_h^g\,\partial S_b^e}\right\rangle_0\\
&+\delta_a^h\left\langle\frac{\partial^3 V_0}{\partial\phi_n^m\partial\phi_f^e\,\partial S_b^g}\right\rangle_0+\delta_a^n\left\langle\frac{\partial^3 V_0}{\partial\phi_h^g\,\partial\phi_f^e\,\partial S_b^m}\right\rangle_0\\
&+12i\delta_b^a\left\langle\frac{\partial^3}{\partial\phi_n^m\partial\phi_h^g\,\partial\phi_f^e}\left[\frac{\partial V_0}{\partial I_4}(\det M-\det M^\dagger)\right]\right\rangle_0.
\end{aligned}
\tag{5.2}
$$

When the indices a and b are unequal, (5.2) amounts to a relation between the ϕ^4 and $S\phi\phi$ coupling constants once V_{SB} is specified. The case $a=b$ brings in the *a priori* unknown objects

$$V_{4jK}=\left\langle\frac{\partial^3 V_0}{\partial I_4\partial I_j\partial I_K}\right\rangle_0$$

in addition to the V_{4j} which entered in the $a=b$ case in (4.2). Again, those ϕ^4 coupling constants which remain unrelated are the ones for vertices with all four particles being isoscalar.

Here, we are interested in calculating $\eta'\to\eta 2\pi$ decay, $\pi\pi$ scattering, and πK scattering so we shall only pick up the $\eta'\eta 2\pi$, π^4, and π^2K^2 four-point terms. The appropriate part of (2.6) is (in isotopic-spin notation)

$$-\mathcal{L}(\phi^4)=\tfrac{1}{2}g_{\eta'}^{(4)}\eta'\eta\vec{\pi}\cdot\vec{\pi}+\tfrac{1}{16}g^{(4)}(\vec{\pi}\cdot\vec{\pi})^2+\tfrac{1}{2}g_K^{(4)}\overline{K}K\vec{\pi}\cdot\vec{\pi}+\cdots, \tag{5.3}$$

where we make the identifications

$$g_{\eta'}^{(4)}=\left\langle\frac{\partial^4 V}{\partial\eta'\partial\eta\partial\phi_1^2\partial\phi_2^1}\right\rangle_0=\sum_{a,b}\frac{\partial\phi_a^a}{\partial\eta'}\,\frac{\partial\phi_b^b}{\partial\eta}\left\langle\frac{\partial^4 V}{\partial\phi_a^a\partial\phi_b^b\partial\phi_1^2\partial\phi_2^1}\right\rangle_0, \tag{5.4}$$

$$g^{(4)}=\left\langle\frac{\partial^4 V}{\partial\phi_1^2\partial\phi_2^1\partial\phi_1^2\partial\phi_2^1}\right\rangle_0, \tag{5.5}$$

$$g_K^{(4)}=\left\langle\frac{\partial^4 V}{\partial\phi_1^3\partial\phi_3^1\partial\phi_1^2\partial\phi_2^1}\right\rangle_0. \tag{5.6}$$

The quantities $\partial\phi_a^a/\partial\eta$ and $\partial\phi_b^b/\partial\eta'$ in (5.4) can be found from the transpose of the matrix in (3.13).

For the case when V_{SB} is given by (2.22), we get the following results for these coupling constants after relating them to the $S\phi\phi$ coupling constants by (5.2) and when possible relating these, in turn, to the particle masses by (4.2):

$$g_{\eta'}^{(4)}=\frac{1}{6\alpha^2}(2\sqrt{2}\cos2\theta_p-\sin2\theta_p)[\epsilon^2-\tfrac{1}{2}(\eta^2+\eta'^2)]+\frac{1}{\alpha\sqrt{6}}(\cos\theta_s-\sqrt{2}\sin\theta_s)g_{\sigma\eta\eta'}+\frac{1}{\alpha\sqrt{6}}(\sin\theta_s+\sqrt{2}\cos\theta_s)g_{\sigma'\eta\eta'}, \tag{5.7}$$

$$g^{(4)}=\frac{1}{3\alpha^2}[(\cos\theta_s-\sqrt{2}\sin\theta_s)^2\sigma^2+(\sin\theta_s+\sqrt{2}\cos\theta_s)^2\sigma'^2]-\pi^2/\alpha^2, \tag{5.8}$$

$$g_K^{(4)}=\frac{1}{2\alpha(\alpha+\alpha_3)}\left[\tfrac{1}{3}\sigma^2(4\sin^2\theta_s-\cos^2\theta_s-(\tfrac{1}{2})^{1/2}\sin2\theta_s)+\tfrac{1}{3}\sigma'^2(4\cos^2\theta_s-\sin^2\theta_s+(\tfrac{1}{2})^{1/2}\sin2\theta_s)+(\kappa^2-\pi^2-K^2)\right]. \tag{5.9}$$

Now let us go to the decay process $\eta'\to\eta\pi^+\pi^-$. This decay is interesting since its Dalitz plot has been determined experimentally.[21] When this amplitude is computed in the tree-diagram approximation[22] there enter, in addition to the four-point "contact" term discussed above, diagrams with intermediate σ, σ', and ϵ lines. All these are shown in Fig. 1. The $S\phi\phi$ type coupling constants involved in these other diagrams are defined in (4.4) and (4.5). After calculating the diagrams of Fig. 1 in the usual way, we find[23]

$$\begin{aligned}T(\eta'(p)\to\eta(q),\pi^+(k_+),\pi^-(k_-))&\\=\frac{1}{\alpha\sqrt{6}}(\cos\theta_s-\sqrt{2}\sin\theta_s)g_{\sigma\eta\eta'}&\left(\frac{\pi^2-\eta^2-\eta'^2-2p\cdot q}{\sigma^2-\eta^2-\eta'^2-2p\cdot q}\right)+\frac{1}{\alpha\sqrt{6}}(\sin\theta_s+\sqrt{2}\cos\theta_s)g_{\sigma'\eta\eta'}\left(\frac{\pi^2-\eta^2-\eta'^2-2p\cdot q}{\sigma'^2-\eta^2-\eta'^2-2p\cdot q}\right)\\+\frac{1}{12\alpha^2}(2\sqrt{2}\cos2\theta_p-\sin2\theta_p)&\Big[2\epsilon^2-(\eta^2+\eta'^2)-(\epsilon^2-\eta^2)(\epsilon^2-\eta'^2)\\&\times\left(\frac{1}{\epsilon^2-\eta'^2-\pi^2-2p\cdot k_+}+\frac{1}{\epsilon^2-\eta'^2-\pi^2-2p\cdot k_-}\right)\Big].\end{aligned} \tag{5.10}$$

It is of some interest to consider the limit of (5.10) wherein the scalar masses σ, σ', and ϵ become infinitely large. This corresponds to the "current-algebra" result. In this limit $g_{\sigma\eta\eta'}$ and $g_{\sigma'\eta\eta'}$ are expected to remain finite since they are not related to particle masses by chiral symmetry. Thus the first two terms of (5.10) do not contribute, and we get

