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[8]Note that the vertical scale is twice that for Fig. 4(a); in order to compare the two figures it should be remembered that the total 3π spectrum must be the same for 4(a) as for 4(b).

[9]The truncation is important only for the $\pi^+\pi_a^-$ masses below the ρ band, as we have found by truncating the lower and upper halves of the ρ independently. For the former the results are similar to those shown.

[10]The incoherent background tends to extend the 3π mass spectrum to higher values, which are difficult to obtain in our double-Regge model. This background, of course, affects the $\pi^+\pi^-$ mass spectrum, but it does not give rise to threshold enhancements. The constant C of Eq. (2.4) has a numerical value of 0.18.

PHYSICAL REVIEW D VOLUME 3, NUMBER 11 1 JUNE 1971

Exact Bounds for K_{l3} Decay Parameters*

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(Received 17 March 1971)

We derive exact bounds for the K_{l3} decay form factors $f_\pm(t)$. Particularly, we find the bound $(m_K{}^2 - m_\pi{}^2)|f_+(0)| \le 16[\frac{1}{3}\pi\Delta(0)]^{1/2}(m_K + m_\pi)^{1/2}(m_K{}^{1/2} + m_\pi{}^{1/2})^{-1}$, where $\Delta(0)$ is the propagator of the divergence of the strangeness-changing current at zero momentum. If we further assume the Hamiltonian of Gell-Mann, Oakes, and Renner in order to estimate $\Delta(0)$, we obtain $|f_+(0)| \lesssim 1.0$. Similarly, an inequality testing the standard K_{l3} soft-pion theorem is found to be well satisfied. In addition, a new inequality involving derivatives of $f_+(t)$ is derived. Taking $\lambda_+ \sim 0.02$, this inequality leads to $|f_-(0)| \lesssim 0.33$.

I. INTRODUCTION AND SUMMARY OF PRINCIPAL RESULTS

According to the standard Cabibbo theory, all the properties of the K_{l3} decays are obtainable from two form factors $f_\pm(t)$ defined by

$$-i\langle\pi^0(p')|V_\mu^{(4-i5)}(0)|K^+(p)\rangle = (4p_0p_0'V^2)^{-1/2}(\tfrac{1}{2})^{1/2}i[(p_\mu + p_\mu')f_+(t) + (p_\mu - p_\mu')f_-(t)], \tag{1.1}$$

with $t = -(p-p')^2$. It is convenient to consider the combination

$$d(t) = (m_K{}^2 - m_\pi{}^2)f_+(t) + tf_-(t), \tag{1.2}$$

which can be obtained from Eq. (1.1) by means of

$$-i\langle\pi^0(p')|\partial_\mu V_\mu^{(4-i5)}(0)|K^+(p)\rangle = (4p_0p_0'V^2)^{-1/2}(\tfrac{1}{2})^{1/2}d(t). \tag{1.3}$$

Recently, Li and Pagels[1] derived the following exact inequality for the derivative of $d(t)$:

$$|d'(0)| \le (8/\sqrt{3})\Delta^{1/2}(0)I^{1/2}, \tag{1.4}$$

where $\Delta(t)$ is defined by

$$\Delta(t) = \tfrac{1}{2}i\int d^4x\, e^{iqx}\langle 0|(\partial_\mu V_\mu^{(4-i5)}(x), \partial_\nu V_\nu^{(4+i5)}(0))_+|0\rangle, \tag{1.5}$$

with $t = -q^2$, and I is the numerical integral

$$I=\int_{t_0}^{\infty} dt \frac{1}{t^2} \frac{1}{(t-t_0)^{1/2}(t-t_1)^{1/2}}, \tag{1.6}$$

with

$$t_0=(m_K+m_\pi)^2, \quad t_1=(m_K-m_\pi)^2. \tag{1.7}$$

If we further assume that the Hamiltonian density is given by[2,3]

$$H(x)=H_0(x)+\epsilon_0 S_0(x)+\epsilon_8 S_8(x), \tag{1.8}$$

where $S_{0,8}(x)$ are members of the $(3,3^*)\oplus(3^*,3)$ representation of $SW(3)=SU^{(+)}(3)\otimes SU^{(-)}(3)$ and $H_0(x)$ is the $SW(3)$ invariant, then it is known[4,5] that

