Threshold Enhancements, Identical Particles, and Diffractive Dissociation*

Bing Lin Young,† J. T. Donohue,‡ and Ronald F. Peierls
Brookhaven National Laboratory, Upton, New York 11973
(Received 22 September 1970)

A discussion of some recent experimental results on threshold enhancements in the reaction $\pi^-p \to \pi^+\pi^-\pi^-p$ is given in terms of a diffractive-dissociation model. In particular, we try to explain why the quasi-two-particle enhancements observed in $\pi_a^-(\pi^+\pi_b^-)$ for different $\pi^+\pi_b^-$ masses subsequently disappear when those $(\pi^+\pi_a^-)$ events in the ρ band are removed. The model is a double-Regge-exchange model symmetrized between the two identical pions. The parameters of the model are adjusted to fit the over-all 2π and 3π spectra, and the question of the enhancements and their disappearance is studied numerically. The model is able to reproduce the observed effects, provided a truncated Breit-Wigner line shape is used for the ρ meson.

I. INTRODUCTION

In a recent study of the reaction

$$\pi^-p \to p\pi^+\pi^-\pi^- \tag{1.1}$$

at 6 GeV/c, Crennell *et al.*[1] have reported enhancements of the three-pion mass spectrum in the regions of the $\rho^0\pi^-$ and $f^0\pi^-$ thresholds. A similar enhancement (L meson) associated with the $K^*(1420)\pi$ threshold has been studied by Barbaro-Galtieri *et al.*[2] in the reaction

$$K^+p \to K^+\pi^+\pi^-p \tag{1.2}$$

at 12 GeV/c. If these enhancements were due to the production and subsequent decay of a quasi-two-particle resonance, the shape of the enhancement should be more or less independent of the mass of the pair of particles forming the ρ (or f, or K^*), whereas if the phenomena observed were threshold enhancements corresponding to a quasi-two-particle threshold, the position of the peak should move upwards as the two-particle mass moves upwards through the ρ (or f or K^*).

The results are shown in Figs. 1 and 2, and lead to opposite conclusions in the two cases. The purpose of this paper is to point out that the behavior in the two reactions is quite compatible with a description of both by a diffractive-dissociation picture leading to a threshold enhancement of the $\rho\pi$, $f\pi$, or $K^*\pi$ system. The apparent discrepancy stems from the existence of two identical particles in the case of reaction (1.1). To simplify the discussion, let us denote the two π^- as π_a^- and π_b^-, respectively. The shaded histogram in Fig. 1 shows all events plotted as a function of the mass of the three-pion system, for various mass ranges of the dipion $\pi^+\pi_a^-$. The unshaded histogram removes all events in which the dipion $\pi^+\pi_b^-$ has a mass in the ρ region while $\pi^+\pi_a^-$ does not. (We shall refer to this as an anti-ρ cut.) This effectively removes the enhancement except in the ρ^0 region for $\pi^+\pi_a^-$.[3] In the $K^+\pi^-$ case, there is no need to make a corresponding cut, since the amount of ρ^0 present is quite small. In Fig. 2 it can be seen that in this case the enhancement is present for all $K^+\pi^-$ mass regions and changes its position from region to region.

In order to see whether these results can be understood, we have constructed a simple phenomenological model for reaction (1.1) embodying Bose symmetry. It is not our aim to show that some particular model can reproduce the effect observed, but rather to show that for any reasonable symmetric model which describes the 2π and uncut 3π spectra, the anti-ρ cut will reduce the threshold enhancements. The essential features to be built into the model are:

(1) The two-particle mass spectrum should be approximately that observed, with a prominent p-wave and lesser d-wave resonances.

(2) The model should predict three-particle threshold enhancements for fixed two-particle masses, again roughly in agreement with experiment.

(3) The amplitude should be symmetric under exchange of the π^-'s.

The model we chose was diffractive dissociation producing the quasi-three-body final state

$$\pi^- + p \to X^0 + \pi_a + p, \qquad X^0 \to \pi^+ + \pi_b^- ,$$

where X^0 denotes ρ^0 or f^0. Bose symmetry is incorporated by adding the contributions of the two diagrams of Fig. 3. In calculating these diagrams, we have taken a Regge-pole model for the exchanged pion and the subsequent rescattering, but we do not believe this to be a crucial feature of the results. The angular distribution of X^0 was taken to be that predicted by elementary pion ex-

