then one has to calculate explicitly the second sum of the right-hand side of this equality by substituting in it the expression of the functional derivative. Thus one gets

$$\sum_{i=1}^{2}\left(\frac{\delta A_\alpha}{\delta \dot{x}_{i\mu}(\alpha_i)}\right)=\sum_{i=1}^{2}\left(\frac{\delta A_\alpha}{\delta \dot{x}_{i\mu}(\tau_i)}\right)$$
$$+c^{2l-1}\left(\int_{\tau_1}^{\alpha_1} d\tau_1' \int_{-\alpha_2}^{\tau_2} d\tau_2' - \int_{\tau_2}^{\alpha_2} d\tau_2' \int_{-\alpha_1}^{\tau_1} d\tau_1'\right)$$
$$\times V_l'([x_1(\tau_1')-x_2(\tau_2')]^2)[\dot{x}_1(\tau_1')\cdot\dot{x}_2(\tau_2')]^{1-l}$$
$$\times[x_1^\mu(\tau_1')-x_2^\mu(\tau_2')]. \quad \text{(A15)}$$

We can now take the limit (A11) of this last expression by considering that the limit holds

$$\lim_{\alpha_1\to\infty;\,\alpha_2\to\infty}\left(\frac{\delta A_\alpha}{\delta \dot{x}_{i\mu}(\tau_i)}\right)=p_i^\mu(\tau_i)\,, \quad \text{(A16)}$$

where $p_i^\mu(\tau_i)$ is defined by the formula (2.5); substituting the expression (A15) in the definition (A11) and making the limit, we obtain the expression of $P^\mu(\tau_1,\tau_2)$ given in Sec. III. By applying this same procedure, one can show that the definition (A12) of $\Omega^{\mu\nu}$ leads to the expression (3.25) of Sec. III. We note that the derivation of the previous results has been only sketched; in fact, the precise definition of the solutions of the equations (A4) has not been discussed, and their dependence on the parameters α_i has been omitted. This method applies to a system of any number of particles.

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Quantization of Evanescent Electromagnetic Waves*

C. K. CARNIGLIA† AND L. MANDEL
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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The problem of the quantization of evanescent waves, which appear in the angular spectrum representation of the electromagnetic field in a half-space, is discussed. Although evanescent waves are associated with material sources, scatterers, etc., we are able to treat the electromagnetic field, including the evanescent waves, effectively as a free field, by making use of the idea of the refractive index of a passive, macroscopically continuous medium. We consider a space which is filled with a homogeneous dielectric to the left of the plane $z=0$, and is empty to the right of the plane. Triplets of incident, reflected, and transmitted waves at the interface form the fundamental orthogonal modes of the space. By expanding the field in terms of these triplet modes, we show that the field Hamiltonian reduces to the sum of independent harmonic-oscillator Hamiltonians. The quantization is therefore straightforward. We introduce the creation and annihilation operators for the triplet wave modes, and encounter Fock states, coherent states, etc., for a field having evanescent wave components. The field commutator at two space-time points in the right half-space is shown to have an explicit contribution from evanescent waves, characterized by an exponential decay to the right and a propagation parallel to the interface. We also examine the problem of atomic excitation by quantized evanescent waves, and show that the results are of the form given by semiclassical treatments.

I. INTRODUCTION

ALTHOUGH evanescent electromagnetic waves have been well known in optics and in the microwave domain for many years, they have tended to be something of a curiosity. They are perhaps most familiar in connection with the total internal reflection of light at a glass-to-air interface, and quantitative features of the evanescent waves produced under these conditions were studied experimentally by Quincke[1] and by Hall[2] as long ago as 1866 and 1902, respectively. In recent years they have been frequently encountered in the context of diffraction, particularly in the angular spectrum representation of the electromagnetic field,[3–9] where the evanescent waves appear as a natural adjunct to the spectrum of homogeneous plane waves. Expansions involving evanescent waves have also proved valuable recently in the treatment of radiation from moving

* Work partially supported by the Air Force Office of Scientific Research.
† Also at the Institute of Optics, University of Rochester, Rochester, N. Y.
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charges,[5,10–12] and it has been shown that, when there is no radiation, the field is describable entirely in terms of evanescent waves. In this context, the evanescent waves may be an alternative to the virtual-photon approach for the representation of the field.[13]

However, in all problems in which evanescent waves appear explicitly, the electromagnetic field has so far been treated as a classical field, and, to the best of our knowledge, no attempt to treat these waves quantum mechanically has been made. It appears that questions regarding the interaction of evanescent waves with atoms cannot now be tackled except by semiclassical methods, despite the fact that there has been some experimental work in this area.[14] In the quantization of the free electromagnetic field, it is customary to expand the field in homogeneous plane waves, and to admit no evanescent components. But this is not a valid procedure if sources, scatterers, apertures, etc., are present, when the field is, in general, not representable by homogeneous plane waves.[15] In that case we are, of course, no longer dealing with a free field, and it might seem that we cannot tackle the problem of quantization without treating the coupled system. Nevertheless, in the following we have succeeded in treating the electromagnetic field, including evanescent waves, effectively as a free field, by making use of the idea of refractive index of a passive, macroscopically continuous medium.

We consider the problem of quantization of the electromagnetic field in a space which is filled with a homogeneous dielectric of refractive index n_0 to the left of the plane $z=0$, and is empty to the right of this plane. Such a space allows the appearance of evanescent waves on the vacuum side of the interface. Instead of introducing the material medium and its interaction with the electromagnetic field explicitly,[16] we allow the material medium to determine the modes of the electromagnetic field, which is then treated as a free field. We show that the evanescent waves may be regarded as a consequence of the spatial phase modulation of the incident and reflected waves at the interface. When each transmitted homogeneous or evanescent wave, together with the incident and reflected waves which give rise to it, is treated as one mode, the field Hamiltonian reduces to the sum of independent harmonic-oscillator Hamiltonians for each mode. The quantization of the field is then straightforward and proceeds in the usual manner. The space-time field commutators are found to contain explicit contributions from evanescent waves, which decay exponentially with distance from the interface.