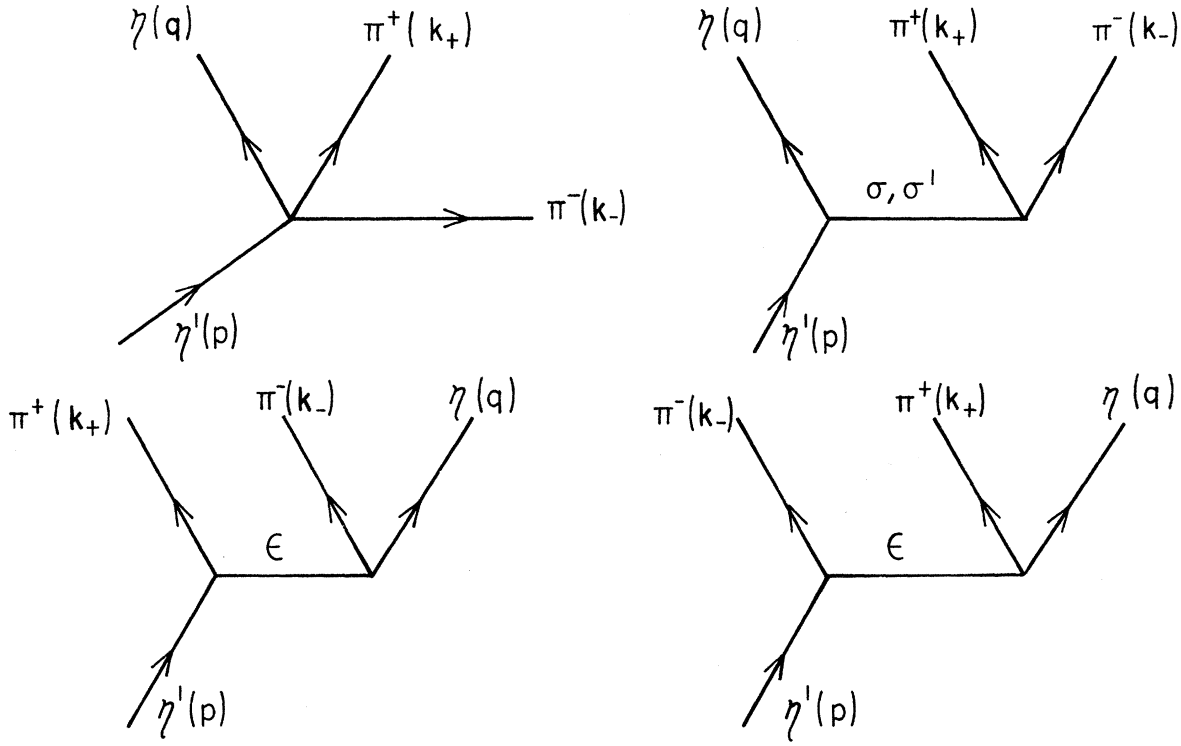

FIG. 1. Diagrams for η' decay.

$$T_{CA}(\eta'\to\eta\pi^+\pi^-)=\frac{2\sqrt{2}\cos 2\theta_p-\sin 2\theta_p}{12\alpha^2}[(\eta^2+\eta'^2-2\pi^2)-2\eta' q_0]. \tag{5.10$'$}$$

The spectrum shape predicted by (5.10′) is essentially the same as that of Schwinger.[22] However, he did not predict the over-all factor.

As it stands, (5.10) contains at least four quantities, namely $g_{\sigma\eta\eta'}$, $g_{\sigma'\eta\eta'}$, θ_s, and σ'^2, about which we have very little idea. However, the imposition of scale invariance on V_0 will enable us to get a handle on these objects and attempt some meaningful comparison with experiment. The discussion is thus deferred until a later section. In preparation for this, though, we expand (5.10) in terms of the η-meson kinetic energy in the η' center-of-mass frame and keep only terms of first order; i.e., we write

$$T(\eta'\to\eta\pi^+\pi^-)=A+BT_\eta+\cdots, \tag{5.11}$$

where $T_\eta=q_0-\eta$ and the coefficients A and B are

$$A=\frac{2}{\sqrt{6}F_\pi}(\cos\theta_s-\sqrt{2}\sin\theta_s)g_{\sigma\eta\eta'}\left[\frac{\pi^2-(\eta'-\eta)^2}{\sigma^2-(\eta'-\eta)^2}\right]+\frac{2}{\sqrt{6}F_\pi}(\sin\theta_s+\sqrt{2}\cos\theta_s)g_{\sigma'\eta\eta'}\left[\frac{\pi^2-(\eta'-\eta)^2}{\sigma'^2-(\eta'-\eta)^2}\right]$$

$$+\frac{1}{3F_\pi^2}(2\sqrt{2}\cos 2\theta_p-\sin 2\theta_p)\left[2\epsilon^2-(\eta^2+\eta'^2)-\frac{2(\epsilon^2-\eta^2)(\epsilon^2-\eta'^2)}{\epsilon^2-\pi^2-\eta\eta'}\right], \tag{5.11$'$}$$

$$B=\frac{2}{\sqrt{6}F_\pi}(\cos\theta_s-\sqrt{2}\sin\theta_s)g_{\sigma\eta\eta'}\frac{2\eta'(\sigma^2-\pi^2)}{[\sigma^2-(\eta'-\eta)^2]^2}+\frac{2}{\sqrt{6}F_\pi}(\sin\theta_s+\sqrt{2}\cos\theta_s)g_{\sigma'\eta\eta'}\frac{2\eta'(\sigma'^2-\pi^2)}{[\sigma'^2-(\eta'-\eta)^2]^2}$$

$$+\frac{1}{3F_\pi^2}(2\sqrt{2}\cos 2\theta_p-\sin 2\theta_p)\frac{2\eta'(\epsilon^2-\eta^2)(\epsilon^2-\eta'^2)}{[\epsilon^2-\pi^2-\eta\eta']^2}. \tag{5.11$''$}$$

The computation of the π-π scattering amplitude is very similar to the computation above. In addition to the "contact" diagram whose coupling constant is given in (5.8), there are diagrams with internal σ and σ' lines. The $\sigma\pi\pi$ and $\sigma'\pi\pi$ coupling constants are given in (4.7). For the process $\pi_i(p_1)+\pi_j(p_2)\to\pi_k(p_1')+\pi_l(p_2')$, we thus have the T amplitude

$$\delta_{ij}\delta_{kl}\left\{\frac{2}{3F_\pi^2}(\cos\theta_s-\sqrt{2}\sin\theta_s)^2\left[\sigma^2-\frac{(\sigma^2-\pi^2)^2}{\sigma^2-s}\right]+\frac{2}{3F_\pi^2}(\sin\theta_s+\sqrt{2}\cos\theta_s)^2\left[\sigma'^2-\frac{(\sigma'^2-\pi^2)^2}{\sigma'^2-s}\right]-\frac{2\pi^2}{F_\pi^2}\right\}$$

$$+\delta_{ik}\delta_{jl}\{(s\to t)\}+\delta_{il}\delta_{jk}\{(s\to u)\}$$

$$\equiv\delta_{ij}\delta_{kl}\left(\frac{T^{(0)}-T^{(2)}}{3}\right)+\delta_{ik}\delta_{jl}\left(\frac{T^{(1)}+T^{(2)}}{2}\right)+\delta_{il}\delta_{jk}\left(\frac{T^{(2)}-T^{(1)}}{2}\right), \tag{5.12}$$

where the decomposition into isospin amplitudes has been shown, and $s=-(p_1+p_2)^2$, $t=-(p_1'-p_1)^2$, $u=-(p_2'-p_1)^2$.

The scattering lengths are found from (5.12) to be

$$a_0=\frac{-1}{32(3.1416)\pi}T^{(0)}(s=4\pi^2,\ t=u=0)$$

$$=\frac{\pi}{8(3.1416)F_\pi^2}\left[\tfrac{7}{2}+(\cos\theta_s-\sqrt{2}\sin\theta_s)^2\left(\frac{9}{2}\frac{\pi^2}{\sigma^2-4\pi^2}+\frac{\pi^2}{3\sigma^2}\right)+(\sin\theta_s+\sqrt{2}\cos\theta_s)^2\left(\frac{9}{2}\frac{\pi^2}{\sigma'^2-4\pi^2}+\frac{\pi^2}{3\sigma'^2}\right)\right], \tag{5.13}$$

$$a_2=\frac{-1}{32(3.1416)\pi}T^{(2)}(s=4\pi^2,\ t=u=0)$$

$$=\frac{-\pi}{8(3.1416)F_\pi^2}\left[1-\frac{\pi^2}{3\sigma^2}(\cos\theta_s-\sqrt{2}\sin\theta_s)^2-\frac{\pi^2}{3\sigma'^2}(\sin\theta_s+\sqrt{2}\cos\theta_s)^2\right]. \tag{5.14}$$

(Note that π in the above formulas stands for the pion mass.) Equations (5.13) and (5.14) are each written in such a way that the first terms on the right-hand side give the usual "current-algebra" result while the remaining terms give a correction. Again we do not presently have enough information to evaluate the correction. Later we will attempt to estimate the correction with the additional assumption that V_0 is scale-invariant.