$$\Delta(0)=-\tfrac{3}{4}\epsilon_8\langle 0|S_8(0)|0\rangle. \tag{1.9}$$

Many estimates of the vacuum expectation value of $S_8(x)$ suggest that it is small, corresponding to the approximate $SU(3)$ invariance of the vacuum state. Indeed, the estimate of Ref. 5 gives

$$\Delta^{1/2}(0)\approx 1.01\, m_\pi f_\pi. \tag{1.10}$$

Using $f_+(0)\approx 0.845$, Eq. (1.4) then implies a rather stringent bound,

$$|\xi+12.3\lambda_+|\lesssim 0.29, \tag{1.11}$$

as has been noted by Li and Pagels.[1] As usual, ξ and λ_+ are defined by

$$\xi=f_-(0)/f_+(0), \quad \lambda_+=m_\pi^2 f_+'(0)/f_+(0). \tag{1.12}$$

Experimental values[6] for ξ and λ_+ are still uncertain but they appear to be consistent with Eq. (1.11) if $|\xi|$ is not too large.

Unfortunately, the method of Li and Pagels is not directly applicable to an evaluation of bounds for $|d(0)|$ itself. The purpose of this note is first to derive an exact upper bound for $|d(0)|$ and second to improve the bound for $|d'(0)|$. We derive, for real t with $t<t_0$,

$$|d(t)|\lesssim 4[\tfrac{1}{3}\pi\Delta(0)]^{1/2} \times\left[1+\left(\frac{t_0}{t_0-t}\right)^{1/2}\right]^2\left[1+\left(\frac{t_0-t_1}{t_0-t}\right)^{1/2}\right]^{-1/2}. \tag{1.13}$$

Using the previous estimate, Eq. (1.10), for $\Delta(0)$ and noting that

$$d(0)=(m_K^2-m_\pi^2)f_+(0),$$

Eq. (1.13) leads to

$$|f_+(0)|\lesssim 1.01, \tag{1.14}$$

which is reasonable in view of the Ademollo-Gatto theorem. Next, we use our inequality (1.13) to test the standard soft-pion theorem[7,8]

$$d(\delta)=(m_K^2-m_\pi^2)[f_K/f_\pi+O(m_\pi^2)] \tag{1.15}$$

at the point $t=\delta\equiv m_K^2-m_\pi^2$. From Eqs. (1.13) and (1.15), we find

$$|f_K/f_\pi|\lesssim 1.43, \tag{1.16}$$

neglecting the small correction due to terms of order m_π^2. Equation (1.16) is experimentally reasonable.

We also derive an inequality involving $d'(0)$,

$$|\gamma d(0)+d'(0)|\lesssim\frac{4}{t_0}[\tfrac{1}{3}\pi\Delta(0)]^{1/2}\left[1+\left(\frac{t_0-t_1}{t_0}\right)^{1/2}\right]^{-1/2} \times\tfrac{1}{2}(A_1+A_2)(1+K^2)^{1/2}, \tag{1.17}$$

where γ is an arbitrary real number and

$$A_1=|1+4t_0\gamma|, \quad A_2=|1-4t_0\gamma|, \quad K=\frac{3A_1+A_2}{A_1+A_2}-(t_0-t_1)^{1/2}[t_0^{1/2}+(t_0-t_1)^{1/2}]^{-1}. \tag{1.18}$$

When we set $\gamma=0$, Eq. (1.17) leads to an inequality for $|d'(0)|$, which is numerically slightly better than but practically almost indistinguishable from that given by Li and Pagels. However, if we choose γ to be $m_\pi^2\gamma=-\lambda_+$, then we find $\gamma d(0)+d'(0)=f_-(0)$ and our inequality gives an upper bound for $|f_-(0)|$ when λ_+ is known. For example, the values $\lambda_+=0.02$, 0.03, and 0.06 give, respectively,

$$|f_-(0)|\lesssim 0.33,\ 0.55,\ \text{and}\ 1.22. \tag{1.19}$$

II. DERIVATION

First, we write down the familiar Lehmann-Källén representation for $\Delta(t)$:

$$\Delta(t)=\int_{t_0}^{\infty} dt' \frac{\rho(t')}{t'-t}, \tag{2.1}$$

where the spectral weight $\rho(t)$ is given by

$$\rho(-q^2)=\tfrac{1}{2}(2\pi)^3\sum_n|\langle 0|\partial_\mu V_\mu^{(4-i5)}(0)|n\rangle|^2\delta^{(4)}(q-p_n). \tag{2.2}$$