Since the evanescent waves constitute only a component of a mode, this approach leads to the point of view that there are no evanescent photons *per se*; there are photons which behave as homogeneous plane waves in the one half-space and as evanescent waves in the other. Although the problem of the dielectric-to-vacuum interface may appear to be a special one, the results should be applicable to other problems involving the field in a half-space. For the actual source giving rise to a homogeneous or evanescent wave in the right half-space is often equivalent to, and may be replaced by, a dielectric together with a pair of homogeneous waves in the left half-space, provided the three waves are coupled via the Fresnel relations for the interface.[17]

We begin by briefly introducing the angular spectrum representation of the classical electromagnetic field, and show under what conditions it leads to the appearance of evanescent waves. We then introduce the transverse electric and transverse magnetic triplet wave modes of the dielectric-to-vacuum interface, and expand the field in terms of these modes. The modes are shown to be orthogonal, so that the field Hamiltonian reduces to quadratures. We quantize the field by treating each mode as a noninteracting harmonic oscillator, and evaluate certain field commutators, which are found to contain explicit contributions from evanescent waves. We show that photon absorption and number operators can be introduced as usual. Finally, we consider the problem of the excitation of an atom in an evanescent wave field, and find that the results are equivalent to those given by semiclassical methods.

II. CLASSICAL ANGULAR SPECTRUM REPRESENTATION AND EVANESCENT WAVES

Let us consider an expansion of the electric field $\mathbf{E}(\mathbf{r},t)$ to the right of some plane located at $z=0$. We suppose that the right half-space is empty, and that sources and scatterers, if any, are located in the left half-space. We first make a two-dimensional spatial Fourier decomposition of $\mathbf{E}(\mathbf{r},t)$ in some plane

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z = constant,

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}dk_1dk_2\mathbf{U}(k_1,k_2,z,t)\exp i(k_1x+k_2y),\quad(1)$$

where $\mathbf{r}\equiv(x,y,z)$, and then Fourier-analyze $\mathbf{U}(k_1,k_2,z,t)$ in time by writing

$$\mathbf{U}(k_1,k_2,z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,\mathbf{u}(k_1,k_2,z,k)e^{-ikt}.\quad(2)$$

We choose our units so that the velocity of light in vacuum is unity. In view of the reality of $\mathbf{E}(\mathbf{r},t)$, it follows from Eq. (1) that

$$\mathbf{U}(-k_1,\,-k_2,\,z,\,t)=\mathbf{U}^*(k_1,k_2,z,t)\quad(3)$$

and from Eq. (2) that

$$\mathbf{u}(-k_1,\,-k_2,\,z,\,-k)=\mathbf{u}^*(k_1,k_2,z,k).\quad(4)$$

With the help of Eqs. (1) and (2), we can now write

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int\mathbf{u}(k_1,k_2,z,k)\times\exp[i(k_1x+k_2y-kt)]dk_1dk_2dk.\quad(5)$$

Since $\mathbf{E}(\mathbf{r},t)$ satisfies the wave equation

$$\nabla^2\mathbf{E}(\mathbf{r},t)-\frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t)=0\quad(6)$$

at all times everywhere in the right half-space, it seems natural to require that the integrand in Eq. (5) satisfies the Helmholtz equation

$$(\nabla^2+k^2)\mathbf{u}(k_1,k_2,z,k)\exp[i(k_1x+k_2y)]=0,\quad(7)$$

throughout the right half-space. Then

$$\left(-k_1^2-k_2^2+\frac{\partial^2}{\partial z^2}+k^2\right)\mathbf{u}(k_1,k_2,z,k)=0,\quad(8)$$

which has the solution

$$\mathbf{u}(k_1,k_2,z,k)=\mathbf{v}(k_1,k_2,k)\exp(ik_3z)+\mathbf{w}(k_1,k_2k)\exp(-ik_3z),\quad(9)$$

for $k_3\neq0$, where

$$k_3=+\sqrt{(k^2-k_1^2-k_2^2)},\quad(10)$$

and may be real or imaginary according as $k_1^2+k_2^2\lessgtr k^2$. When $k_3=0$, the solution of Eq. (8) grows linearly with z to ∞, and is therefore not an acceptable solution, unless it is a constant. From condition (4) we find

$$\mathbf{v}(-k_1,\,-k_2,\,-k)\exp(ik_3z)+\mathbf{w}(-k_1,\,-k_2,\,-k)\times\exp(-ik_3z)=\mathbf{v}^*(k_1,k_2,k)\exp(-ik_3^*z)+\mathbf{w}(k_1,k_2,k)\exp(ik_3^*z).$$

When k_3 is real, this leads to

$$\mathbf{w}^*(k_1,k_2,k)=\mathbf{v}(-k_1,\,-k_2,\,-k),\quad(11)$$

while, when k_3 is imaginary, we have

$$\begin{aligned}\mathbf{v}^*(k_1,k_2,k)&=\mathbf{v}(-k_1,\,-k_2,\,-k),\\ \mathbf{w}^*(k_1,k_2,k)&=\mathbf{w}(-k_1,\,-k_2,\,-k).\end{aligned}\quad(12)$$

With the help of relations (9)–(12), we can now rewrite Eq. (5) in the form

$$\begin{aligned}\mathbf{E}(\mathbf{r},t)=&\frac{1}{(2\pi)^3}\int_{k_1^2+k_2^2<k^2}[\mathbf{v}(k_1,k_2,k)\exp i(k_1x+k_2y+k_3z-kt)+\text{c.c.}]dk_1dk_2dk\\&+\frac{\frac{1}{2}}{(2\pi)^3}\int_{k_1^2+k_2^2>k^2}[\mathbf{v}(k_1,k_2,k)\exp(-|k_3|z)\exp i(k_1x+k_2y-kt)+\text{c.c.}]dk_1dk_2dk\\&+\frac{\frac{1}{2}}{(2\pi)^3}\int_{k_1^2+k_2^2>k^2}[\mathbf{w}(k_1,k_2,k)\exp(|k_3|z)\exp i(k_1x+k_2y-kt)+\text{c.c.}]dk_1dk_2dk.\quad(13)\end{aligned}$$

The first integral represents contributions to $\mathbf{E}(\mathbf{r},t)$ from ordinary homogeneous, plane waves. The second and third integrals represent contributions from waves which propagate in a direction parallel to the xy plane and decay or grow exponentially in the z direction. Since the contribution from the third integral represents a field that becomes infinite as $z\to\infty$, we put $\mathbf{w}(k_1,k_2,k)=0$ and discard this term. Finally, we rewrite the double-sided k integral as an integral over the positive-frequency range. We may discard the homogeneous waves travelling to the left, if there are no sources or scatterers on the right.[18] We then have

$$\begin{aligned}\mathbf{E}(\mathbf{r},t)=&\frac{1}{(2\pi)^3}\int_0^{\infty}dk\int_{k_1^2+k_2^2<k^2}dk_1dk_2[\mathbf{v}(k_1,k_2,k)\exp i(k_1x+k_2y+k_3t-kt)+\text{c.c.}]\\&+\frac{1}{(2\pi)^3}\int_0^{\infty}dk\int_{k_1^2+k_2^2>k^2}dk_1dk_2[\mathbf{v}(k_1,k_2,k)\exp(-|k_3|z)\exp i(k_1x+k_2y-kt)+\text{c.c.}].\quad(14)\end{aligned}$$

[18] It is worth noting that this is generally not possible when a representation including only homogeneous plane waves is used. See Ref. 15.