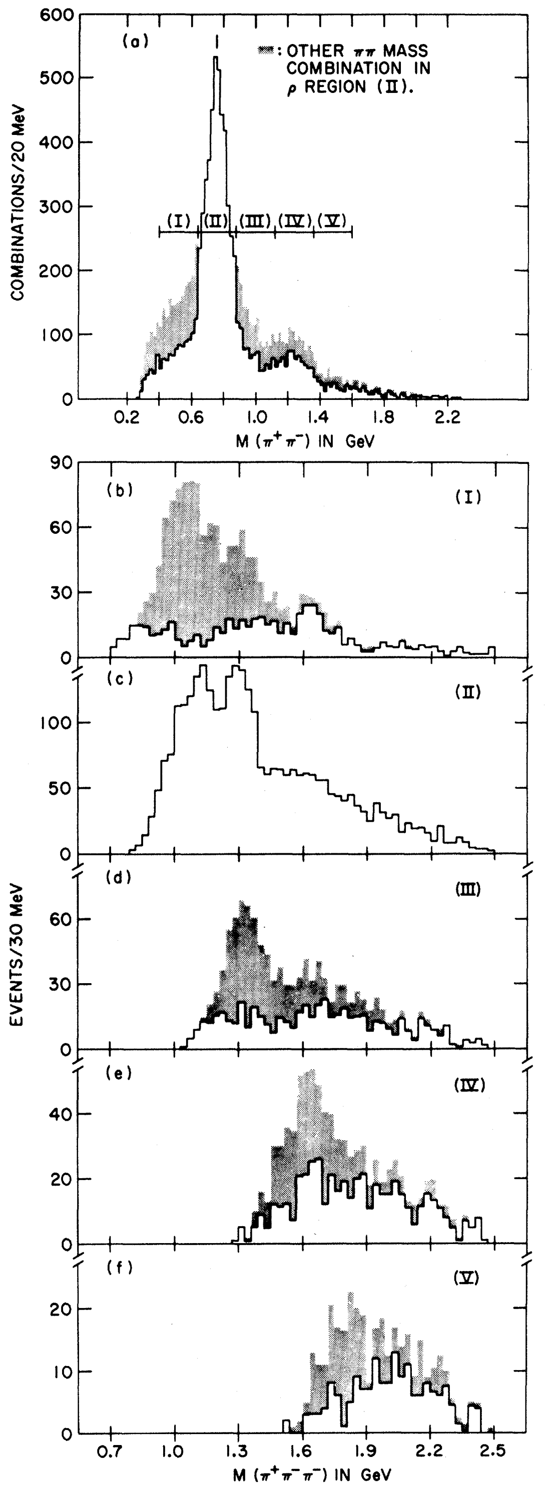

FIG. 1. Experimental data of Crennell *et al.*, Ref. 1, showing the $\pi^+\pi^-$ mass spectrum, and the $\pi^+\pi^-\pi^-$ mass spectra for fixed intervals in $\pi^+\pi^-$ mass. Shaded events are those for which the other $\pi^+\pi^-$ combination occurs in the ρ-mass region. When the shaded events are removed threshold enhancements vanish.

change, namely, $\propto|P_l(\cos\theta)|^2$, and the integration over phase space was done by Monte Carlo methods. This double-Regge model can account for the observed three-pion enhancements, and the parameters of the ρ and f^0 can be chosen so as to approximate the experimental $\pi^+\pi^-$ mass spectrum.[4] The effect of the anti-ρ cut is then studied, and we find that the observed reduction of threshold enhancements is reproduced. The sensitivity of the effect to some details of the model is also investigated.