Because of the positivity of $\rho(t)$, we can compute a lower bound for it by restricting the summation over $|n\rangle$ to the K-π intermediate states only. Then, as has been noted by Li and Pagels, we discover for $t\geqslant t_0$ the inequality

$$|d(t)|^2\leqslant\tfrac{64}{3}\pi^2 t(t-t_0)^{-1/2}(t-t_1)^{-1/2}\rho(t). \tag{2.3}$$

Now, the usual consideration shows that $d(t)$ is the boundary value of a real analytic function [which we shall simply write as $d(t)$ again] at the cut $t_0\leqslant t<\infty$. The reality condition implies

$$d^*(t^*)=d(t) \tag{2.4}$$

in the cut plane. Equation (2.3) enables us to compute an upper bound of $|d(t)|$ on the cut. If $\Delta(0)$ is known, then we can estimate

$$\int_{t_0}^{\infty} dt \frac{1}{t^2}(t-t_0)^{1/2}(t-t_1)^{1/2}|d(t)|^2 \leq \tfrac{64}{3}\pi^2\Delta(0). \tag{2.5}$$

It is now natural to question if we can get some information regarding $|d(0)|$ when $\Delta(0)$ is known. As we shall see shortly, the answer is affirmative. To that end we shall assume that $d(t)$ satisfies a dispersion relation with finite numbers of subtractions. As a matter of fact, the validity of Eq. (2.5) strongly suggests that it may satisfy an unsubtracted dispersion relation, but we need not assume so for our purpose.

In order *not* to introduce unknown parameters, we will make use of the following method. Let $\phi(t)$ be an arbitrary analytic function of t with a possible cut at the interval $t_0 \leq t < \infty$. Moreover, we require $\phi(t)$ to go to zero sufficiently fast at infinity. Then, a function $G(t) \equiv \phi(t)d(t)$ is an analytic function of t with cut at $t_0 \leq t < \infty$ and will satisfy the unsubtracted dispersion relation

$$\phi(t)d(t) = \frac{1}{2\pi i}\int_{\text{cut}} d\xi \frac{1}{\xi - t}\phi(\xi)d(\xi), \tag{2.6}$$

where the integration is over both upper and lower cuts at $t_0 \leq \xi < \infty$. As an example of $\phi(t)$ satisfying these required conditions, we could choose

$$\phi(t) = [(t-t_0)^{1/2} + (t-c)^{1/2}]^{-n},$$

where c is an arbitrary real constant larger than t_0 and n is a sufficiently large positive number. Our method is essentially a generalization[9] of the continuous-moment sum rule.

If we fix the value of t by setting $t = a < t_0$, Eq. (2.6) is now written as

$$d(a) = \frac{1}{2\pi i}\int_{\text{cut}} d\xi \frac{1}{\xi - a} f(\xi)d(\xi), \tag{2.7}$$

where we set

$$f(\xi) = \frac{1}{\phi(a)}\phi(\xi). \tag{2.8}$$

From our construction, $f(\xi)$ is an *arbitrary* analytic function of ξ in the cut plane, provided that it satisfies the condition

$$f(a) = 1 \tag{2.9}$$

and that the integral in Eq. (2.7) converges. This fact is crucial to what follows.

It is more convenient to map our cut ξ plane into the interior of the unit circle, $|z| < 1$, by the following conformal transformation:

$$(\xi - t_0)^{1/2} = i\alpha\frac{1+z}{1-z}, \quad \alpha = (t_0 - a)^{1/2} > 0. \tag{2.10}$$

By this transformation, it is easy to check that the upper and lower cuts in the ξ plane are respectively mapped on lower and upper semicircle of $|z| = 1$, and the three points $\xi = \infty$, a, and t_0 are projected onto $z = 1, 0$, and -1, respectively. Defining

$$D(z) \equiv d(\xi), \quad F(z) \equiv f(\xi), \tag{2.11}$$

Eq. (2.7) is now written as

$$d(a) = -\frac{1}{2\pi}\int_0^{2\pi} d\theta \cot\tfrac{1}{2}\theta D(e^{i\theta})F(e^{i\theta}), \tag{2.12}$$

where we parametrized $z = e^{i\theta}$ on the circle.