The terms contributed by the second integral, which are characterized by exponential decay in the z direction, are known as evanescent waves. They appear whenever $\mathbf{v}(k_1,k_2,k)\neq 0$ for $k_1^2+k_2^2>k^2$. Since $\mathbf{v}(k_1,k_2,k)$ is simply a three-dimensional Fourier transform of the field in the plane $z=0$, we see that evanescent waves are expected to appear, loosely speaking, whenever there are spatial modulations of the field in the plane $z=0$ with periodicities shorter than about a wavelength. Such a modulation could be brought about by a diffraction grating of sufficiently small line spacing. But the most familiar example of such modulation occurs at a dielectric-air interface, when a light beam is incident on the interface from the dielectric side, at an angle greater than the critical angle. Since the refractive index n of the dielectric is normally greater than unity, it is possible to satisfy the conditions $k_1^2+k_2^2<n^2k^2$ and $k_1^2+k_2^2>k^2$ simultaneously at the interface. There is therefore a phase modulation of the field with spatial periodicity which is longer than the wavelength in glass, but shorter than the wavelength in air. Accordingly evanescent waves appear on the air side of the interface.

Let us briefly examine the vectorial properties of the representation. From the divergence condition for the field in free space and Eq. (14), it follows immediately that

$$\mathbf{k}\cdot\mathbf{v}(k_1,k_2,k)=0\,, \tag{15}$$

where $\mathbf{k}$ is the wave vector, real or complex, with components k_1, k_2, k_3. So long as $\mathbf{k}$ is real and the corresponding wave is homogeneous, Eq. (15) implies transversality in the usual sense, in that the real and imaginary parts of $\mathbf{v}(k_1,k_2,k)$ are normal to $\mathbf{k}$. However, when k_3 becomes imaginary and the wave is evanescent, these conclusions no longer hold. In particular, $\mathbf{v}(k_1,k_2,k)$ can be proportional to a real vector only if this vector lies in the xy plane.

Although each component $\mathbf{v}(k_1,k_2,k)\exp i(\mathbf{k}\cdot\mathbf{r}-kt)$, with $\mathbf{k}$ real or complex, in the expansion in Eq. (14) is a possible solution of the Helmholtz equation for the right half-space, and may therefore be regarded as a "mode" of the field, these modes are not orthogonal in the usual sense. The scalar product of two different modes integrated with respect to $\mathbf{r}$ over the half-space does not vanish. As a result, the expression for the energy of the field in the right half-space in terms of modes is complicated, and does not reduce to the sum of contributions from each mode.

It is not difficult to see the origin of this complication. For the field in the right half-space is not a free field in the usual sense, but may be generated by sources in the left half-space. The field throughout space is given by the solution of the inhomogeneous wave equation and is therefore coupled to the sources, scatterers, etc. It is clear that the mere disregard of the left half-space does not eliminate the sources and scatterers.

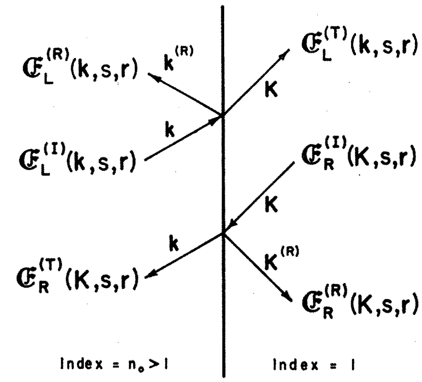

FIG. 1. Illustrating the notation for the incident, reflected, and transmitted components of each mode. All modes are labeled by the wave vector of the incident wave. For waves incident from the left the wave vector $\mathbf{k}$ is in the dielectric; for waves incident from the right the wave vector $\mathbf{K}$ is in vacuum. Although electric fields were chosen for illustration, the notation is similar for the magnetic fields.

In order to describe a field having evanescent components and yet avoid the explicit introduction of sources, we will make use of the idea of a macroscopically continuous medium of refractive index n, which is located in the left half-space. Evidently this is an abstraction which takes the place of certain secondary sources or scatterers. However, a passive medium allows a simple connection to be made between the fields inside and outside the medium, and therefore permits us to express the energy of the field throughout all space in terms of the component modes. By including the left half-space in the expansion of the field, we can formally dispense with sources and scatterers and treat the field as a "free" field. We shall see that the formalism will then allow quantization of this free field in a straightforward manner.

III. MODES OF INTERFACE

We consider a space which is filled with a nonmagnetic, transparent, homogeneous, isotropic medium of refractive index n_0 to the left of the plane $z=0$, and is empty everywhere to the right of this plane. Then the refractive index function $n(\mathbf{r})$ has the property

$$\begin{aligned} n(\mathbf{r})&=n_0 \quad \text{for} \quad z<0\\ &=1 \quad \text{for} \quad z>0\,. \end{aligned} \tag{16}$$

As is well known, a plane wave incident on the interface from the left or the right will, in general, give rise to a reflected and a transmitted wave, and we label these three components by superscripts I, R, T, respectively (see Fig. 1). It is convenient to make a spectral decomposition of each wave, and, in addition, to decompose the incident beam into transverse electric (TE) and transverse magnetic (TM) components, which behave somewhat differently.