In Sec. II we give details of the model. The re-

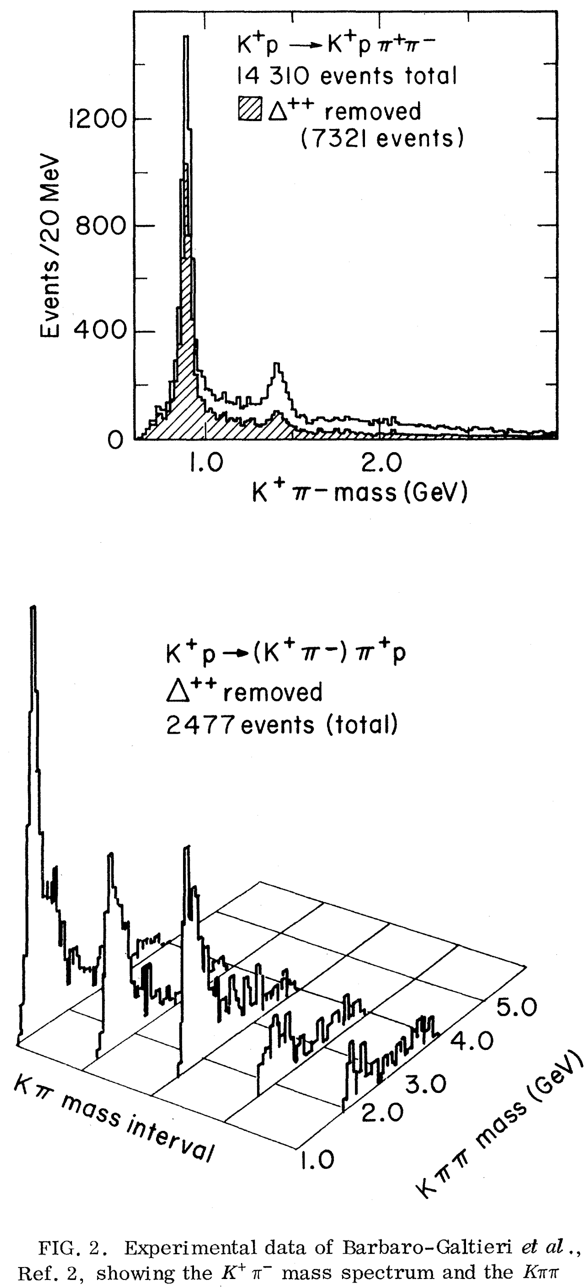

FIG. 2. Experimental data of Barbaro-Galtieri *et al.*, Ref. 2, showing the $K^+\pi^-$ mass spectrum and the $K\pi\pi$ mass spectra for fixed intervals in $K^+\pi^-$ mass.

sults of the calculation are presented in Sec. III and the conclusions in Sec. IV. The integration over phase space is discussed in the Appendix.

II. DESCRIPTION OF MODEL

The properties we seek in the model are simplicity, Bose symmetry, and the ability to describe the $\pi^+\pi^-$ and uncut 3π mass spectra. It is well known that double-Regge-pole exchange for the quasi-three-body $\rho\pi p$ final state with π and Pomeranchuk exchange gives reasonable agreement with experiment, and that it predicts roughly the $A_1(\rho\pi)$ enhancement near 1100 MeV.[4,5] We generalize this slightly to include the f^0 resonance, which enables us to describe approximately the observed $\pi^+\pi_a^-$ mass spectrum. The invariant mass squared of the $\pi^+\pi_b^-$ combination depends linearly on the decay cosine of the π^+ in the rest frame of the resonance; therefore we must specify the decay angular distributions of the ρ^0 and f^0. We choose the decay amplitude $\propto P_l(\cos\theta)$ for a resonance of spin l, where $\cos\theta$ is measured relative to the incident beam. (This corresponds to the density-matrix element $\rho_{00}=1$ in the Gottfried-Jackson or t-channel frame.[6]) The choice is not critical.

For the graphs shown in Fig. 3 we define the kinematic variables:

$$
\begin{aligned}
t_{14} &= (k_4 - k_1)^2\,,\\
t_{5a3} &= (k_5 - k_3 - k_a)^2\,,\\
t_{5b3} &= (k_5 - k_3 - k_b)^2\,,\\
s_3 &= (k_a + k_b + k_3)^2\,,\\
s_{1b} &= (k_1 + k_b)^2\,,\\
s_{1a} &= (k_1 + k_a)^2\,.
\end{aligned}
\tag{2.1}
$$

Then with the decay amplitude $\propto P_l(\cos\theta)$ we obtain for the amplitude

$$
A_a = \left[\sum_{l=1}^{2} \frac{\eta_l^a (q_{a3}/m_{a3})^l P_l(\cos\theta_{a3})}{(k_a+k_3)^2 - m_l^2 + i\Gamma_l m_l}\right] \times \xi_\pi(t_{5a3})(s_3/s_0)^{\alpha_\pi(t)} s_{1b}\,\sigma_{\pi N}(t_{14})\,, \tag{2.2}
$$

where

$$
\xi_\pi(t) = \frac{1+e^{-i\pi\alpha_\pi(t)}}{\sin\pi\alpha_\pi(t)}\,,
$$

$$
\alpha_\pi(t) = t - m_\pi^{\,2}\,, \tag{2.3}
$$

and

$$
\sigma_{\pi N}(t) = \text{const} \times e^{4t}\,.
$$

The quantities q_{a3} and $\cos\theta_{a3}$ refer to the three-momentum and angle of particle a in the frame where $\vec{k}_a + \vec{k}_3 = 0$, and m_{a3} is the energy $(k_a)_0 + (k_3)_0$ in that frame. The parameters η_l can be used to adjust the strength and the shape of the ρ and f^0 resonances. For the Regge scale parameter s_0, we use 2 GeV^2. The masses and widths used are[7]

$$
\begin{aligned}
m_\rho &= 760 \text{ MeV}\,,\\
\Gamma_\rho &= 160 \text{ MeV}\,,\\
m_{f^0} &= 1246 \text{ MeV}\,,\\
\Gamma_{f^0} &= 150 \text{ MeV}\,.
\end{aligned}
$$

The amplitudes A_a and A_b correspond to the two graphs shown in Fig. 3. Various values of the Regge and resonance parameters were used to generate the mass spectra, and the effects of Bose symmetry were investigated by using independently $|A_a|^2$, $|A_b|^2$, $|A_a|^2+|A_b|^2$ for the matrix element squared, in addition to $|A_a+A_b|^2$. Since the A_a amplitude does not have any $\pi^+\pi_b^-$ ρ signal, it can be used to study how the anti-ρ cut removes non-ρ events which happen to reflect into the ρ band. (Of course this cannot be studied experimentally.) On the other hand, the A_b amplitude is dominated by the ρ signal in $\pi^+\pi_b^-$, so that the desired effects of the anti-ρ cut can be observed. The difference between $|A_a+A_b|^2$ and $|A_a|^2+|A_b|^2$ gives information about effects of the strongly model-dependent term $\mathrm{Re}A_aA_b^*$. [Note that multiplying A_a by $e^{if(a,b)}$, where $f(a,b)$ is an arbitrary real function of the kinematic variables, will change neither $|A_a|^2$ nor $|A_b|^2$, but may drastically change $\mathrm{Re}A_aA_b^*$.] It