In terms of the new variable, $F(z)$ is an analytic function inside $|z| < 1$, subject only to the constraints that

$$F(0) = 1 \tag{2.13}$$

and the integral on the right-hand side of Eq. (2.12) is finite.

Let $k(\theta)$ be an arbitrary positive function of θ on the circle. We can then majorize the integral of Eq. (2.12) by means of the Schwarz inequality to obtain

$$|d(a)| \leq \frac{1}{2\pi}\left[\int_0^{2\pi} d\theta \frac{\cot^2(\frac{1}{2}\theta)}{k(\theta)}|D(e^{i\theta})|^2\right]^{1/2} \times \left[\int_0^{2\pi} d\theta k(\theta)|F(e^{i\theta})|^2\right]^{1/2}. \tag{2.14}$$

Choosing

$$k(\theta) = |\cot\tfrac{1}{2}\theta|\cos^2(\tfrac{1}{2}\theta) \times \left[\left(1 + \frac{t_0}{t_0 - a}\tan^2(\tfrac{1}{2}\theta)\right)^2 \Big/ \left(1 + \frac{t_0 - t_1}{t_0 - a}\tan^2(\tfrac{1}{2}\theta)\right)^{1/2}\right] \tag{2.15}$$

and noting the reality condition Eq. (2.4), we find

$$\int_0^{2\pi} d\theta \frac{1}{k(\theta)}\cot^2(\tfrac{1}{2}\theta)|D(e^{i\theta})|^2 \leq \tfrac{128}{3}\pi^2\Delta(0), \tag{2.16}$$

where we used the inequality Eq. (2.5) after we remapped the integral on the unit circle into the original cut plane. Setting

$$J = \frac{1}{2\pi}\int_0^{2\pi} d\theta k(\theta)|F(e^{i\theta})|^2, \tag{2.17}$$

we find

$$|d(a)| \leq 8[\tfrac{1}{3}\pi\Delta(0)]^{1/2}J^{1/2}. \tag{2.18}$$

It is important to remember now that $F(z)$ is an arbitrary analytic function of z for $|z| < 1$ such that $F(0) = 1$. We may forget about the condition that J must be finite, since then Eq. (2.18) is trivially satisfied. Hence, our task is to discover an appropriate $F(z)$ which minimizes the integral J. At this point, we may remark that the method of Li and Pagels is equivalent to setting $F(z) \equiv 1$ identically. Because J becomes divergent with this choice $F(z) = 1$, their method is not applicable to an evaluation of bounds for $|d(0)|$. At any rate, $k(\theta)$ given by Eq. (2.15) is singular at $\theta=0$ and 2π, so that it is not summable on the circle. To eliminate this deficiency, we set

$$F(z)=(1-z)^{\beta}G(z) \tag{2.19}$$

for a positive β. We then have

$$J=\frac{1}{2\pi}\int_0^{2\pi} d\theta\, w(\theta)\,|G(e^{i\theta})|^2 , \tag{2.20}$$

where $w(\theta)$ is given by

$$w(\theta)=k(\theta)(2\sin\tfrac{1}{2}\theta)^{2\beta} \quad (\beta>0)\,. \tag{2.21}$$

Note that $w(\theta)$ is a positive summable function of θ on the unit circle, because of the multiplicative factor $\sin^{2\beta}(\frac{1}{2}\theta)$. Also, $G(z)$ is an arbitrary analytic function of z for $|z|<1$ such that $G(0)=1$. We can now minimize the integral J by means of the well-known Szegö theorem[10,11] which states

$$\operatorname{Inf}\ \frac{1}{2\pi}\int_0^{2\pi} d\theta\, w(\theta)\,|1+a_1z+a_2z^2+\cdots+a_Nz^N|^2$$
$$=\exp\left[\frac{1}{2\pi}\int_0^{2\pi} d\theta \ln w(\theta)\right] \quad (z=e^{i\theta})\,, \tag{2.22}$$

where the infimum must be taken over all complex numbers $a_1,\ldots,a_N$ and arbitrary non-negative integer N. Therefore, we compute

$$\operatorname{Inf} J=\tfrac{1}{4}\left[1+\left(\frac{t_0}{t_0-a}\right)^{1/2}\right]^4\left[1+\left(\frac{t_0-t_1}{t_0-a}\right)^{1/2}\right]^{-1}. \tag{2.23}$$