We denote the wave vectros for one spectral component of the wave inside the medium and in the vacuum by $\mathbf{k}$ and $\mathbf{K}$, respectively, and note that they are connected by the formulas[17]

$$K_1=k_1, \quad (17)$$

$$K_2=k_2, \quad (18)$$

$$K=k/n_0, \quad (19)$$

$$K_3=\pm\sqrt{(K^2-K_1^2-K_2^2)}, \quad (20)$$

$$k_3=\pm\sqrt{(n_0^2K^2-k_1^2-k_2^2)}, \quad (21)$$

when n_0 is real. K is of course the frequency of the light, but whereas $\mathbf{k}$ is always a real vector, $\mathbf{K}$ will be complex when $K_1^2+K_2^2>K^2$. We adopt the convention that the positive sign is chosen in Eq. (20) when k_3 is positive and the negative sign when k_3 is negative. As is well known, the complex amplitudes of the electric and magnetic fields $\mathfrak{E}$ and $\mathfrak{B}$ of the incident, reflected, and transmitted waves are connected via the Fresnel relations,[17] which become, for a TE wave incident from the left,

$$\mathfrak{E}_L^{(I)}(\mathbf{k},1,\mathbf{r})=\frac{1}{\sqrt{2}n_0}\boldsymbol{\varepsilon}\exp(i\mathbf{k}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (22)$$

$$\mathfrak{E}_L^{(R)}(\mathbf{k},1,\mathbf{r})=\frac{1}{\sqrt{2}n_0}\boldsymbol{\varepsilon}\frac{k_3-K_3}{k_3+K_3}\exp(i\mathbf{k}^{(R)}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (23)$$

$$\mathfrak{E}_L^{(T)}(\mathbf{k},1,\mathbf{r})=\frac{1}{\sqrt{2}n_0}\boldsymbol{\varepsilon}\frac{2k_3}{k_3+K_3}\exp(i\mathbf{K}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0. \quad (24)$$

We have written $\mathbf{k}^{(R)}$ for the wave vector $(k_1, k_2, -k_3)$ of the reflected wave. $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(\mathbf{k})$ is a real unit vector lying in the plane $z=0$, which is orthogonal to both $\mathbf{k}$ and $\mathbf{K}$ and characterizes the polarization of the wave. The scale factors $1/(\sqrt{2}n_0)$ are introduced for later convenience in the normalization. The label 1 in $\mathfrak{E}_L^{(I)}(\mathbf{k},1,\mathbf{r})$, etc., identifies the waves as TE waves, and the suffiix L indicates the incidence from the left. Notice that we have chosen to label all three waves $\mathfrak{E}_L^{(I)}(\mathbf{k},1,\mathbf{r})$, $\mathfrak{E}_L^{(R)}(\mathbf{k},1,\mathbf{r})$, $\mathfrak{E}_L^{(T)}(\mathbf{k},1,\mathbf{r})$ by the wave vector $\mathbf{k}$ of the *incident* wave, even though they represent waves propagating in three different directions, in order to emphasize that these three waves belong together (see Fig. 1). Indeed, they form an elementary "mode" of the system under consideration, for the functions formed by adding the three wave components,

$$\mathfrak{E}_L(\mathbf{k},1,\mathbf{r})\equiv\mathfrak{E}_L^{(I)}(\mathbf{k},1,\mathbf{r})+\mathfrak{E}_L^{(R)}(\mathbf{k},1,\mathbf{r})+\mathfrak{E}_L^{(T)}(\mathbf{k},1,\mathbf{r}), \quad (25)$$

evidently satisfy the Helmholtz equation

$$\nabla^2\mathfrak{E}_L(\mathbf{k},1,\mathbf{r})+K^2n^2(\mathbf{r})\mathfrak{E}_L(\mathbf{k},1,\mathbf{r})=0, \quad (26)$$

and can therefore be used for the representation of solutions of Eq. (26).[18]

We must now briefly discuss some properties of the refractive index n_0. In general it will be a complex function of the optical frequency K. As is well known, the causality requirement imposes constraints on the allowed forms of $n_0(K)$, so that the real and imaginary parts of $n_0(K)-1$ are coupled by Hilbert transform dispersion relations.[19] But the introduction of an imaginary component of the refractive index is somewhat unfortunate for our purpose, since it involves energy dissipation and prevents waves launched from infinity with finite amplitude from arriving at the interface. For this reason, we will make the simplifying assumption that the imaginary part of the refractive index vanishes over all frequencies in the optical region or below with which we shall be concerned, and does not become nonzero until much higher frequencies, say in the x-ray region, are reached. Under these conditions the real part of the index will be nearly constant over the frequencies of interest, and we may treat n_0 as a real constant which is greater than unity. Such an assumption is certainly valid at this stage. Later on, when we encounter integrals over frequencies ranging to infinity, we shall have to re-examine the implications.

The magnetic fields associated with the foregoing electric fields follow immediately from Maxwell's equation

$$\nabla\times\mathbf{E}(\mathbf{r},t)=-(\partial/\partial t)\mathbf{B}(\mathbf{r},t), \quad (27)$$

when the real fields are obtained from Eqs. (22) to (24) by multiplying by the time factor $\exp(-iKt)$ and adding the complex conjugate. In the corresponding notation the magnetic fields are given by

$$\mathfrak{B}_L^{(I)}(\mathbf{k},1,\mathbf{r})=\frac{1}{\sqrt{2}}(\boldsymbol{\kappa}\times\boldsymbol{\varepsilon})\exp(i\mathbf{k}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (28)$$

$$\mathfrak{B}_L^{(R)}(\mathbf{k},1,\mathbf{r})=\frac{1}{\sqrt{2}}(\boldsymbol{\kappa}^{(R)}\times\boldsymbol{\varepsilon})\frac{k_3-K_3}{k_3+K_3}\exp(i\mathbf{k}^{(R)}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (29)$$

$$\mathfrak{B}_L^{(T)}(\mathbf{k},1,\mathbf{r})=\frac{1}{\sqrt{2}n_0}(\mathbf{c}\times\boldsymbol{\varepsilon})\frac{2k_3}{k_3+K_3}\exp(i\mathbf{K}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0, \quad (30)$$

[19] See, for example, Jan Hilgevoord, *Dispersion Relations and Causal Description* (North-Holland, Amsterdam, 1960).

where $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}^{(R)}$ are unit vectors in the directions of $\mathbf{k}$ and $\mathbf{k}^{R)}$, respectively, and $\mathbf{c}$ is a (possibly complex) unit vector in the direction of (possibly complex) $\mathbf{K}$. Since $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}^{(R)}$ are real, $\mathfrak{B}_L^{(I)}$ and $\mathfrak{B}_L^{(R)}$ are both proportional to real vectors. However, the situation is a little more complicated for $\mathfrak{B}_L^{(T)}$. When K_3 is real, $\mathfrak{B}_L^{(T)}$ is also proportional to a real vector. But when K_3 is imaginary, $\mathfrak{B}_L^{(T)}$ is a complex vector, whose real and imaginary parts point in different directions. It is once again convenient to denote the sum of the three foregoing magnetic fields by

$$\mathfrak{B}_L(\mathbf{k},1,\mathbf{r})\equiv\mathfrak{B}_L^{(I)}(\mathbf{k},1,\mathbf{r})+\mathfrak{B}_L^{(R)}(\mathbf{k},1,\mathbf{r})+\mathfrak{B}_L^{(T)}(\mathbf{k},1,\mathbf{r}). \quad (31)$$