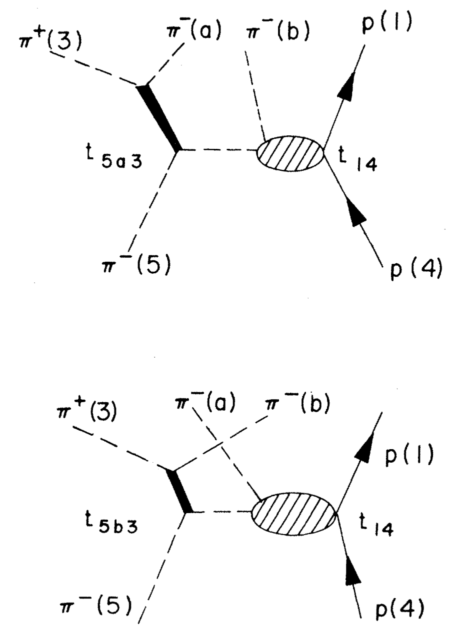

FIG. 3. The diffractive-dissociation (Reggeized Deck effect) graphs used to calculate the $\pi^+\pi_a^-$ and $\pi^+\pi^-\pi^-$ mass spectra. The momenta of the particles are k_i, $i=1$–5, and k_a, k_b. Dashed lines denote pions, solid lines protons.

would be an unpleasant feature of the model if the experimental results could be reproduced only by carefully adjusting the phase of A_a and A_b. Finally, the question of line shape for the ρ meson was investigated. The matrix element squared for the reaction is then

$$\sigma = |A_a + A_b|^2 + C \,, \tag{2.4}$$

where C denotes a constant (phase-space) background.

To obtain the $\pi^+\pi_a^-$ and 3π mass spectra, random events were computer-generated in the multidimensional phase space and then weighted by the transition probability (2.4) in forming the histograms.

It is well known that for a wide resonance the Breit-Wigner amplitude is appreciable for masses far from the position of the resonance pole. We found that by truncating the ρ shape, that is, cutting off the tails of the Breit-Wigner form, we obtained a better fit to the data. To do this the parameter η_1^a was taken as

$$\eta_1^a = \exp\left\{-\left[\frac{(k_3+k_a)^2 - m_\rho^{\,2}}{2m_\rho\Gamma_\rho}\right]^2\right\} \,. \tag{2.5}$$

Near the pole position this shape function is $\simeq 1$, so that the Breit-Wigner curve describes the data, while far from the pole, the shape function cuts off more strongly than the Breit-Wigner form.

III. RESULTS

The 2π and 3π mass spectra generated by the Monte Carlo calculation are displayed in Figs. 4–6 as three-dimensional plots, the two horizontal axes being the 2π and 3π mass, with the number of events as the vertical axis. The vertical scale is arbitrary and differs from curve to curve. We attach no significance to the absolute normalization: it was simply adjusted to fit the data. The $\pi^+\pi_a^-$ mass spectra with and without the anti-ρ cut are shown, while the $\pi^+\pi^-\pi^-$ mass spectra are shown for various intervals of $\pi^+\pi_a^-$ mass. The intervals are 240 MeV, and the central value of the $\pi^+\pi_a^-$ mass interval is shown beneath the corresponding 3π spectrum. For example, the first mass interval, from 400 to 640 MeV, is denoted by 0.52. Those portions of the spectra which remain after performing the anti-ρ cut are shown in the histograms with vertical lines accented.

In Figs. 4(a), 4(b), and 4(c) are shown the histograms corresponding to $|A_a|^2$, $|A_b|^2$, and $|A_a+A_b|^2$, respectively, with the parameters of Eq. (2.2) having the values $\eta_\rho = 1$, $\eta_{f^0} = 0.8$. From Fig. 4(a) it is clear that the model produces three-pion mass enhancements for a given 2π mass, and also that the effect of the anti-ρ cut is not very severe, as indeed it should not be. The threshold enhancements are seen to persist even after the anti-ρ cut. The situation in Fig. 4(b) is quite different.[8] Once again the threshold enhancements for fixed 2π mass are observed, but here the effect of the anti-ρ cut is large and the threshold enhancements are sharply