Note that the infimum does not depend[12] upon the arbitrary constant β we introduced in Eq. (2.19). In computing Eq. (2.23) we used the elementary formula

$$\int_0^{2\pi} d\theta \ln[(1+c^2\tan^2(\tfrac{1}{2}\theta)] = 4\pi\ln(1+c) \quad (c\geq 0)\,. \tag{2.24}$$

Summarizing, we find finally

$$|d(a)|\leq 4\,[\tfrac{1}{3}\pi\Delta(0)]^{1/2}\left[1+\left(\frac{t_0}{t_0-a}\right)^{1/2}\right]^2 \times\left[1+\left(\frac{t_0-t_1}{t_0-a}\right)^{1/2}\right]^{-1/2}. \tag{2.25}$$

When we replace a by t, this gives Eq. (1.13) of Sec. I.

It is worth mentioning that we can also derive Eq. (2.25) by a method more simple and mathematically more rigorous than used here, as we shall see in the Appendix. Also, we shall there prove that our inequality is the best one we can obtain. Unfortunately, the method does not seem to be easily generalizable for inequalities involving $d'(t)$, and therefore we have here used a slightly more awkward method in deriving Eq. (2.25).

To find an inequality for $d'(0)$, we differentiate both sides of Eq. (2.6) with respect to t and then set $t=a<t_0$ to obtain

$$d'(a)+\gamma d(a)=\frac{1}{2\pi i}\int_{\text{cut}} d\xi\,\frac{1}{(\xi-a)^2}f(\xi)d(\xi)\,, \tag{2.26}$$

where γ is given by

$$\gamma=\frac{1}{\phi(a)}\phi'(a)=f'(a)\,. \tag{2.27}$$

Repeating the same procedure as before, we find

$$|\gamma d(a)+d'(a)|\leq 8[\tfrac{1}{3}\pi\Delta(0)]^{1/2}J^{1/2}\frac{1}{t_0-a}, \tag{2.28}$$

where J is defined now by

$$J=\frac{1}{2\pi}\int_0^{2\pi} d\theta\, k(\theta)\sin^4(\tfrac{1}{2}\theta)\,|F(e^{i\theta})|^2 , \tag{2.29}$$

while $k(\theta)$ is still given by Eq. (2.15).

An important difference in the present case as compared to the previous one is that $F(z)$ is an arbitrary analytic function of z in $|z|<1$, which now satisfies two constraints,

$$F(0)=1, \quad F'(0)=-4(t_0-a)\gamma\,. \tag{2.30}$$

Before going into details, we remark that the result, Eq. (1.4), of Li and Pagels is obtainable from Eq. (2.28) by setting $\gamma=0$ and $F(z)\equiv 1$. Because of the new constraint for $F'(0)$, we cannot directly apply the Szegö theorem. However, we can still evaluate an infimum of J as follows. Let us set

$$F(z)=[1-4(t_0-a)\gamma z]G(z) \tag{2.31}$$

and

$$w(\theta)=k(\theta)\sin^4(\tfrac{1}{2}\theta) \times[1+16(t_0-a)^2\gamma^2-8(t_0-a)\gamma\cos\theta]\,. \tag{2.32}$$

Then, we have to minimize

$$J=\frac{1}{2\pi}\int_0^{2\pi} d\theta\, w(\theta)\,|G(e^{i\theta})|^2 \tag{2.33}$$

with respect to an arbitrary analytic function $G(z)$ subject to the conditions

$$G(0)=1, \quad G'(0)=0\,. \tag{2.34}$$

Since the positive function $w(\theta)$ given by Eq. (2.32) is summable on the unit circle, we can apply a generalization[13] of the Szegö theorem which states that

$$\operatorname{Inf}\frac{1}{2\pi}\int_0^{2\pi} d\theta\, w(\theta)\,|1+a_2z^2+\cdots+a_Nz^N|^2$$
$$=|g(0)|^2+|g'(0)|^2 \quad (z=e^{i\theta})\,, \tag{2.35}$$

where $g(z)$ is given by

$$g(z)=\exp\left(\frac{1}{4\pi}\int_0^{2\pi} d\theta\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln w(\theta)\right). \tag{2.36}$$