The relations (22)–(24) and (28)–(30) hold for transverse electric waves incident from the left, but we can, of course, write down similar equations for transverse magnetic waves, and for waves which are incident from the right. Each of these triplets forms another fundamental mode. We use the label 2 for the transverse magnetic components and the suffix R for the modes excited by waves incident from the right, which are labelled by the wave vector $\mathbf{K}$. Unlike the first set of modes, this last set contains only homogeneous plane waves, and all the wave vectors are real, by virtue of the fact that the waves are passing from a low-index to a higher-index medium. The complex amplitudes of the other mode functions are given by

$$\mathfrak{B}_L^{(I)}(\mathbf{k},2,\mathbf{r})=\frac{1}{\sqrt{2}}\boldsymbol{\varepsilon}\exp(i\mathbf{k}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (32)$$

$$\mathfrak{B}_L^{(R)}(\mathbf{k},2,\mathbf{r})=\frac{1}{\sqrt{2}}\boldsymbol{\varepsilon}\frac{k_3-n_0^2K_3}{k_3+n_0^2K_3}\exp(i\mathbf{k}^{(R)}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (33)$$

$$\mathfrak{B}_L^{(T)}(\mathbf{k},2,\mathbf{r})=\frac{1}{\sqrt{2}}\boldsymbol{\varepsilon}\frac{2k_3}{k_3+n_0^2K_3}\exp(i\mathbf{K}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0, \quad (34)$$

$$\mathfrak{E}_L^{(I)}(\mathbf{k},2,\mathbf{r})=-\frac{1}{\sqrt{2}n_0}(\boldsymbol{\kappa}\times\boldsymbol{\varepsilon})\exp(i\mathbf{k}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (35)$$

$$\mathfrak{E}_L^{(R)}(\mathbf{k},2,\mathbf{r})=-\frac{1}{\sqrt{2}n_0}(\boldsymbol{\kappa}^{(R)}\times\boldsymbol{\varepsilon})\frac{k_3-n_0^2K_3}{k_3+n_0^2K_3}\exp(i\mathbf{k}^{(R)}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (36)$$

$$\mathfrak{E}_L^{(T)}(\mathbf{k},2,\mathbf{r})=-\frac{1}{\sqrt{2}}(\mathbf{c}\times\boldsymbol{\varepsilon})\frac{2k_3}{k_3+n_0^2K_3}\exp(i\mathbf{K}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0 \quad (37)$$

for the TM waves incident from the left,

$$\mathfrak{E}_R^{(I)}(\mathbf{K},1,\mathbf{r})=\frac{1}{\sqrt{2}}\boldsymbol{\varepsilon}\exp(i\mathbf{K}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0, \quad (38)$$

$$\mathfrak{E}_R^{(R)}(\mathbf{K},1,\mathbf{r})=\frac{1}{\sqrt{2}}\boldsymbol{\varepsilon}\frac{K_3-k_3}{K_3+k_3}\exp(i\mathbf{K}^{(R)}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0, \quad (39)$$

$$\mathfrak{E}_R^{(T)}(\mathbf{K},1,\mathbf{r})=\frac{1}{\sqrt{2}}\boldsymbol{\varepsilon}\frac{2K_3}{K_3+k_3}\exp(i\mathbf{k}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (40)$$

$$\mathfrak{B}_R^{(I)}(\mathbf{K},1,\mathbf{r})=\frac{1}{\sqrt{2}}(\mathbf{c}\times\boldsymbol{\varepsilon})\exp(i\mathbf{K}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0, \quad (41)$$

$$\mathfrak{B}_R^{(R)}(\mathbf{K},1,\mathbf{r})=\frac{1}{\sqrt{2}}(\mathbf{c}^{(R)}\times\boldsymbol{\varepsilon})\frac{K_3-k_3}{K_3+k_3}\exp(i\mathbf{K}^{(R)}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0, \quad (42)$$

$$\mathfrak{B}_R^{(T)}(\mathbf{K},1,\mathbf{r})=\frac{n_0}{\sqrt{2}}(\boldsymbol{\kappa}\times\boldsymbol{\varepsilon})\frac{2K_3}{K_3+k_3}\exp(i\mathbf{k}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0 \quad (43)$$

for the TE waves incident from the right, and

$$\mathfrak{B}_R^{(I)}(\mathbf{K},2,\mathbf{r})=\frac{1}{\sqrt{2}}\boldsymbol{\varepsilon}\exp(i\mathbf{K}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0, \quad (44)$$

$$\mathfrak{B}_R^{(R)}(\mathbf{K},2,\mathbf{r})=\frac{1}{\sqrt{2}}\boldsymbol{\varepsilon}\frac{n_0^2K_3-k_3}{k_3+n_0^2K_3}\exp(i\mathbf{K}^{(R)}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0, \quad (45)$$

$$\mathfrak{B}_R^{(T)}(\mathbf{K},2,\mathbf{r}) = \frac{1}{\sqrt{2}}\boldsymbol{\varepsilon}\frac{2n_0^2K_3}{n_0^2K_3+k_3}\exp(i\mathbf{k}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (46)$$

$$\mathfrak{E}_R^{(I)}(\mathbf{K},2,\mathbf{r}) = -\frac{1}{\sqrt{2}}(\mathbf{c}\times\boldsymbol{\varepsilon})\exp(i\mathbf{K}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0, \quad (47)$$

$$\mathfrak{E}_R^{(R)}(\mathbf{K},2,\mathbf{r}) = -\frac{1}{\sqrt{2}}(\mathbf{c}^{(R)}\times\boldsymbol{\varepsilon})\frac{n_0^2K_3-k_3}{n_0^2K_3+k_3}\exp(i\mathbf{K}^{(R)}\cdot\mathbf{r}) \quad \text{for } z\geqslant 0$$
$$=0 \quad \text{for } z<0, \quad (48)$$

$$\mathfrak{E}_R^{(T)}(\mathbf{K},2,\mathbf{r}) = -\frac{1}{\sqrt{2}n_0}(\boldsymbol{\kappa}\times\boldsymbol{\varepsilon})\frac{2n_0^2K_3}{n_0^2K_3+k_3}\exp(i\mathbf{k}\cdot\mathbf{r}) \quad \text{for } z<0$$
$$=0 \quad \text{for } z\geqslant 0, \quad (49)$$

for the TM waves incident from the right. $\mathbf{K}^{(R)}$ is the vector with components $(K_1, K_2, -K_3)$ and $\mathbf{c}^{(R)}$ is the unit vector in the direction of $\mathbf{K}^{(R)}$. As before, we use the shortened notation $\mathfrak{E}_L(\mathbf{k},2,\mathbf{r})$, $\mathfrak{E}_R(\mathbf{K},1,\mathbf{r})$, etc., for the sum of the corresponding complex amplitudes.