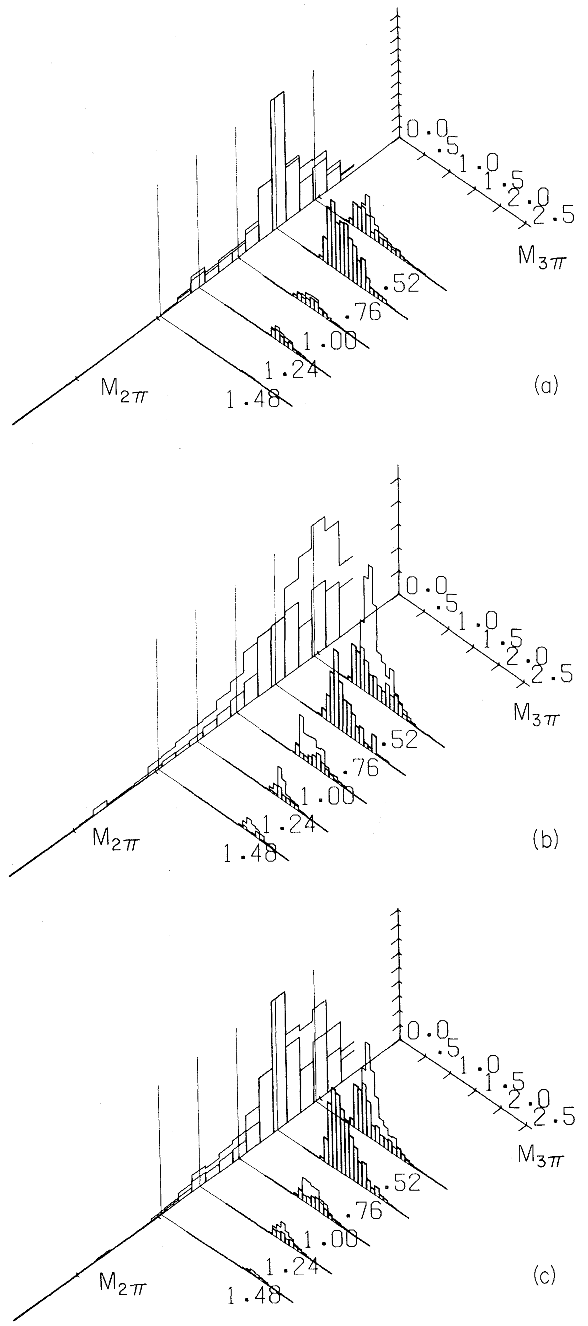

FIG. 4. Three-dimensional plots (number of events as a function of $M_{\pi^+\pi_a^-}$ and $M_{\pi^+\pi^-\pi^-}$) as calculated from the amplitude of Eq. (2.2). The histogram of events which survive the anti-ρ cut are shown with vertical lines accented. (a) $|A_a|^2$. (b) $|A_b|^2$. (c) $|A_a+A_b|^2$. The masses are in GeV.

reduced. In Fig. 4(c) the symmetrized amplitude is used, $|A_a+A_b|^2$, and the results are intermediate between those of Figs. 4(a) and 4(b). In the uncut spectra the threshold enhancements are seen. For the $\pi^+\pi_a^-$ mass intervals centered at 1.0, 1.24, and 1.48 GeV, the threshold enhancements do not survive the anti-ρ cut, but for the low-mass interval, centered at 0.52 there is still some threshold enhancement even though the anti-ρ cut is appreciable. This feature is in contrast to the experimental data, where the anti-ρ cut effectively destroys the 3π mass bumps for all $\pi^+\pi_a^-$ mass slices.

In order to obtain better agreement between experiment and the model, we investigated the effect of truncating the Breit-Wigner resonant line shapes as described in Eq. (2.5).[9] The results are shown in Figs. 5(a), 5(b), and 5(c), which are similar to Figs. 4(a), 4(b), and 4(c). The parameters η_ρ and η_{f0} are given by Eq. (2.5), with numerical values 1.0 and 0.8, respectively, at the peak. In the $\pi^+\pi_a^-$ mass spectrum of Fig. 5(a), one sees only the ρ and f^0; consequently the threshold enhancements are significant only in the second and fourth 3π mass spectra. The effect of the anti-ρ cut is negligible. From Fig. 5(b) one sees that the number of events which survive the anti-ρ cut is small, the cut being much more effective than for Fig. 4(b). In the symmetrized amplitude, Fig. 5(c), the spectra generated by the model are found to be in reasonable agreement with the experimental results of Figs. 1 and 2. In particular, threshold enhancements are seen for all $\pi^+\pi_a^-$ mass spectra, but these enhancements are destroyed by making the anti-ρ cut. Finally, in Fig. 6 we show the prediction for the symmetrized, truncated-resonance, amplitude with a small incoherent 3π background.[10] The main contribution of this background is to 2π and 3π high-mass regions, causing them to better resemble experiment.