Using the formula

$$\int_0^{2\pi} d\theta \cos\theta \ln[1+c^2\tan^2(\tfrac{1}{2}\theta)] = -4\pi\frac{c}{1+c} \quad (c \geq 0),$$

$$\int_0^{2\pi} d\theta \sin\theta \ln[1+c^2\tan^2(\tfrac{1}{2}\theta)] = 0,$$

we compute

$$|\gamma d(a) + d'(a)| \leq [\tfrac{1}{3}\pi\Delta(0)]^{1/2}\frac{1}{t_0-a}\left[1+\left(\frac{t_0}{t_0-a}\right)^{1/2}\right]^2 \times\left[1+\left(\frac{t_0-t_1}{t_0-a}\right)^{1/2}\right]^{-1/2}\frac{A_1+A_2}{2}(1+K^2)^{1/2}, \tag{2.37}$$

where A_1, A_2, and K are

$$A_1 = |1+4(t_0-a)\gamma|, \quad A_2 = |1-4(t_0-a)\gamma|,$$
$$K = \frac{A_1-A_2}{A_1+A_2} + 4\frac{(t_0)^{1/2}}{(t_0)^{1/2}+(t_0-a)^{1/2}} - \frac{(t_0-t_1)^{1/2}}{(t_0-t_1)^{1/2}+(t_0-a)^{1/2}}. \tag{2.38}$$

When we set $a=0$, Eq. (2.37) reduces to Eq. (1.17). Our upper bound for $|d'(0)|$, which is obtainable from Eq. (2.37) by setting $\gamma=0$, is in principle an improvement over that given by Li and Pagels. However, both give essentially indistinguishable numerical results.

The method outlined above is applicable to a variety of problems. For example, let us consider the Lehmann-Källén representation for the hadronic electromagnetic current $j_\mu(x)$,

$$\langle 0|[j_\mu(x), j_\nu(y)]|0\rangle = \int_0^\infty dm^2\left(\delta_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu\right) \times\rho(m^2)\Delta(x-y,m), \tag{2.39}$$

where the spectral weight $\rho(m^2)$ is defined by

$$(2\pi)^3\sum_n \langle 0|j_\mu(0)|n\rangle\langle n|j_\nu(0)|0\rangle\delta^{(4)}(p_n-k) = (\delta_{\mu\nu}+\frac{1}{m^2}k_\mu k_\nu)\rho(m^2) \quad (k^2=-m^2<0). \tag{2.40}$$

Again, due to the positivity of the spectral weight $\rho(m^2)$, we can obtain a lower bound by taking only the two-pion intermediate state,

$$\rho(t) \geq \frac{1}{48\pi^2}t^{-1/2}(t-4\mu^2)^{3/2}|F_\pi(t)|^2 \quad (t\geq 4\mu^2), \tag{2.41}$$

where μ is the pion mass and $F_\pi(t)$ is the electromagnetic form factor of a positive pion:

$$\langle\pi^+(p')|j_\mu(0)|\pi^+(p)\rangle = (4p_0p_0'V^2)^{-1/2}(p_\mu+p_\mu')F_\pi(t) \quad [t=-(p-p')^2]. \tag{2.42}$$

Repeating essentially the same procedure, it is not difficult to derive, for $t<t_0=4\mu^2$,

$$|F_\pi(t)| \leq 2[3\pi\Delta_n(0)]^{1/2}[(t_0-t)^{1/2}+t_0^{1/2}]^{n+1/2} \times(t_0-t)^{-5/4}, \tag{2.43}$$

where $t_0=4\mu^2$, n is an arbitrary non-negative number, and $\Delta_n(0)$ is defined by

$$\Delta_n(0) = \int_{t_0}^\infty dt\, t^{-n}\rho(t). \tag{2.44}$$

When we set $t=0$ and notice

$$F_\pi(0)=1, \tag{2.45}$$

Eq. (2.43) leads to the inequality

$$\int_{t_0}^\infty dt\frac{1}{t^n}\rho(t) \geq \frac{1}{384\pi}(4\mu)^{4-2n}. \tag{2.46}$$

In particular, if we set $n=1$, the left-hand side of Eq. (2.46) represents the Schwinger term and we find that it must satisfy the absolute inequality

$$\int_{t_0}^\infty dt\frac{1}{t}\rho(t) \geq \frac{1}{24\pi}\mu^2. \tag{2.47}$$

Applications to other problems will be given elsewhere.