We have now established a set of modes, each of which consists of a triplet of waves, including evanescent waves in some cases, which may be used for the representation of an arbitrary "source-free" optical field in the combined vacuum and the dielectric half-spaces. Each frequency component of such an electromagnetic field satisfies a Helmholtz equation of the form (26).

If these modes are also orthogonal when integrated over the whole space, they should lead to a compact expression for the energy of the electromagnetic field. We shall now briefly examine this question, and find that the modes we have established do indeed satisfy an orthonormality condition.

IV. ORTHOGONALITY AND MODE EXPANSION

By forming products of two mode functions, and integrating over all space with the weight function $n^2(\mathbf{r})$, we may readily show that (see Appendix)

$$\int \mathfrak{E}_L^*(\mathbf{k},s,\mathbf{r})\cdot\mathfrak{E}_L(\mathbf{k}',s',\mathbf{r})n^2(\mathbf{r})d^3x = \tfrac{1}{2}(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \quad (50)$$

$$\int \mathfrak{E}_R^*(\mathbf{K},s,\mathbf{r})\cdot\mathfrak{E}_R(\mathbf{K}',s,'\mathbf{r})n^2(\mathbf{r})d^3x = \tfrac{1}{2}(2\pi)^3\delta^3(\mathbf{K}-\mathbf{K}')\delta_{ss'}, \quad (51)$$

$$\int \mathfrak{E}_L^*(\mathbf{k},s,\mathbf{r})\cdot\mathfrak{E}_R(\mathbf{K}',s',\mathbf{r})n^2(\mathbf{r})d^3x = 0, \quad (52)$$

which is the usual expression of orthogonality. Similar relations hold also for the magnetic fields, except that the weight function is then unity. But since the electric and magnetic fields are generally encountered together, it is convenient to express the orthogonality condition in a form which combines both. We then find the following sets of relations:

$$\int [\mathfrak{E}_L^*(\mathbf{k},s,\mathbf{r})\cdot\mathfrak{E}_L(\mathbf{k}',s',\mathbf{r})n^2(\mathbf{r}) + \mathfrak{B}_L^*(\mathbf{k},s,\mathbf{r})\cdot\mathfrak{B}_L(\mathbf{k}',s',\mathbf{r})]d^3x = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \quad (53)$$

$$\int [\mathfrak{E}_R^*(\mathbf{K},s,\mathbf{r})\cdot\mathfrak{E}_R(\mathbf{K}',s',\mathbf{r})n^2(\mathbf{r}) + \mathfrak{B}_R^*(\mathbf{K},s,\mathbf{r})\cdot\mathfrak{B}_R(\mathbf{K}',s',\mathbf{r})]d^3x = (2\pi)^3\delta^3(\mathbf{K}-\mathbf{K}')\delta_{ss'}, \quad (54)$$

$$\int [\mathfrak{E}_L^*(\mathbf{k},s,\mathbf{r})\cdot\mathfrak{E}_R(\mathbf{K}',s',\mathbf{r})n^2(\mathbf{r}) + \mathfrak{B}_L^*(\mathbf{k},s,\mathbf{r})\cdot\mathfrak{B}_R(\mathbf{K}',s',\mathbf{r})]d^3x = 0, \quad (55)$$

$$\int [\mathfrak{E}_L(\mathbf{k},s,\mathbf{r})\cdot\mathfrak{E}_L(\mathbf{k}',s',\mathbf{r})n^2(\mathbf{r}) + \mathfrak{B}_L(\mathbf{k},s,\mathbf{r})\cdot\mathfrak{B}_L(\mathbf{k}',s',\mathbf{r})]d^3x = 0, \quad (56)$$

$$\int [\mathfrak{E}_R(\mathbf{K},s,\mathbf{r})\cdot\mathfrak{E}_R(\mathbf{K}',s',\mathbf{r})n^2(\mathbf{r}) + \mathfrak{B}_R(\mathbf{K},s,\mathbf{r})\cdot\mathfrak{B}_R(\mathbf{K}',s',\mathbf{r})]d^3x = 0, \quad (57)$$

$$\int [\mathfrak{E}_L(\mathbf{k},s,\mathbf{r})\cdot\mathfrak{E}_R(\mathbf{K}',s',\mathbf{r})n^2(\mathbf{r}) + \mathfrak{B}_L(\mathbf{k},s,\mathbf{r})\cdot\mathfrak{B}_R(\mathbf{K}',s',\mathbf{r})]d^3x = 0. \quad (58)$$

Equations (53) and (54) express the usual orthonormality of the modes generated by left-going waves, and by right-going waves, among themselves. It is interesting to note that the relations are the same whether or not evanescent waves appear in the right half-space. Equation (55) expresses orthogonality between any mode produced by a left-going wave and one produced by a right-going wave. The remaining equations, together with their complex conjugates, are also needed for the evaluation of products of real fields.

By combining all possible modes with arbitrary amplitudes, we may form a representation of an arbitrary electromagnetic field, each Fourier component of which satisfies the Helmholtz equation of the form (26). For any such field, we may write

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int_{k_3>0} d^3k \sum_{s=1}^{2}\left(\frac{K}{\epsilon_0}\right)^{1/2}[u(\mathbf{k},s)\mathfrak{E}_L(\mathbf{k},s,\mathbf{r})e^{-iKt}+\text{c.c.}] + \frac{1}{(2\pi)^3}\int_{K_3<0} d^3K \sum_{s=1}^{2}\left(\frac{K}{\epsilon_0}\right)^{1/2}[v(\mathbf{K},s)\mathfrak{E}_R(\mathbf{K},s,\mathbf{r})e^{-iKt}+\text{c.c.}] \quad (59)$$

and

$$\mathbf{B}(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int_{k_3>0}d^3k\sum_{s=1}^{2}\left(\frac{K}{\epsilon_0}\right)^{1/2}[u(\mathbf{k},s)\mathfrak{B}_L(\mathbf{k},s,\mathbf{r})e^{-iKt}+\text{c.c.}]+\frac{1}{(2\pi)^3}\int_{K_3<0}d^3K\sum_{s=1}^{2}\left(\frac{K}{\epsilon_0}\right)^{1/2}[v(\mathbf{K},s)\mathfrak{B}_R(\mathbf{K},s,\mathbf{r})e^{-iKt}+\text{c.c.}],\quad(60)$$

where ϵ_0 is the vacuum dielectric constant, and the factor $(K/\epsilon_0)^{1/2}$ is introduced for later convenience. We have labelled the complex amplitudes of the modes generated by right-going and left-going waves $u(\mathbf{k},s)$ and $v(\mathbf{K},s)$, respectively, in order to emphasize the difference between these modes.