Finally, for the case of πK scattering we must compute the "contact" diagram, whose coupling constant is given in (5.9) as well as diagrams with internal σ, σ', and κ lines. The T matrix for $K+\pi_i\to K+\pi_j$ is (in isotopic-spin notation)

$$\delta_{ij}\left[g_K^{(4)}-\tfrac{1}{2}g_{\kappa K\pi}{}^2\left(\frac{1}{\kappa^2-s}+\frac{1}{\kappa^2-u}\right)-\frac{g_{\sigma K\bar{K}}g_{\sigma\pi\pi}}{\sigma^2-t}-\frac{g_{\sigma'K\bar{K}}g_{\sigma'\pi\pi}}{\sigma'^2-t}\right]-\tfrac{1}{2}[\tau_i,\tau_j]\left[\tfrac{1}{2}g_{\kappa K\pi}{}^2\left(\frac{1}{\kappa^2-u}-\frac{1}{\kappa^2-s}\right)\right]$$

$$\equiv\delta_{ij}\left(\frac{T^{(1/2)}+2T^{(3/2)}}{3}\right)-\tfrac{1}{2}[\tau_i,\tau_j]\left(\frac{T^{(1/2)}-T^{(3/2)}}{3}\right), \tag{5.15}$$

where the isospin decomposition has been displayed and the τ's are the Pauli matrices.

Evaluating (5.15) at threshold, substituting in our formulas for the coupling constants, and multiplying by $-[8(3.1416)(\pi+K)]^{-1}$ gives the scattering lengths as

$$a_{1/2}=\frac{-1}{16(3.1416)(\pi+K)}\left\{\left(\frac{\kappa^2-\pi^2}{\alpha_1+\alpha_3}\right)^2\left[\frac{1}{\kappa^2-(K-\pi)^2}-\frac{3}{\kappa^2-(K+\pi)^2}\right]\right.$$

$$\left.+\frac{1}{\alpha(\alpha+\alpha_3)}\left[\kappa^2-\frac{\pi^2K^2}{3\sigma^2}\left(4\sin^2\theta_s-\cos^2\theta_s-\frac{1}{\sqrt{2}}\sin 2\theta_s\right)-\frac{\pi^2K^2}{3\sigma'^2}\left(4\cos^2\theta_s-\sin^2\theta_s+\frac{1}{\sqrt{2}}\sin 2\theta_s\right)\right]\right\}, \tag{5.16}$$

$$a_{3/2}=\frac{-1}{8(3.1416)(\pi+K)}\left\{-\left(\frac{\kappa^2-\pi^2}{\alpha+\alpha_3}\right)^2\frac{1}{\kappa^2-(K-\pi)^2}\right.$$

$$\left.+\frac{1}{2\alpha(\alpha+\alpha_3)}\left[\kappa^2-\frac{\pi^2K^2}{3\sigma^2}\left(4\sin^2\theta_s-\cos^2\theta_s-\frac{1}{\sqrt{2}}\sin 2\theta_s\right)-\frac{\pi^2K^2}{3\sigma'^2}\left(4\cos^2\theta_s-\sin^2\theta_s+\frac{1}{\sqrt{2}}\sin 2\theta_s\right)\right]\right\}. \tag{5.17}$$

We will estimate the numerical values of these scattering lengths after determining θ_s and σ' from scale invariance of V_0.

VI. SCALE INVARIANCE

There has been a great deal of recent interest[8] in the invariance (or more correctly, approximate invariance) of the strong interactions under scale transformations of the space-time coordinates, i.e., transformations of the form

$$X_\mu \to \beta X_\mu , \tag{6.1}$$

where β is some real constant. The main reason for this interest[24] seems to be the experimental fact that high-energy inelastic electroproduction amplitudes display an approximate scale-invariant behavior. On the other hand, the way in which this approximate invariance should be implemented in terms of field theory or "current-algebra" models has not at all been settled even though many different proposals have been made.[8]

The simplest approach to this topic (and the one we shall investigate here) is to proceed in exact analogy to classical physics. In a classical model the scale invariance of a theory just says that no constants having any dimensions should enter into the Lagrangian, and the whole problem becomes an exercise in dimensional analysis. The dimensions of all fields can be read off from the "free" Lagrangian density which has the dimensions of an energy density or (mass)^4. Thus our fields ϕ and S should each have the dimension (mass)^1. Note that as in the previous case of chiral invariance it is more convenient to impose the scale symmetry on the Lagrangian written in terms of the original fields as in (2.1) rather than the Lagrangian written in terms of the "physical" fields as in (2.6). Actually it is easy to see that any scale symmetry which is present in (2.1) becomes severely mangled when expressed[25] in terms of the "physical fields."

For the Lagrangian density (2.1), *exact* scale invariance would mean, by dimensional arguments, that $V=V_0+V_{SB}$ be a homogeneous function of order four in the fields ϕ and S. Now the particular choice of V_{SB} given in (2.22) obviously violates this criterion. Since this choice of V_{SB} seems to be a fairly reasonable one, it is tempting to require *only V_0 to be scale-invariant.*[26] *Then the same term which breaks chiral $SU(3)\times SU(3)$ would also break scale invariance.* Obviously, other more complicated assumptions can be made but we will test this simplest case here.

At first thought, it would appear that requiring V_0 to be a homogeneous function of order four limits us to a linear combination of $(I_1)^2$ and I_2, where I_1 and I_2 are the invariants defined in (2.2). However, in a theory of the present type this is definitely not true. The reason is that the invariants I_j of (2.2) are expressed in terms of the original unphysical fields which have nonvanishing "equilibrium"-point values. Thus $\langle I_j\rangle_0 \neq 0$, and we must expand

$$I_j=\langle I_j\rangle_0+\mathrm{Tr}\left(\left\langle\frac{\partial I_j}{\partial\phi}\right\rangle_0\tilde{\phi}\right)+\mathrm{Tr}\left(\left\langle\frac{\partial I_j}{\partial S}\right\rangle_0\tilde{S}\right)+\cdots$$

so that $(I_j)^{-1}$ does not blow up for the "ground" state. The result of this is that terms like $(I_1)^{-1}I_3$, $(I_1)^{-1}(I_4)^2$, as well as an infinity of others, are allowed. An easy way to proceed in general is just to use Euler's theorem on homogeneous functions for V_0. This immediately gives us the basic equation

$$\mathrm{Tr}\left(\phi\frac{\partial V_0}{\partial\phi}+S\frac{\partial V_0}{\partial S}\right)=4V_0, \tag{6.2}$$

since V_0 is a homogeneous function of order 4 in ϕ and S.

Equation (6.2) is analogous to the basic equations (2.18) and (2.21) in the sense that it is a "generating" relation from which all the consequences of the scale invariance of V_0 on masses, couplings, etc., can be obtained simply by differentiation with respect to ϕ and S.

In the literature,[8] most of the work on scale invariance has used the energy-momentum tensor $\Theta_{\mu\nu}$ as a starting point. Although the simplicity of the present approach using (6.2) can hardly be improved upon, it may be helpful for the purposes of comparing with other work or going beyond the framework of this model to study this alternate method. This is done in Appendix C, where it is shown how our relations may be derived by calculating the trace of $\Theta_{\mu\nu}$. The situation is analogous to the chiral-summetry case where an alternate approach involves using the current divergences (see Appendix A).

Now let us take up the consequences[27] of (6.2). Differentiating it with respect to the scalar field and evaluating the result at the "equilibrium" point gives

$$\sum_{a=1}^{3}\alpha_a\left\langle\frac{\partial^2 V_0}{\partial S_a^a S_b^b}\right\rangle_0=-3\left\langle\frac{\partial V_{SB}}{\partial S_b^b}\right\rangle_0 \quad (b=1,2,3). \tag{6.3}$$

Equations (6.3) are very interesting in that they provide information on the σ mass, σ' mass, and scalar mixing angle, θ_s, all of which were left completely free after the imposition of chiral symmetry on V_0 (see the discussion in Sec. III).