Note added in proof. After this paper had been written, the author received a preprint by Li and Pagels [Phys. Rev. D (to be published)], in which they derive Eq. (1.13) by the method of Meiman.

ACKNOWLEDGMENT

The author would like to express his gratitude to Dr. D. Palmer for reading the manuscript.

APPENDIX

As we have mentioned in Sec. II, we can derive the upper bound for $|d(t)|$ in a mathematically more rigorous way as follows. Suppose $f(z)$ is analytic inside the unit circle, and p is a fixed positive number. If we have

$$I_p(\gamma, f) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, |f(\gamma e^{i\theta})|^p \leq M < \infty$$

for all $0\leq\gamma<1$, then we say $f(z)$ belongs to the class H^p.

Any function $f(z)$ belonging to $H^1(p=1)$ is known to admit the decomposition[10]

$$f(z) = cB(z)S(z)F(z), \tag{A1}$$

where c is a constant with unit modulus, and $B(z)$, $S(z)$, and $F(z)$ are given by

$$B(z)=z^m\prod_{n=1}^{\infty}\left(\frac{\alpha_n^*}{|\alpha_n|}\ \frac{\alpha_n-z}{1-\alpha_n^*z}\right),$$

$$S(z)=\exp\left[-\int_0^{2\pi}d\mu(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}\right], \tag{A2}$$

$$F(z)=\exp\left[\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln|f(e^{i\theta})|\right].$$

The function $B(z)$ is the so-called Blaschke product, where α_n are the zero points of $f(z)$ inside the circle satisfying the condition

$$|\alpha_n|<1,\quad \sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty. \tag{A3}$$

$S(z)$ is the singular function with singular non-negative measure $\mu(\theta)$, and $F(z)$ is said to be an outer function.

From Eqs. (A1) and (A2), we find

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\ln|f(e^{i\theta})| =\ln|f(0)|+\sum_n\ln|\alpha_n|^{-1}+\int d\mu(\theta), \tag{A4}$$

provided $f(0)\neq 0$. Since $|\alpha_n|<1$ and $\int d\mu(\theta)\geq 0$, this gives the Jensen inequality[10]

$$\ln|f(0)|\leq\frac{1}{2\pi}\int_0^{2\pi}d\theta\ln|f(e^{i\theta})|. \tag{A5}$$

Unfortunately, our function $D(z)$ defined in Sec. II does not necessarily belong to the class H^1 since $D(z)$ may have a singularity at $z=1$ corresponding to a possible divergence of $d(t)$ at infinity. However, we can circumvent this difficulty as follows. Assume now that $D(z)$ is analytic inside $|z|<1$, and that we can find an outer H^2 function $g(z)$ such that the product $f(z)=g(z)D(z)$ belongs to the class H^1. Then, applying the Jensen inequality to $f(z)$, we have

$$\ln|D(0)|+\ln|g(0)| \leq\frac{1}{2\pi}\int_0^{2\pi}d\theta[\ln|D(e^{i\theta})|+\ln|g(e^{i\theta})|].$$

However, any outer H^2 function $g(z)$ must satisfy the identity[10,11]

$$\ln|g(0)|=\frac{1}{2\pi}\int_0^{2\pi}d\theta\ln|g(e^{i\theta})|. \tag{A6}$$

Therefore, we conclude that

$$\ln|D(0)|\leq\frac{1}{2\pi}\int_0^{2\pi}d\theta\ln|D(e^{i\theta})|. \tag{A7}$$

Hence, the Jensen inequality is also valid for such functions $D(z)$.

Now, in our case, $d(t)$, defined in Sec. II, is assumed to satisfy a dispersion relation with a finite number of subtractions. This implies

$$\int_{t_0}^{\infty}dt\frac{1}{t^n}|d(t)|<\infty \tag{A8}$$

for a sufficiently large positive integer n. Now, we map the cut t plane into the interior of the unit circle $|z|=1$ by Eq. (2.10). In terms of the corresponding function $D(z)$, the condition Eq. (A8) is then equivalent to a statement that the function $f(z)=(1+z)(1-z)^nD(z)$ for a sufficiently large positive integer n must belong to H^1. Since $g(z)=(1+z)\times(1-z)^n$, for a positive integer n, is outer and belongs to H^2, we conclude that $D(z)$ satisfies the Jensen inequality Eq. (A7).