We shall not here enter into the question of completeness of the set of modes with respect to solutions of the Helmholtz equation, which appears to be a difficult problem. However, if validity of the expansions (59) and (60) is assumed, the amplitudes $u(\mathbf{k},s)$ and $v(\mathbf{K},s)$ for any given field may readily be derived. Thus, on taking scalar products of both sides of Eqs. (59) and (60) with $n^2(\mathbf{r})\mathfrak{E}_L^*(\mathbf{k}',s',\mathbf{r})$ and $\mathfrak{B}_L^*(\mathbf{k}',s',\mathbf{r})$, respectively, and integrating over all space, we find with the help of Eqs. (53)–(58)

$$u(\mathbf{k}',s')=(\epsilon_0/K')^{1/2}e^{iK't}\int[n^2(\mathbf{r})\mathbf{E}(\mathbf{r},t)\cdot\mathfrak{E}_L^*(\mathbf{k}',s',\mathbf{r})+\mathbf{B}(\mathbf{r},t)\cdot\mathfrak{B}_L^*(\mathbf{k}',s',\mathbf{r})]d^3x.\quad(61)$$

Similarly we have

$$v(\mathbf{K}',s')=(\epsilon_0/K')^{1/2}e^{iK't}\int[n^2(\mathbf{r})\mathbf{E}(\mathbf{r},t)\cdot\mathfrak{E}_R^*(\mathbf{K}',s',\mathbf{r})+\mathbf{B}(\mathbf{r},t)\cdot\mathfrak{B}_R^*(\mathbf{K}',s',\mathbf{r})]d^3x.\quad(62)$$

The amplitudes appearing in the expansions (59) and (60) can therefore be found. In particular, we may use the expansions to represent a field which is composed only of evanescent waves in the right half-space, by taking $v(\mathbf{K},s)=0$, and $u(\mathbf{k},s)=0$ for $(k_1^2+k_2^2)(n_0^2-1)<k_3^2$.

Finally let us consider the energy $\mathcal{H}$ of the electromagnetic field in the whole space. As usual, for a nonmagnetic medium this is given by the integral

$$\mathcal{H}=\frac{1}{2}\int[\mathbf{D}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)+(1/\mu_0)\mathbf{B}^2(\mathbf{r},t)]d^3x,\quad(63)$$

in which $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are given by Eqs. (59) and (60), and the electric displacement $\mathbf{D}(\mathbf{r},t)$ may be derived from $\mathbf{E}(\mathbf{r},t)$ by multiplying by $\epsilon_0 n^2(\mathbf{r})$. On introducing the mode expansions under the integral in Eq. (63), and making use of the various orthogonality conditions (53)–(58), we readily obtain the result

$$\mathcal{H}=\frac{1}{(2\pi)^3}\int_{k_3>0}d^3k\sum_s K|u(\mathbf{k},s)|^2+\frac{1}{(2\pi)^3}\int_{K_3<0}d^3K\sum_s K|v(\mathbf{K},s)|^2.\quad(64)$$

This shows that the total energy of the electromagnetic field is expressible as the sum of contributions from independent harmonic oscillators, one for each mode. The situation is therefore exactly the same as for the free field in vacuum, although we must not lose sight of the fact that the modes are very different in this case, and that each mode consists of three waves, one of which (for $k_3>0$) may be an evanescent wave.

V. QUANTIZATION OF FIELD

The simple expression (64) for the electromagnetic energy now leads to a straightforward procedure for quantizing the field, which is essentially the same as that for the free field in vacuum.[20] We regard the field as a collection of independent quantum oscillators. The complex amplitudes $u(\mathbf{k},s)$, $u^*(\mathbf{k},s)$ and $v(\mathbf{K},s)$, $v^*(\mathbf{K},s)$ are replaced by Hilbert-space operators[21] $\hat{u}(\mathbf{k},s)$, $\hat{u}^\dagger(\mathbf{k},s)$ and $\hat{v}(\mathbf{K},s)$, $\hat{v}^\dagger(\mathbf{K},s)$, which can be given the usual interpretation of annihilation and creation operators for quantum excitations, or photons, labelled by the mode $\mathbf{k}$, s and $\mathbf{K}$, s, respectively. Since the different harmonic oscillators are independent, all operators belonging to different modes commute. In addition, all annihilation operators and all creation operators commute among themsleves. We therefore have the following commutation relations:

$$\begin{aligned}
[\hat{u}(\mathbf{k},s),\hat{u}(\mathbf{k}',s')]&=0=[\hat{u}^\dagger(\mathbf{k},s),\hat{u}^\dagger(\mathbf{k}',s')],\\
[\hat{v}(\mathbf{K},s),\hat{v}(\mathbf{K}',s')]&=0=[\hat{v}^\dagger(\mathbf{K},s),\hat{v}^\dagger(\mathbf{K}',s')],\\
[\hat{u}(\mathbf{k},s),\hat{v}(K',s')]&=0=[\hat{u}^\dagger(\mathbf{k},s),\hat{v}^\dagger(\mathbf{K}',s')],\\
[\hat{u}(\mathbf{k},s),\hat{v}^\dagger(K',s')]&=0=[\hat{u}^\dagger(\mathbf{k},s),\hat{v}(\mathbf{K}',s')],
\end{aligned}\quad(65)$$

while

$$[\hat{u}(\mathbf{k},s),\hat{u}^\dagger(\mathbf{k}',s')]=f(\mathbf{k},s)\delta_{ss'}\delta^3(\mathbf{k}-\mathbf{k}'),\quad(66)$$

$$[\hat{v}(\mathbf{K},s),v^\dagger(\mathbf{K}',s')]=g(\mathbf{K},s)\delta_{ss'}\delta^3(K-K'),\quad(67)$$

in which the functions $f(\mathbf{k},s)$ and $g(\mathbf{K},s)$ are assumed to be c numbers, but are undetermined as yet. The expression for the energy of the quantized field can now be written in the normally ordered form

$$\mathcal{H}_{\text{op}}=\sum_s\frac{1}{(2\pi)^3}\left[\int_{k_3>0}d^3k\,K\hat{u}^\dagger(\mathbf{k},s)\hat{u}(\mathbf{k},s)+\int_{K_3<0}d^3K\,K\hat{v}^\dagger(\mathbf{K},s)v(\mathbf{K},s)\right],\quad(68)$$

[20] See, for example, W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill, New York, 1964).