To make our initial test of the assumption that V_0 be scale-invariant, we check (6.3) for the symmetry breaking term given by (2.22). Note that because of isotopic-spin invariance only two of the three equations in (6.3) are different from each other. After using the scalar analog of (3.14) to express the quantities $\langle\partial^2 V_0/\partial S_a^a\partial S_b^b\rangle_0$ in terms of σ, σ', and θ_s, we may write these two equations as

$$\frac{6A_1}{\alpha}=\tfrac{1}{2}(\sigma^2+\sigma'^2)-\frac{1}{6\sqrt{2}}[\sqrt{2}\,(1+2w)\cos 2\theta_s - (w-4)\sin 2\theta_s](\sigma^2-\sigma'^2),$$

$$\frac{6A_3}{\alpha}=\tfrac{1}{2}w(\sigma^2+\sigma'^2)+\tfrac{1}{6}[(w-4)\cos 2\theta_s+\sqrt{2}\,(2w+1)\sin 2\theta_s](\sigma^2-\sigma'^2), \qquad (6.4)$$

where w [see (2.10)] is the quantity which characterizes the $SU(3)$ noninvariance of the "ground" state. The angle θ_s may be eliminated from (6.4) to give the following relation between σ^2 and σ'^2:

$$(w^2+2)(\sigma\sigma')^2=\frac{-36}{\alpha^2}[2(A_1)^2+(A_3)^2]+\frac{6}{\alpha}(2A_1+wA_3)(\sigma^2+\sigma'^2). \qquad (6.5)$$

The quantities w, A_1/α, and A_3/α appearing above can all be computed from the pseudoscalar-mass spectrum in our model. Corresponding to the numerical values in (3.11), we may thus write (6.5) as

$$\sigma'^2\simeq\frac{61.2-\sigma^2}{1-\sigma^2/38.5}, \qquad (6.6)$$

where masses are expressed in multiples of the π^0 mass. Let us assume for definiteness that $\sigma^2<\sigma'^2$. Then (6.6) shows that $\sigma^2<38.5$ and $\sigma'^2>61.2$, or equivalently the lower of the two scalar isosinglets, must be less massive than 840 MeV, while the higher one must be more massive than 1050 MeV. The mixing angle θ_s may also be calculated from (6.4). Below, a chart is given showing the predictions of σ'^2 and θ_s for various values of σ^2.

σ^2	σ'^2	θ_s
0	61.2	127°
10	69.2	118°
20	85.0	107°
31.7	166.5	90°
38.5	∞	75°

Generally it is believed[20] that there is an extremely broad isoscalar resonance in the vicinity of the ρ meson ($\sigma^2=31.7$). If this is identified with σ, we are led to expect the σ' to be around 1740 MeV. Furthermore the σ' is predicted to be the member of the $SU(3)$ octet while the lower mass σ should be essentially a singlet. [See the scalar analog of (3.12).]

Another set of predictions from scale invariance of V_0 may be found by differentiating (6.2) twice with respect to the pseudoscalar field and evaluating the result at the "equilibrium" point. This gives the following set of relations between the $S\phi\phi$ coupling constants and the pseudoscalar-meson masses:

$$\sum_a \alpha_a\left\langle\frac{\partial^3 V_0}{\partial S_a^a\,\partial\phi_e^f\,\partial\phi_g^h}\right\rangle_0=2\left\langle\frac{\partial^2 V_0}{\partial\phi_e^f\,\partial\phi_g^h}\right\rangle_0. \qquad (6.7)$$

An amusing consequence of (6.7) is that those $S\phi\phi$ coupling constants which could not be related to the others by the imposition of chiral symmetry on V_0 [i.e., the ones defined in (4.5)] now are required to satisfy the following equations:

$$Xg_{\sigma\eta\eta}+Yg_{\sigma'\eta\eta}=2\eta^2/\alpha, \qquad (6.8)$$

$$Xg_{\sigma\eta'\eta'}+Yg_{\sigma'\eta'\eta'}=2\eta'^2/\alpha, \qquad (6.9)$$

$$Xg_{\sigma\eta\eta'}+Yg_{\sigma'\eta\eta'}=0, \qquad (6.10)$$

where

$$X=(\tfrac{2}{3})^{1/2}(1-w)\cos\theta_s-(\tfrac{1}{3})^{1/2}(2+w)\sin\theta_s,$$
$$Y=(\tfrac{2}{3})^{1/2}(1-w)\sin\theta_s+(\tfrac{1}{3})^{1/2}(2+w)\cos\theta_s, \qquad (6.11)$$

and we have assumed that V_{SB} is given by (2.22). These equations are not completely academic since (6.10) is useful in connection with our computation of η' decay in Sec. V.

It would first seem that (6.7) also gives some new restrictions on the $\sigma\pi\pi$, $\sigma'\pi\pi$, $\sigma K\bar{K}$, and $\sigma' K\bar{K}$ coupling constants. However, these relations are easily seen to be no different from the chiral-symmetry predictions of (4.2) when taken together with our previous result (6.3).

Finally we note that, in general, scale invariance relates n-point vertices where at least one incoming line is an *isoscalar* scalar particle to the $(n-1)$-point vertex without this incoming line. Specifically, differentiating (6.2) p times with respect to the scalar field, q times (q even) with respect to the pseudoscalar field, and evaluating the result at the "equilibrium" point gives

$$\sum_{a=1}^{3}\alpha_a\left\langle\frac{\partial^{p+q+1}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_q}^{t_q}\partial S_{m_1}^{n_1}\cdots\partial S_{m_p}^{n_p}\partial S_a^a}\right\rangle_0=(p+q-4)\left\langle\frac{\partial^{p+q}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_q}^{t_q}\partial S_{m_1}^{n_1}\cdots\partial S_{m_p}^{n_p}}\right\rangle_0. \qquad (6.12)$$

VII. APPLICATIONS OF SCALE-INVARIANCE RELATIONS

We found in Secs. IV and V that just imposing chiral $SU(3)\times SU(3)$ on V_0 was not sufficient to give us all the information needed to calculate the σ and σ' widths, $\eta'\to\eta 2\pi$ decays, or $\pi\pi$ and πK scatterings. In the last section some of this information became available on the assumption that V_0 is *also* scale-invariant. Here we shall use this information to make some estimate on the processes mentioned.

A. $\eta' \to \eta 2\pi$ Decay

Experimentally, the width of this decay is very small and all that can be said quantitatively[28] is that it is less than the resolution width of the apparatus – around 10 MeV. However, there are more data available from angular-distribution measurements. The best fit[21] to the Dalitz plot is given for the matrix element

$$1+ay, \tag{7.1}$$

where $a=-0.11\pm 0.05$ and the variable y is defined in terms of the η kinetic energy T_η and the decay Q value as

$$y=\frac{\eta+2\pi}{\pi}\frac{T_\eta}{Q}-1\,. \tag{7.2}$$

This means that the quantities A and B introduced in (5.10) should satisfy

$$B/A \simeq -0.615 \tag{7.3}$$

in units of inverse π^0 masses.

The theoretical predictions for A and B are given in (5.11′) and (5.11″). The σ mass appearing in these equations is taken at the ρ mass ($\sigma^2 \simeq 31.7$) from crude experimental indications. Then scale invariance (Sec. VI) predicts $\sigma'^2=166.5$ and $\theta_s=90°$. (We note that slightly different choices of σ^2 should not make much difference in the results.) Furthermore, scale invariance relates $g_{\sigma'\eta\eta'}$ to $g_{\sigma\eta\eta'}$ by (6.10). Thus the only unknowns appearing in (5.11′) and (5.11″) are, say, $g_{\sigma'\eta\eta'}$ and the value of the ϵ (scalar isovector particle) mass. We shall proceed as follows: A value of ϵ^2 will be assumed and the value of $g_{\sigma'\eta\eta'}$ will be adjusted to fit the experimental result (7.3). Then the width will be predicted[29] as listed in Table I.

Our results are evidently not in disagreement with the present experimental situation; a better experimental bound on Γ would pin down the ϵ mass more closely, granted our assumptions.

For comparison we note that the "current-algebra" result (5.10′) predicts $B/A \simeq -1.9(\pi^0)^{-1}$ [which seems to disagree with (7.3)] and $\Gamma \simeq 0.1$ MeV.

TABLE I. Predicted width.

ϵ^2 ($\pi_0{}^2$)	$\Gamma(\eta'\to\eta\pi^+\pi^-)$ (MeV)	$g_{\sigma'\eta\eta'}$ (π_0)
35	7.1	+304
45	5.5	−65.4
50.4	4.2	−79.4
75	2.1	−83.0
100	1.5	−79.8
200	1.0	−75.6

B. Scattering Lengths

The formulas for the $I=0$ and $I=2$ $\pi\pi$ scattering lengths are (5.13) and (5.14). Taking the values of σ^2, σ'^2, and θ_s as above enables us to evaluate these quantities *including* the effects of corrections to the "current-algebra" results. We find $a_0=0.151$ in units of inverse π^0 and $a_2=-0.038$ in the same units. The correction term to the "current-algebra" term for a_0 is positive and about 10% of a_0 while the correction terms for a_2 is positive and only about 2% of a_2 in magnitude.