After these preliminaries, let us consider the following mathematical problem. Let $\rho(\theta)$ be a non-negative function on the circle such that $\ln\rho(\theta)$ is summable. Suppose that the quantity

$$I_p=\left[\frac{1}{2\pi}\int_0^{2\pi}d\theta\,\rho(\theta)|D(e^{i\theta})|^p\right]^{1/p}\quad (p>0) \tag{A9}$$

is given and $D(z)$ satisfies the Jensen inequality Eq. (A7). Then can we find an upper bound for $|D(0)|$? The answer is yes, with the result

$$|D(0)|\leq I_p\exp\left[-\frac{1}{2\pi p}\int_0^{2\pi}d\theta\ln\rho(\theta)\right]\quad (p>0). \tag{A10}$$

Also, we can prove that this is the best inequality possible.

The proof is elementary. We can rewrite the Jensen inequality Eq. (A7) as

$$\ln|D(0)|\leq\frac{1}{2\pi p}\int_0^{2\pi}d\theta\ln[\rho(\theta)|D(e^{i\theta})|^p] -\frac{1}{2\pi p}\int_0^{2\pi}d\theta\ln\rho(\theta). \tag{A11}$$

We now note the validity of the well-known inequality[10]

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\ln|h(\theta)|\leq\ln\left[\frac{1}{2\pi}\int_0^{2\pi}d\theta|h(\theta)|\right] \tag{A12}$$

for a summable function $h(\theta)$. Hence, we can rewrite Eq. (A11) as

$$\ln|D(0)|\leq\frac{1}{p}\ln\left[\frac{1}{2\pi}\int_0^{2\pi}d\theta\,\rho(\theta)|D(e^{i\theta})|^p\right] -\frac{1}{2\pi p}\int_0^{2\pi}d\theta\ln\rho(\theta).$$

This is nothing but the inequality Eq. (A10). Also from this derivation it is obvious that the equality in Eq. (A10) is possible if and only if we have

$$D(z)=I_p\exp\left[-\frac{1}{2\pi p}\int_0^{2\pi}d\theta\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln\rho(\theta)\right]. \tag{A13}$$

Therefore, our inequality is the best one we can obtain.

The result of Sec. II is reproduced easily if we choose $p = 2$ and

$$w(\theta) \equiv \frac{1}{\rho(\theta)} = |\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta|\left(1+\frac{t_0}{t_0-a}\tan^2(\tfrac{1}{2}\theta)\right)^2 \times\left(1+\frac{t_0-t_1}{t_0-a}\tan^2(\tfrac{1}{2}\theta)\right)^{-1/2}, \qquad (A14)$$

so that we have

$$I_2 \leqslant 8[\tfrac{1}{3}\pi\Delta(0)]^{1/2}. \qquad (A15)$$

Hence, Eq. (A10) gives

$$|D(0)| \leqslant 8[\tfrac{1}{3}\pi\Delta(0)]^{1/2}\exp\left[\frac{1}{4\pi}\int_0^{2\pi} d\theta \ln w(\theta)\right].$$

This is nothing but the bound Eq. (2.25).

Unfortunately, this technique does not seem to be easily generalizable to the evaluation of bounds for $|D'(0)|$.

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*Work supported in part by the U. S. Atomic Energy Commission.

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[12]This is a special case of an identity Eq. (A6) of the Appendix. Since $g(z) = (1-z)^\beta$ $(\beta > 0)$ is outer and belongs to H^2 class, we must have

$$\int_0^{2\pi} d\theta \ln|g(e^{i\theta})| = 2\pi \ln|g(0)| = 0,$$

as we may easily check.

[13]See Ref. 11, pp. 21 and 22. The function $g(z)$ defined by Eq. (2.36) is an outer H^2 function since $\ln w(\theta)$ is summable. Also, it satisfies $w(\theta) = |g(e^{i\theta})|^2$, so that Eq. (2.35) follows.