The importance of the interference term was checked by generating histograms using $|A_a|^2 + |A_b|^2$. Such spectra are almost indistinguishable from those of Figs. 4(c) and 5(c), and we shall not reproduce them here. It is enough to note that the observed effect is not critically sensitive to the

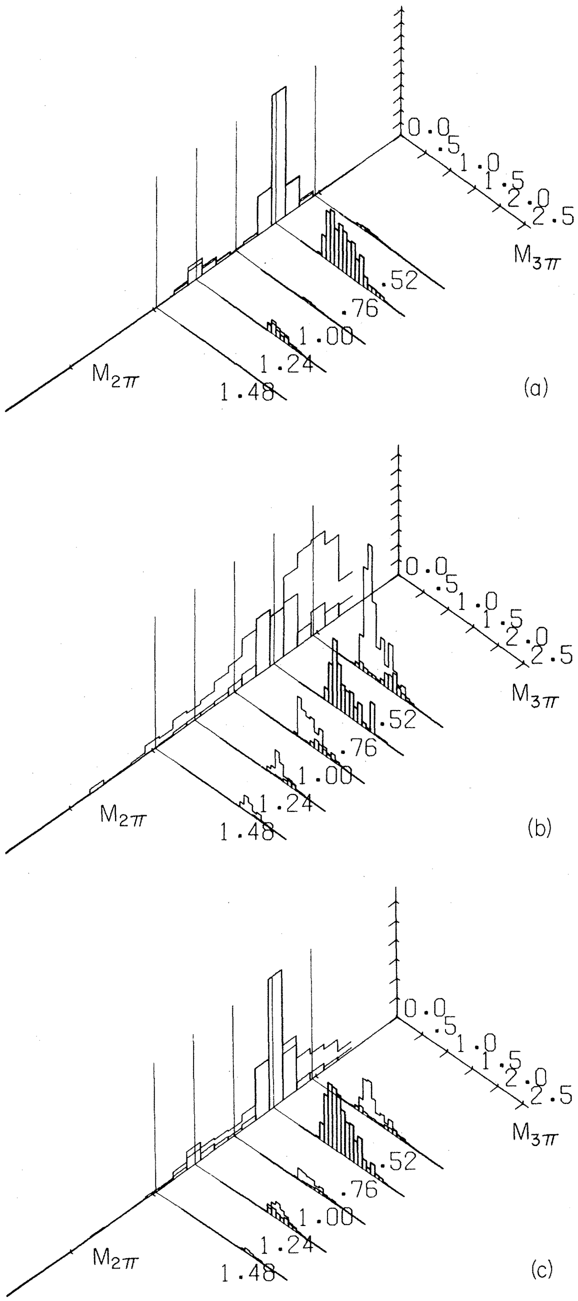

FIG. 5. Three-dimensional plots of the mass spectra (masses in GeV) calculated from Eq. (2.2) using the truncated Breit-Wigner form of Eq. (2.5). (a) $|A_a|^2$. (b) $|A_b|^2$. (c) $|A_a+A_b|^2$.

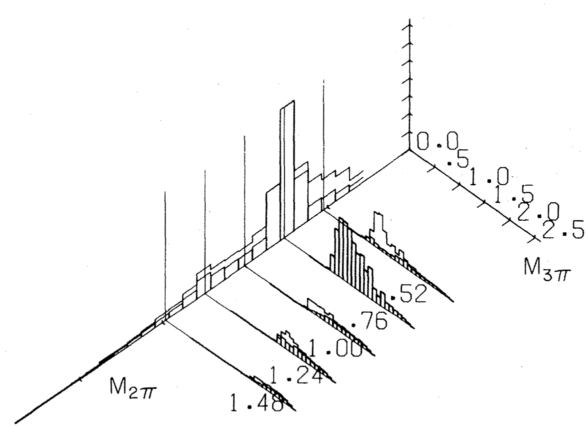

FIG. 6. Three-dimensional plot of the mass spectra (masses in GeV) using the truncated Breit-Wigner form, as in Fig. 5, but also including a small incoherent background.

interference between A_a and A_b. Thus the identity of the two pions enters only through the equality of $|A_a|$ and $|A_b|$ in corresponding regions. In addition we have investigated also the importance of our assumption that the ρ and f^0 decay amplitudes are $P_1(\cos\theta)$ and $P_2(\cos\theta)$, respectively. This assumption is not crucial to our results, since qualitatively similar results are obtained when a ρ distribution is used which is isotropic or one which is strongly peaked forward.