The quantities σ^2, σ'^2, and θ_s were also previously the only unknown ones in the πK scattering-length formulas (5.16) and (5.17). Taking the values stated above for these objects gives the predictions $a_{1/2}=0.13$ inverse π^0 mass units and $a_{3/2}\approx -0.050$ inverse π^0 mass units.

C. σ and σ' Decays

If we accept the values $\sigma^2=31.7$ (π_0 masses)^2, $\sigma'^2=166.5$, and $\theta_s=90°$, we can estimate most of these decays from the $S\phi\phi$ coupling constants given in (4.7). We find $\Gamma(\sigma\to\pi\pi)\simeq 830$ MeV. Such an extremely broad resonance is in rough qualitative agreement with fits to the $I=0$ $\pi\pi$ phase shifts. Perhaps we should think of the σ not as a resonance in the conventional sense but as a way of describing the low-energy continuum.

The same situation holds for σ', even more so. We find $\Gamma(\sigma'\to\pi\pi)\simeq 5500$ MeV, even though its mass is expected to be only about 1740 MeV. Furthermore, $\Gamma(\sigma'\to K\bar{K})\simeq 720$ MeV. $\Gamma(\sigma'\to\eta\eta)$ is also calculable from the results for $g_{\sigma'\eta\eta'}$ in Sec. VII A when the value of ϵ^2 is decided upon.

The width estimations above followed just from a first-order calculation of the $S\phi\phi$ vertices. It is conceivable that sizeable corrections would arise if higher-order diagrams were taken into account (see also Ref. 19).

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APPENDIX A

It may be helpful to show how formulas like the ones we have obtained can be gotten by the more usual approach of considering the current divergences. For simplicity let us assume that V_{SB} is given by (2.22). It is convenient to introduce the matrices

$$\underline{A}=\begin{pmatrix}A_1 & 0 & 0\\ 0 & A_2 & 0\\ 0 & 0 & A_3\end{pmatrix},\quad \underline{\alpha}=\begin{pmatrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\\ 0 & 0 & \alpha_3\end{pmatrix}. \tag{A1}$$

Now if either a "vector" or an "axial-vector" infinitesimal transformation is made, we have from the Lagrangian equations of motion

$$\delta\mathfrak{L}=\partial_\mu \mathrm{Tr}\left(\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi)}\delta\phi+\frac{\partial\mathfrak{L}}{\partial(\partial_\mu S)}\delta S\right)$$
$$=-\partial_\mu \mathrm{Tr}(\partial_\mu\phi\delta\phi+\partial_\mu S\delta S). \tag{A2}$$

If the transformation above is given by (2.11), we define the *nonet* matrix of vector currents, V_μ, by $\delta\mathfrak{L}=+i\ \mathrm{Tr}(\partial_\mu V_\mu E_V)$ so that we can identify

$$V_\mu=i(\phi\overleftrightarrow{\partial}_\mu\phi+S\overleftrightarrow{\partial}_\mu S).$$

Using (2.5) and (A1) this becomes

$$V_\mu=i(\phi\overleftrightarrow{\partial}_\mu\phi+\tilde{S}\overleftrightarrow{\partial}_\mu\tilde{S}+[\underline{\alpha},\partial_\mu\tilde{S}]). \tag{A3}$$

On the other hand, $\delta\mathfrak{L}$ is also given by

$$-\mathrm{Tr}\left(\frac{\partial V_{\mathrm{SB}}}{\partial S}\delta S\right)=2\mathrm{Tr}([S,\underline{A}]E_V),$$

so that we have the "partial conservation" relation:

$$\partial_\mu V_\mu=-2i[\tilde{S},\underline{A}]. \tag{A4}$$

When the transformation in (A2) is the "axial-vector" one of (2.13), we similarly define the *nonet* matrix of axial-vector currents, P_μ, by $\delta\mathfrak{L}=+i\,\mathrm{Tr}(\partial_\mu P_\mu E_A)$. Then we find

$$P_\mu=\tilde{S}\overleftrightarrow{\partial}_\mu\phi-\phi\overleftrightarrow{\partial}_\mu\tilde{S}+[\underline{\alpha},\partial_\mu\phi]_+ \tag{A5}$$

$$\partial_\mu P_\mu=2[\underline{A},\phi]_++12i\frac{\partial V_0}{\partial I_4}(\det M-\det M^\dagger)\times 1. \tag{A6}$$

The last term in (A6) may be expanded in a similar way as (2.6) to give

$$\partial_\mu P_\mu=2[\underline{A},\phi]_+-24V_4\alpha_1\alpha_2\alpha_3\mathrm{Tr}(\underline{\alpha}^{-1}\phi)\times 1+\cdots. \tag{A7}$$

As an example of computing mass formulas, let us consider the π^+ case. Equation (A5) gives

$$(P_1{}^2)_\mu=(\alpha_1+\alpha_2)\partial_\mu\phi_1^2+\cdots, \tag{A8}$$

while (A7) gives

$$\partial_\mu(P_1{}^2)_\mu=2(A_1+A_2)\phi_1^2. \tag{A9}$$

Taking the matrix element of the divergence of both sides of (A8) in lowest order gives

$$\sqrt{2q_0}\langle 0|\partial_\mu(P_1{}^2)_\mu|\pi^+(q)\rangle=(\alpha_1+\alpha_2)\pi_+{}^2. \tag{A10}$$

Taking the matrix element of both sides of (A9) gives

$$\sqrt{2q_0}\langle 0|\partial_\mu(P_1{}^2)_\mu|\pi^+(q)\rangle=2(A_1+A_2). \tag{A11}$$

Equating (A10) and (A11) gives

$$\pi_+{}^2=2\frac{A_1+A_2}{\alpha_1+\alpha_2},$$

which is just the mass formula in (3.8).

As an example of computing coupling constants let us consider the $\kappa K\pi$ case. From (A3) we have

$$(V_1{}^3)_\mu=i\,\pi^+\overleftrightarrow{\partial}_\mu K^0+i(\alpha_1-\alpha_3)\partial_\mu\kappa^++\cdots, \tag{A12}$$

while from (A4),

$$\partial_\mu(V_1{}^3)_\mu=2i(A_1-A_3)\kappa^+. \tag{A13}$$

Taking the matrix element of the divergence of both sides of (A12) in lowest order gives

$$[(2P_0)(2P_0')]^{1/2}\langle\pi^-(P')|\partial_\mu(V_1{}^3)_\mu|K^0(P)\rangle$$
$$=i(K^2-\pi^2)+i(\alpha_3-\alpha_1)q^2$$
$$\times[(2P_0)(2P_0')]^{1/2}\langle\pi^-(P')|\kappa^+|K^0(P)\rangle, \tag{A14}$$

where $q_\mu=P_\mu-P_\mu'$. Taking the matrix element of both sides of (A13) gives

$$[(2P_0)(2P_0')]^{1/2}\langle\pi^-(P')|\partial_\mu(V_1{}^3)_\mu|K^0(P)\rangle$$
$$=2i(A_1-A_3)[(2P_0)(2P_0')]^{1/2}\langle\pi^-(P')|\kappa^+|K^0(P)\rangle. \tag{A15}$$

Equating (A14) and (A15) gives for the vertex function in question

$$[(2P_0)(2P_0')]^{1/2}\langle\pi^-(P')|\kappa^+|K^0(P)\rangle\equiv-\frac{g_{\kappa K\pi}}{q^2+\kappa^2}$$
$$=-\frac{K^2-\pi^2}{\alpha_3-\alpha_1}\left[q^2+2\left(\frac{A_3-A_1}{\alpha_3-\alpha_1}\right)\right]^{-1}, \tag{A16}$$

so that we may identify, with the help of (3.9)

$$g_{\kappa K\pi}=\frac{K^2-\pi^2}{\alpha_3-\alpha_1}. \tag{A17}$$

In computing (A17) we used the κ-type vector current. We could also use either the K-type or the π-type axial-vector current. This gives three alternative expressions:

$$g_{\kappa K\pi}=\frac{K^2-\pi^2}{\alpha_3-\alpha_1}=\frac{\kappa^2-K^2}{\alpha_1+\alpha_2}=\frac{\kappa^2-\pi^2}{\alpha_1+\alpha_3}. \tag{A18}$$

The identity of all three expressions in (A18) is easily verified for the case when V_{SB} is given by (2.22).

If the left-hand side of (A14) is parametrized in the usual way as

$$f_+(P+P')_\mu + f_-(P-P')_\mu\,,$$

we find from the above the results

$$f_+ = -\frac{1}{\sqrt{2}}\,,\quad f_- = \left(-\frac{1}{\sqrt{2}}\right)\frac{K^2-\pi^2}{(P-P')^2+\kappa^2}\,,$$

$$\xi \equiv \frac{f_-(0)}{f_+(0)} = \frac{K^2-\pi^2}{\kappa^2} \simeq \tfrac{1}{4}\,.$$

APPENDIX B

Coupling Constants of Arbitrary Order

It is straightforward to write down the relation between any vertex with n lines and vertices with $(n-1)$ lines.

Differentiating the basic equations (2.18) p times with respect to the scalar field, q times (q even) with respect to the pseudoscalar fields, and evaluating the result at the equilibrium point gives

$$\begin{aligned}
(\alpha_b-\alpha_a)&\left\langle\frac{\partial^{p+q+1}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_q}^{t_q}\,\partial S_{m_1}^{n_1}\cdots\partial S_{m_p}^{n_p}\,\partial S_b^a}\right\rangle_0\\
&=\sum_{i=1}^{p}\left(\delta_a^{m_i}\left\langle\frac{\partial^{p+q}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_q}^{t_q}\,\partial S_{m_1}^{n_1}\cdots\partial S_b^{n_i}\cdots\partial S_{m_p}^{n_p}}\right\rangle_0-\delta_{n_i}^b\left\langle\frac{\partial^{p+q}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_q}^{t_q}\,\partial S_{m_1}^{n_1}\cdots\partial S_{m_i}^{a}\cdots\partial S_{m_p}^{n_p}}\right\rangle_0\right)\\
&\quad+\sum_{j=1}^{q}\left(\delta_a^{r_j}\left\langle\frac{\partial^{p+q}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_b^{t_j}\cdots\partial\phi_{r_q}^{t_q}\,\partial S_{m_1}^{n_1}\cdots\partial S_{m_p}^{n_p}}\right\rangle_0-\delta_{t_j}^b\left\langle\frac{\partial^{p+q}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_j}^{a}\cdots\partial\phi_{r_q}^{t_q}\partial S_{m_1}^{n_1}\cdots\partial S_{m_p}^{n_p}}\right\rangle_0\right).
\end{aligned}\tag{B1}$$

The mass formula (3.3) corresponds to $q=0$, $p=1$ in (B1).

Similarly, differentiating the "axial-vector" invariance equation (2.21) p times with respect to the scalar field and q times (q odd) with respect to the pseudoscalar field results in

$$\begin{aligned}
(\alpha_a+\alpha_b)\left\langle\frac{\partial^{p+q+1}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_q}^{t_q}\,\partial S_{m_1}^{n_1}\cdots\partial S_{m_p}^{n_p}\,\partial\phi_b^a}\right\rangle_0&=-\sum_{i=1}^{p}\left(\delta_a^{m_i}\left\langle\frac{\partial^{p+q}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_q}^{t_q}\,\partial S_{m_1}^{n_1}\cdots\partial S_{m_{i-1}}^{n_{i-1}}\,\partial S_{m_{i+1}}^{n_{i+1}}\cdots\partial S_{m_p}^{n_p}\,\partial\phi_b^{n_i}}\right\rangle_0\right.\\
&\quad\left.+\,\delta_{n_i}^b\left\langle\frac{\partial^{p+q}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_q}^{t_q}\partial S_{m_1}^{n_1}\cdots\partial S_{m_{i-1}}^{n_{i-1}}\,\partial S_{m_{i+1}}^{n_{i+1}}\cdots\partial S_{m_p}^{n_p}\partial\phi_{m_i}^{a}}\right\rangle_0\right)+\sum_{j=1}^{q}\left(\delta_a^{r_j}\left\langle\frac{\partial^{p+q}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_{j-1}}^{t_{j-1}}\partial\phi_{r_{j+1}}^{t_{j+1}}\cdots\partial\phi_{r_q}^{t_q}\partial S_{m_1}^{n_1}\cdots\partial S_{m_p}^{n_p}\,\partial S_b^{t_j}}\right\rangle_0\right.\\
&\quad\left.+\,\delta_{t_j}^b\left\langle\frac{\partial^{p+q}V_0}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_{j-1}}^{t_{j-1}}\partial\phi_{r_{j+1}}^{t_{j+1}}\cdots\partial\phi_{r_q}^{t_q}\partial S_{m_1}^{n_1}\cdots\partial S_{m_p}^{n_p}\partial S_{r_j}^{a}}\right\rangle_0\right)+12i\,\delta_a^b\left\langle\frac{\partial^{p+q}}{\partial\phi_{r_1}^{t_1}\cdots\partial\phi_{r_q}^{t_q}\partial S_{m_1}^{n_1}\cdots\partial S_{m_p}^{n_p}}\left(\frac{\partial V_0}{\partial I_4}\,(\det M-\det M^\dagger)\right)\right\rangle_0
\end{aligned}\tag{B2}$$

The choices ($p=0$, $q=1$), ($p=1$, $q=1$), and ($p=0$, $q=3$) correspond to our formulas (3.6), (4.2), and (5.2), respectively.

Note that the left-hand side of (B1) vanishes for $a=b$ while the right-hand side of (B2) contains additional *a priori* unknown quantities in this case.

APPENDIX C

Energy-Momentum Tensor

The energy-momentum tensor $\Theta_{\mu\nu}$ computed from the Lagrangian (2.1) in the ordinary way does not satisfy the tracelessness condition $\Theta_{\mu\mu}=0$ when $V_0=V_{SB}=0$. This has led to the introduction[30] of a "new, improved" $\Theta_{\mu\nu}$ which has the same physical consequences as the old one but which does satisfy tracelessness in the appropriate limit.

Here we use the fact that the equations of motion of a system remain unchanged when we add a four-divergence to the Lagrangian density. We define a "new, improved" Lagrangian such that when $\Theta_{\mu\nu}$ is computed in the *ordinary way*, it will be the "new, improved" $\Theta_{\mu\nu}$. Thus we add a four-divergence to (2.1) to get

$$\begin{aligned}
\mathcal{L} &= -\tfrac{1}{2}\mathrm{Tr}(\partial_\mu\phi\,\partial_\mu\phi)-\tfrac{1}{2}\mathrm{Tr}(\partial_\mu S\partial_\mu S)\\
&\quad -V+\tfrac{1}{3}\partial_\mu\mathrm{Tr}(\phi\,\partial_\mu\phi+S\partial_\mu S)\\
&= -\tfrac{1}{6}\mathrm{Tr}(\partial_\mu\phi\,\partial_\mu\phi)-\tfrac{1}{6}\mathrm{Tr}(\partial_\mu S\partial_\mu S)\\
&\quad -V+\tfrac{1}{3}\mathrm{Tr}(\phi\Box\phi+S\Box S)\,,
\end{aligned}\tag{C1}$$

where in the second step we have shown that it is possible to regard $\mathcal{L}$ as a function of the fields, the derivatives of the fields, and the d'Alembertians of the fields, i.e.,

$$\mathcal{L}=\mathcal{L}(\phi,\partial_\mu\phi,\Box\phi;S,\partial_\mu S,\Box S). \tag{C2}$$

The equations of motion for a Lagrangian of the form (C2) follow from a variational principle and are

$$\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}+\Box\frac{\partial\mathcal{L}}{\partial\Box\phi}=0,$$

$$\frac{\partial\mathcal{L}}{\partial S}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu S)}+\Box\frac{\partial\mathcal{L}}{\partial\Box S}=0. \tag{C3}$$

For the Lagrangian (C1), Eqs. (C3) yield, as expected,

$$\Box\phi=\frac{\partial V}{\partial\phi},\ \Box S=\frac{\partial V}{\partial S}. \tag{C4}$$