IV. CONCLUSIONS

We have shown that a simple diffractive-dissociation-effect model incorporating Bose symmetry can reproduce the results of a recent experiment on the reaction $\pi^- p \to \pi^-\pi^-\pi^+ p$. In particular, we can understand the observation of threshold enhancements in $M_{\pi^+\pi^-\pi^-}$ for fixed $M_{\pi^+\pi^-}$, and the subsequent vanishing of the effect when the anti-ρ cut is enforced. It was found necessary to use a truncated form for the ρ and f^0 resonant line shapes in order to obtain good agreement with the experimental data. The effect does not depend crucially on the interference term in the symmetrized amplitude. A simple qualitative explanation is as follows: Suppose that the ρ dominates the $\pi^+\pi^-$ spectrum to such an extent (here we neglect the f^0) that in each $\pi^+\pi^-\pi^-$ event one of the $\pi^+\pi^-$ combinations lies in the ρ. By the usual Reggeized Deck effect, there is a very broad $\pi^+\pi^-\pi^-$ enhancement beginning at mass $= m_\rho + m_\pi$. When one considers a $\pi^+\pi^-$ mass interval not containing the ρ, nearly all of the signal comes from the other $\pi^+\pi^-$ combination being in the ρ, and one sees a portion of the broad $\pi^-\rho$ enhancement. However, when the anti-ρ cut is applied, there is almost no signal left, and consequently no threshold enhancement. While slightly oversimplified, this is the basic conclusion of our calculation. This suggests that the observed $\pi^+\pi^-$ spectrum contains few events for which neither $\pi^+\pi^-$ combination lies near the ρ or f^0. In contrast, the observed $K^+\pi^-$ spectrum of Ref. 2 may be considered as having genuine $K^+\pi^-$ events which are neither $K^{*0}(1420)$ nor crossed ρ, and therefore contribute to threshold enhancements via a mechanism such as double-Regge-pole exchange.

ACKNOWLEDGMENTS

We should like to thank Dr. Kwan Wu Lai for suggesting this problem and for helpful discussions.

APPENDIX

Integration Over the Four-Body Phase Space

We specify the seven independent variables for the phase-space integration as follows: The invariant volume element is

$$d\rho = \Big[\prod_i d^4k_i\,\delta(k_i^{\,2} - m_i^{\,2})\Big]\delta^{(4)}\Big(\sum_j k_j - k_4 - k_5\Big)\,, \tag{A1}$$

where $i, j = 1, 2, a, b$. We now write

$$\begin{aligned} k_2 &= k_a + k_b \equiv (m_2, \vec{0})\,,\\ k_2 &= k_a - k_b \equiv (\epsilon_0, \vec{\epsilon})\,, \end{aligned} \tag{A2}$$

working explicitly in the center-of-mass system of particles a and b. Then

$$\begin{aligned} d^4k_a\, d^4k_b\, \delta(k_a^{\,2} - m_a^{\,2})\delta(k_b^{\,2} - m_b^{\,2}) &= \int \big[d^4k_2 d\epsilon_0 \epsilon^2 d\epsilon\, d\Omega_0\, \delta(\varphi_+ - m_a^{\,2})\delta(\varphi_- - m_b^{\,2})\\ &\qquad \times \delta(k_2^{\,2} - m_2^{\,2})\big]\, dm_2^{\,2}\,, \end{aligned} \tag{A3}$$

where $\varphi_\pm = \frac{1}{2}[(m_2 \pm \epsilon_0)^2 - \epsilon^2]$. The integration over ϵ_0, ϵ can be done explicitly, the two δ functions giving simply a Jacobian factor

$$\left(\frac{\partial(\varphi_+, \varphi_-)}{\partial(\epsilon, \epsilon_0)}\right)^{-1} = \tfrac{1}{2} m_2 \epsilon\,.$$

Thus

$$\begin{aligned} d\rho &\equiv \frac{\epsilon}{8m_2}\, dm_2^{\,2} d\Omega_0 \left[\prod_{i=1}^{3} d^4k_i\, \delta(k_i^{\,2} - m_i^{\,2})\right]\\ &\qquad \times \delta^{(4)}\left(\sum_{i=1}^{3} k_i - k_4 - k_5\right)\,. \end{aligned} \tag{A4}$$

Now let $k_i \equiv (E_i, \vec{k}_i)$ and use the E_1, E_2, E_3, $\vec{k}_3$ integrations to eliminate the mass-shell and total 3-momentum δ functions, and we obtain

$$d\rho \equiv \frac{\epsilon}{8m_2}\, dm_2^{\,2} d\Omega_0\, \frac{d^3k_1 d^3k_2}{E_1 E_2 E_3}\, \delta(E_1 + E_2 + E_3 - E_4 - E_5)\,. \tag{A5}$$

Now choose the polar axis for $\vec{k}_1$ along $\vec{k}_4$ and the polar axis for $\vec{k}_2$ along $\vec{k}_1$, in the over-all center-of-mass system, and we find that

$$\frac{\partial \cos\theta_2}{\partial E_3} = \frac{E_3}{k_1 k_2}\,.$$

Finally, writing $k dk_1 = E_1 dE_1$, $k_2 dk_2 = E_2 dE_2$, and integrating over the one arbitrary azimuth angle, we obtain

$$d\rho \equiv \frac{\pi\epsilon}{4m_2}\, dm_2 d\cos\theta_0 d\cos\phi_0 dE_1 dE_2 d\cos\theta_1 d\cos\phi_2\,, \tag{A6}$$

where

$$\epsilon = \frac{(m_2^{\,2} - m_a^{\,2} - m_b^{\,2})^2 - 4m_a^{\,2} m_b^{\,2}}{4m_2^{\,2}}\,.$$

(θ_0, ϕ_0) is the direction of $\vec{k}_a$ in the a, b rest system, θ is the polar angle of $\vec{k}_1$ relative to $\vec{k}_4$ in the over-all c.m. system, and ϕ_2 is the relative azimuth of $\vec{k}_4$ and $\vec{k}_2$ about the direction of $\vec{k}_1$.