The (possibily unsymmetrical) energy-momentum tensor $T_{\mu\nu}$ is derived from (C2) in the ordinary way by requiring that $\mathcal{L}$ have no *explicit* dependence on the space-time coordinates, i.e., $\partial\mathcal{L}/\partial x_\mu=0$. This yields

$$0=\frac{\partial\mathcal{L}}{\partial x_\mu}=\frac{d\mathcal{L}}{dx_\mu}-\mathrm{Tr}\left(\frac{\partial\mathcal{L}}{\partial\phi}\partial_\mu\phi+\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\partial_\nu\phi+\frac{\partial\mathcal{L}}{\partial\Box\phi}\partial_\mu\Box\phi\right)-\mathrm{Tr}\left(\frac{\partial\mathcal{L}}{\partial S}\partial_\mu S+\frac{\partial\mathcal{L}}{\partial(\partial_\nu S)}\partial_\mu\partial_\nu S+\frac{\partial\mathcal{L}}{\partial\Box S}\partial_\mu\Box S\right). \tag{C5}$$

Rewriting (C5) with the help of (C3) gives

$$\frac{d}{dx_\nu}\left\{\delta_{\mu\nu}\mathcal{L}-\mathrm{Tr}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\phi+\frac{\partial\mathcal{L}}{\partial\Box\phi}\partial_\mu\partial_\nu\phi-\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\Box\phi}\right)\partial_\mu\phi\right]-\mathrm{Tr}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu S)}\partial_\mu S+\frac{\partial\mathcal{L}}{\partial\Box S}\partial_\mu\partial_\nu S-\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\Box S}\right)\partial_\mu S\right]\right\}$$

$$\equiv\frac{d}{dx_\nu}\{T_{\mu\nu}\}=0, \tag{C6}$$

where it is also shown how $T_{\mu\nu}$ is to be identified. For the Lagrangian (C1), computation of $T_{\mu\nu}$ according to (C6) gives a result symmetrical on ($\mu\to\nu$) interchange, so that $T_{\mu\nu}$ is the same as $\Theta_{\mu\nu}$ here. We have then

$$\Theta_{\mu\nu}=-V\delta_{\mu\nu}+\tfrac{1}{6}\mathrm{Tr}[-\delta_{\mu\nu}\partial_\rho\phi\partial_\rho\phi+4\partial_\mu\phi\partial_\nu\phi-2\phi\partial_\mu\partial_\nu\phi+2\delta_{\mu\nu}\phi\Box\phi-\delta_{\mu\nu}\partial_\rho S\partial_\rho S+4\partial_\mu S\partial_\nu S-2S\partial_\mu\partial_\nu S+2\delta_{\mu\nu}S\Box S]. \tag{C7}$$

The 4-trace of this expression is, using the equations of motion (C3),

$$\Theta_{\mu\mu}=-4V+\mathrm{Tr}(\phi\Box\phi+S\Box S)$$

$$=-4V+\mathrm{Tr}\left(\phi\frac{\partial V}{\partial\phi}+S\frac{\partial V}{\partial S}\right). \tag{C8}$$

It is obvious that $\Theta_{\nu\nu}=0$ for $V=0$, so that $\Theta_{\mu\nu}$ is certainly qualified as a "new improved" tensor. Furthermore, when V is scale-invariant (i.e., a homogeneous function of order four in ϕ and S), $\Theta_{\mu\mu}$ also vanishes by Euler's theorem on homogeneous functions. In that event, (C8) is the same as (6.2) in the text. On the other hand, if we write $V=V_0+V_{SB}$ where only V_0 is scale-invariant, we have the relation

$$\Theta_{\mu\mu}=\mathrm{Tr}\left(\phi\frac{\partial V_{SB}}{\partial\phi}+S\frac{\partial V_{SB}}{\partial S}\right)-4V_{SB}. \tag{C9}$$

Now let us consider the question as to how relations like the ones given in Sec. VI of the text can be derived from matrix elements of the energy-momentum tensor. The method is an analog of the method used in Appendix A to derive consequences of broken chiral symmetry from current divergences. In that case the matrix element of the divergence of a current as computed from the direct definition of the current in terms of physical fields [e.g., (A5)] was equated to the matrix element of the same current divergence as computed in terms of V_{SB}, e.g., (A6). In the present case, the 4-trace $\Theta_{\mu\mu}$ is the analog of the current divergence. The direct definition of $\Theta_{\mu\nu}$ in terms of the physical fields is, from (C7),

$$\Theta_{\mu\nu}=-\delta_{\mu\nu}\left[\langle V\rangle_0+\tfrac{1}{2}\sum\left(\left\langle\frac{\partial^2V}{\partial\phi_a^b\,\partial\phi_c^d}\right\rangle_0\phi_a^b\phi_c^d+\left\langle\frac{\partial^2V}{\partial S_a^b\,\partial S_c^d}\right\rangle_0\tilde{S}_a^b\tilde{S}_c^d\right)+\cdots\right]$$

$$+\tfrac{1}{6}\mathrm{Tr}[-\delta_{\mu\nu}\partial_\rho\phi\partial_\rho\phi+4\partial_\mu\phi\partial_\nu\phi-2\phi\partial_\mu\partial_\nu\phi+2\delta_{\mu\nu}\phi\Box\phi-\delta_{\mu\nu}\partial_\rho\tilde{S}\partial_\rho\tilde{S}+4\partial_\mu\tilde{S}\partial_\nu\tilde{S}-2\tilde{S}\partial_\mu\partial_\nu\tilde{S}+2\delta_{\mu\nu}\tilde{S}\Box\tilde{S}]$$

$$+\tfrac{1}{3}\mathrm{Tr}(\delta_{\mu\nu}\underline{\alpha}\Box\tilde{S}-\underline{\alpha}\partial_\mu\partial_\nu\tilde{S}), \tag{C10}$$

where $\underline{\alpha}$ is the matrix introduced in (A1) of Appendix A. From (C10) the direct definition of $\Theta_{\mu\mu}$ is

$$\Theta_{\mu\mu}=-4\left[\langle V\rangle_0+\tfrac{1}{2}\sum\left(\left\langle\frac{\partial^2 V}{\partial\phi_a^b\partial\phi_c^d}\right\rangle_0\phi_a^b\phi_c^d+\left\langle\frac{\partial^2 V}{\partial S_a^b\partial S_c^d}\right\rangle_0\tilde{S}_a^b\tilde{S}_c^d\right)+\cdots\right]+\mathrm{Tr}(\phi\Box\phi+\tilde{S}\Box\tilde{S})+\mathrm{Tr}(\underline{\alpha}\Box\tilde{S}). \tag{C11}$$

On the other hand, the expression for $\Theta_{\mu\mu}$ when V_{SB} is given by (2.22) is found from (C9) to be [31]

$$\Theta_{\mu\mu}=6\mathrm{Tr}(\underline{A}\,\underline{\alpha})+6\mathrm{Tr}(\underline{A}\tilde{S}), \tag{C12}$$

where the matrix $\underline{A}$ is also defined in (A1) of Appendix A. We evaluate the matrix element $\sqrt{2q_0}\langle 0|\Theta_{\mu\mu}|\tilde{S}_b^b(q)\rangle$ from (C12) to give

$$\sqrt{2q_0}\langle 0|\Theta_{\mu\mu}|\tilde{S}_b^b(q)\rangle=6A_b\quad(b=1,2,3). \tag{C13}$$

We also evaluate to lowest order this same matrix element from (C11), noting that only the last term on the right-hand side, which may be written as

$$\sum_{a=1}^{3}\alpha_a\left\langle\frac{\partial^2 V}{\partial S_b^b\,\partial S_a^a}\right\rangle_0\tilde{S}_b^b+\cdots,$$

contributes. Then we get

$$\sqrt{2q_0}\langle 0|\Theta_{\mu\mu}|\tilde{S}_b^b(q)\rangle=\sum_{a=1}^{3}\alpha_a\left\langle\frac{\partial^2 V}{\partial S_b^b\,\partial S_a^a}\right\rangle_0\quad(b=1,2,3). \tag{C14}$$

Equating (C13) and (C14) results in (6.3) of the text when V_{SB} is given by (2.22).

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[1]A list of references is given in our previous paper: J. Schechter and Y. Ueda, Phys. Rev. D 3, 168 (1971). We designate this reference as paper I.

[2]M. Lévy, Nuovo Cimento 52A, 23 (1967).