*Work supported in part by the U. S. Atomic Energy Commission.

†Permanent address: Iowa State University, Ames, Iowa.

‡Present address: Laboratoire de Physique Thêorique, Facultê des Sciences de Nice, Nice, France.

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[3]There is some $A_2(\rho\pi)$ signal in the data which we do not attempt to reproduce.

[4]Reggeized Deck effects have been studied by E. L. Berger, Phys. Rev. 166, 1525 (1968); 179, 1567 (1969); C. D. Froggatt and G. Ranft, Phys. Rev. Letters 23, 943 (1969); C. C. Shih and B. L. Young, Phys. Rev. D 1, 2631 (1970).

[5]We have neglected double-Regge graphs other than the π-exchange graphs of Fig. 2. N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. Letters 19, 614 (1967), and Berger, Ref. 4, have argued that this is not unreasonable.

[6]For elementary pion exchange this choice would be correct, and empirically it is a reasonable approximation to the data.

[7]The widths chosen for the ρ and f^0 are larger than the accepted values, but it is well known that in strong interactions, the ρ typically has an observed width of ~150 MeV. When we study the question of truncation of the Breit-Wigner forms, the effective widths are about 25% smaller than the input widths.

[8]Note that the vertical scale is twice that for Fig. 4(a); in order to compare the two figures it should be remembered that the total 3π spectrum must be the same for 4(a) as for 4(b).

[9]The truncation is important only for the $\pi^+\pi_a^-$ masses below the ρ band, as we have found by truncating the lower and upper halves of the ρ independently. For the former the results are similar to those shown.

[10]The incoherent background tends to extend the 3π mass spectrum to higher values, which are difficult to obtain in our double-Regge model. This background, of course, affects the $\pi^+\pi^-$ mass spectrum, but it does not give rise to threshold enhancements. The constant C of Eq. (2.4) has a numerical value of 0.18.

PHYSICAL REVIEW D VOLUME 3, NUMBER 11 1 JUNE 1971

Exact Bounds for K_{l3} Decay Parameters*

Susumu Okubo

Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

(Received 17 March 1971)

We derive exact bounds for the K_{l3} decay form factors $f_\pm(t)$. Particularly, we find the bound $(m_K{}^2 - m_\pi{}^2)|f_+(0)| \leq 16[\frac{1}{3}\pi\Delta(0)]^{1/2}(m_K + m_\pi)^{1/2}(m_K{}^{1/2} + m_\pi{}^{1/2})^{-1}$, where $\Delta(0)$ is the propagator of the divergence of the strangeness-changing current at zero momentum. If we further assume the Hamiltonian of Gell-Mann, Oakes, and Renner in order to estimate $\Delta(0)$, we obtain $|f_+(0)| \lesssim 1.0$. Similarly, an inequality testing the standard K_{l3} soft-pion theorem is found to be well satisfied. In addition, a new inequality involving derivatives of $f_+(t)$ is derived. Taking $\lambda_+ \sim 0.02$, this inequality leads to $|f_-(0)| \lesssim 0.33$.

I. INTRODUCTION AND SUMMARY OF PRINCIPAL RESULTS

According to the standard Cabibbo theory, all the properties of the K_{l3} decays are obtainable from two form factors $f_\pm(t)$ defined by

$$-i\langle\pi^0(p')|V_\mu^{(4-i5)}(0)|K^+(p)\rangle = (4p_0p_0'V^2)^{-1/2}(\tfrac{1}{2})^{1/2}i[(p_\mu + p_\mu')f_+(t) + (p_\mu - p_\mu')f_-(t)], \tag{1.1}$$

with $t = -(p-p')^2$. It is convenient to consider the combination

$$d(t) = (m_K{}^2 - m_\pi{}^2)f_+(t) + tf_-(t), \tag{1.2}$$

which can be obtained from Eq. (1.1) by means of

$$-i\langle\pi^0(p')|\partial_\mu V_\mu^{(4-i5)}(0)|K^+(p)\rangle = (4p_0p_0'V^2)^{-1/2}(\tfrac{1}{2})^{1/2}d(t). \tag{1.3}$$

Recently, Li and Pagels[1] derived the following exact inequality for the derivative of $d(t)$:

$$|d'(0)| \leq (8/\sqrt{3})\Delta^{1/2}(0)I^{1/2}, \tag{1.4}$$

where $\Delta(t)$ is defined by

$$\Delta(t) = \tfrac{1}{2}i\int d^4x\, e^{iqx}\langle 0|(\partial_\mu V_\mu^{(4-i5)}(x), \partial_\nu V_\nu^{(4+i5)}(0))_+|0\rangle, \tag{1.5}$$

with $t = -q^2$, and I is the numerical integral