[3]M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[4]J. A. Cronin, Phys. Rev. 161, 1483 (1967).

[5]W. A. Bardeen and B. W. Lee, Phys. Rev. 177, 2389 (1968).

[6]Some recent papers which have come to our attention are C. Cecchini and G. Cicogna, University of Pisa report (unpublished), and N. Papastamatiou, H. Umezawa, and D. Welling, Phys. Rev. D 3, 2267 (1971).

[7]J. Goldstone, Nuovo Cimento 19, 154 (1961); Y. Nambu, Phys. Rev. Letters 4, 380 (1960).

[8]A representative selection of recent work which also contains other references is G. Mack and A. Salam, Ann. Phys. (N.Y.) 53, 174 (1969); K. Wilson, Phys. Rev. 179, 1499 (1969); M. Gell-Mann, Cal Tech. Report No. CALT-68-244 (unpublished); C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970); D. J. Gross and J. Wess, Phys. Rev. D 2, 753 (1970); H. A. Kastrup, Nucl. Phys. B15, 179 (1970); S. P. de Alwis and P. S. O'Donnell, Phys. Rev. D 2, 1023 (1970); P. Carruthers, *ibid.* 2, 2265 (1970); J. Ellis, P. H. Weisz, and B. B. Zumino, CERN Report No. TH 1253 (unpublished); H. Kleinert and P. H. Weisz, CERN Report No. TH 1236 (unpublished); M. Chen, *ibid.* 3, 1025 (1971); M. A. B. Bég, J. Bernstein, D. J. Gross, R. Jackiw, and A. Sirlin, Phys. Rev. Letters 25, 1231 (1970); J. H. Lowenstein and B. Schroer, University of Pittsburgh Report No. NYO-3829-61 (unpublished).

[9]See, for example, L. K. Pande, Phys. Rev. Letters 25, 777 (1970); W. F. Palmer, Ohio State University Report No. 00-1545-82 (unpublished).

[10]M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

[11]One simple example is

$$V_{\mathrm{SB}}=C\left(S_1^1+S_2^2+\frac{1}{2\alpha}S_3^3S_3^3\right),$$

which clearly breaks $SU(3)$ but which satisfies

$$\left\langle\frac{\partial V_{\mathrm{SB}}}{\partial S_1^1}\right\rangle_0=\left\langle\frac{\partial V_{\mathrm{SB}}}{\partial S_3^3}\right\rangle_0=C\quad\text{when } w=1.$$

[12]Note that (3.6) and (3.3) are equivalent to Eqs. (24) and (25) of paper I when the extremum condition (22) or (22′) is used.

[13]This point is discussed in detail by R. Dashen, Princeton University report (unpublished).

[14]V. Mathur and S. Okubo, Phys. Rev. D 1, 2046 (1970); S. Okubo and V. Mathur, *ibid.* 2, 394 (1970).

[15]To compare with Mathur and Okubo, we shall take the isotopic-spin limit of our expressions, and for simplicity consider V_0 to have $U(3)\times U(3)$ symmetry so that $V_4=0$. Then the positivity of the π, η', K, and κ squared masses gives the four inequalities

$$\frac{A_1}{\alpha_1}>0,\ \frac{A_3}{\alpha_3}>0,\ \frac{A_1+A_3}{\alpha_1+\alpha_3}>0,\ \text{and}\ \frac{A_1-A_3}{\alpha_1-\alpha_3}>0.$$

Introducing $R=A_3/A_1$ and $w=\alpha_3/\alpha_1$ gives in addition to $A_1/\alpha_1>0$ the following:

$$\frac{R}{w}>0,\ \frac{1+R}{1+w}>0,\ \frac{1-R}{1-w}>0.$$

These three inequalities are all satisfied in the following four regions of the R-w plane:

(1) $0<R<1$, $0<w<1$;
(2) $R>1$, $w>1$;
(3) $-1<R<0$, $-1<w<0$;
(4) $R<-1$, $w<-1$.

Mathur and Okubo work in the a-b plane, where a and b are related to our R and w by $a=(1-R)/(2+R)$ and $b=(1-w)/(2+w)$. Our region (1) corresponds to their III, our (2) to their IV, our (3) to their II, and finally our (4) to their I, V, VI, and VII. The reason they get four regions instead of one in the last case is just because of a different choice of variables.

[16]The identification of the currents of this model with the Cabibbo currents gives $2\alpha=F_\pi\simeq 1.01\pi^0$ mass units and $F_K=\alpha+\alpha_3$, where F_π and F_K are the pion and kaon decay constants defined as in Sec. III of paper I.

[17]One way is to use the following formula which comes from diagonalization:

$$\tan 2\theta_p=\frac{-2\sqrt{2}\,(m_{11}+m_{12}-m_{13}-m_{33})}{m_{33}-m_{11}-m_{12}-8m_{13}}\ ,$$

where

$$m_{ab}=\left\langle\frac{\partial^2 V}{\partial\phi_a^a\,\partial\phi_b^b}\right\rangle_0\ .$$

The m_{ab}'s are calculable from our other formulas (see paper I).

[18]When $a\neq b$, equivalent relations to (4.2) can be found by differentiating (2.18) twice with respect to the pseudoscalar field.

[19]Two other possibilities would instruct us to (1) use a different form for V_{SB} or introduce interactions with derivatives, and (2) calculate the widths beyond the tree approximation, including the "triangle" diagrams for example. It seems likely that the introduction of vector mesons in this model will add derivative interactions which will decrease the widths. See, for example, G. Kramer, Phys. Rev. 177, 2515 (1969).

[20]E. Malamud and P. E. Schlein, in *Proceedings of a Conference on the ππ and Kπ Interactions at Argonne National Laboratory, 1969*, edited by F. Loeffler and E. Malamud (Argonne National Laboratory, Argonne, 1969), p. 107.

[21]Alan Rittenberg, Ph. D. thesis, LRL Report No. UCRL-18863, 1969 (unpublished). See also G. London *et al.*, Phys. Rev. 143, 1034 (1966).

[22]This process has been computed in the nonlinear theory by J. A. Cronin, Ref. 4, and J. Schwinger, Phys. Rev. 167, 1432 (1968). See also D. P. Majumdar, Phys. Rev. Letters 21, 502 (1968).

[23]Our T amplitude is defined in terms of the S matrix as

$$S=1-i(2\pi)^4\delta^{(4)}(P_f-P_i)\left(\frac{1}{\sqrt{2P_0}}\,\frac{1}{\sqrt{2q_0}}\,\frac{1}{\sqrt{2k_{+0}}}\,\frac{1}{\sqrt{2k_{-0}}}\right)T\ .$$

[24]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).

[25]See also P. Carruthers, Ref. 8.

[26]In this connection see also the discussion of M. Gell-Mann, Ref. 8.

[27]The most immediate consequence is found by evaluating (6.2) at the equilibrium point. This gives

$$\langle V_0\rangle_0=-\frac{1}{4}\sum_a \alpha_a\left\langle\frac{\partial V_{SB}}{\partial S_a^a}\right\rangle_0\ .$$

For the case when V_{SB} is given by (2.22),

$$\langle V_0\rangle_0=\frac{1}{2}\sum_a \alpha_a A_a\ ,$$

and the ground-state energy density is

$$\langle V\rangle_0=\langle V_0\rangle_0+\langle V_{SB}\rangle_0=-\tfrac{3}{2}\sum_a \alpha_a A_a\ .$$

[28]We would like to thank Professor M. Goldberg for discussion of this point.

[29]We neglected the T_η term for estimating the width and used the formula

$$\Gamma(\eta'\to\eta\pi^+\pi^-)=\frac{A^2}{64\eta'(3.1416)^3}\iint d\omega_+\,d\omega_-\ ,$$

where the double integral is over the allowed π^+ and π^- energy region. Numerically, $\iint d\omega_+ d\omega_-\simeq 0.23$, in units of squared pion masses. Note that isotopic-spin invariance predicts $\Gamma(\eta'\to\eta\pi^0\pi^0)=\frac{1}{2}\Gamma(\eta'\to\eta\pi^+\pi^-)$.

[30]For example, C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).

[31]Note that $\langle\Theta_{\mu\mu}\rangle_0$ is not zero but is $6\sum_{a=1}^{3}\alpha_a A_a$. This corresponds to nonzero ground-state energy for this system. See also Ref. 27.