[21] We use the caret ∧ to denote a Hilbert-space operator, or, when this is not available, the subscript op.

in which possible zero-point contributions have been omitted. As usual, the operators

$$[1/(2\pi)^3]d^3k\hat{u}^\dagger(\mathbf{k},s)\hat{u}(\mathbf{k},s)/\hbar$$

and

$$[1/(2\pi)^3]d^3K\hat{v}^\dagger(\mathbf{K},s)\hat{v}(\mathbf{K},s)/\hbar$$

behave as number operators for the excitations, or photons, of type $\mathbf{k}$, s and $\mathbf{K}$, s, within the differential ranges d^3k and d^3K, respectively. The eigenstates of these operators are the Fock states of the field, which we can form in the customary way by allowing the operators $\hat{u}^\dagger(\mathbf{k},s)$, $\hat{v}^\dagger(\mathbf{K},s)$ to operate on the vacuum state $|\{0\}\rangle$.[22] Thus

$$|\mathbf{k}_1,s_1,\ldots,\mathbf{k}_n,s_n;K_1,\sigma_1,\ldots,K_m,\sigma_m\rangle = [1/(\sqrt{n!}\sqrt{m!})]\hat{u}^\dagger(\mathbf{k}_1s_1)\cdots\hat{u}^\dagger(\mathbf{k}_n,s_n) \times\hat{v}^\dagger(\mathbf{K}_1,\sigma_1)\cdots\hat{v}^\dagger(\mathbf{K}_m,\sigma_m)|\{0\}\rangle, \quad (69)$$

with the normalization

$$\langle\mathbf{K}_M',\sigma_M',\ldots,\mathbf{K}_1',\sigma_1';\mathbf{k}_N',s_N',\ldots,\mathbf{k}_1',s_1'|\mathbf{k}_1,s_1,\ldots,\mathbf{k}_n,s_n;\mathbf{k}_1,\sigma_1,\ldots,K_m,\sigma_m\rangle = \delta_{nN}\delta_{mM}\frac{1}{n!m!}\sum_P \delta^3(\mathbf{k}_1-\mathbf{k}_1')\delta_{s_1s_1'}\cdots\delta^3(\mathbf{k}_n-\mathbf{k}_n')\delta_{s_ns_n'}\delta^3(\mathbf{K}_1-\mathbf{K}_1')\delta_{\sigma_1\sigma_1'}\cdots\delta^3(\mathbf{K}_m-\mathbf{K}_m')\delta_{\sigma_m\sigma_m'}, \quad (70)$$

where $\sum_P$ denotes the sum over all $n!$ permutations of the $\mathbf{k}$, s modes and all $m!$ permutations of the $\mathbf{K}$, σ modes.

Similarly, we can define coherent states of the field,[23] labelled by a set of complex functions $u(\mathbf{k},s)$, $v(\mathbf{K},s)$, as the eigenstates of the annihilation operators $\hat{u}(\mathbf{k},s)$ and $\hat{v}(\mathbf{K},s)$:

$$\begin{aligned}\hat{u}(\mathbf{k},s)|\{u(\mathbf{k},s)\},\{v(\mathbf{K},s)\}\rangle&=u(\mathbf{k},s)|\{u(\mathbf{k},s)\},\{v(\mathbf{K},s)\}\rangle,\\ \hat{v}(\mathbf{K},s)|\{u(\mathbf{k},s)\},\{v(\mathbf{K},s)\}\rangle&=v(\mathbf{K},s)|\{u(\mathbf{k},s)\},\{v(\mathbf{K},s)\}\rangle.\end{aligned} \quad (71)$$

The notation $|\{u(\mathbf{k},s)\},\{v(\mathbf{K},s)\}\rangle$ is meant to emphasize that the states are functionals of $u(\mathbf{k},s)$, $v(\mathbf{K},s)$. We can also make "diagonal" representations of the density operator for the state of the field in terms of coherent states, in the usual way.[23–25]

The formalism is therefore strictly parallel to the usual formalism for the quantization of the free electromagnetic field in vacuum, except for the fact that the fundamental modes are different, and that each mode, although labelled by one wave vector and one polarization index, always stands for three waves coupled via the Fresnel formulas. As is to be expected, this difference becomes important when we treat the behavior of the field in configuration space, via the mode expansions.

VI. CONFIGURATION-SPACE FIELD COMMUTATORS

By making use of the mode expansions (59) and (60), in which the fields $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, and the mode amplitudes $u(\mathbf{k},s)$, $v(\mathbf{K},s)$, are replaced by their corresponding Hilbert-space operators, together with the commutation rules (65)–(67), we can form commutators of the $\hat{E}_i(\mathbf{r},t)$ and $\hat{B}_j(\mathbf{r},t)$ operators. Since we are particularly interested in the contributions from evanescent waves, we shall be mainly concerned with the fields in the empty right half-space, for which

$$\mathfrak{E}_L(\mathbf{k},s,\mathbf{r})=\mathfrak{E}_L^{(T)}(\mathbf{k},s,\mathbf{r}), \quad (72)$$

$$\mathfrak{B}_L(\mathbf{k},s,\mathbf{r})=\mathfrak{B}_L^{(T)}(\mathbf{k},s,\mathbf{r}), \quad (73)$$

and

$$\mathfrak{E}_R(\mathbf{K},s,\mathbf{r})=\mathfrak{E}_R^{(I)}(\mathbf{K},s,\mathbf{r})+\mathfrak{E}_R^{(R)}(\mathbf{K},s,\mathbf{r}), \quad (74)$$

$$\mathfrak{B}_R(\mathbf{K},s,\mathbf{r})=\mathfrak{B}_R^{(I)}(\mathbf{K},s,\mathbf{r})+\mathfrak{B}_R^{(R)}(\mathbf{K},s,\mathbf{r}). \quad (75)$$

With the help of Eqs. (59) and (65)–(67), and (24), (37)–(39), (47), and (48), we then obtain the following expression for the electric field commutators at two space-time points in the right half-space:

$$[\hat{E}_i(\mathbf{r},t),\hat{E}_j(\mathbf{r}',t')]=\frac{1}{(2\pi)^6\epsilon_0}\int_{k_3>0}d^3k\left[\frac{K}{2n_0^2}f(\mathbf{k},1)\epsilon_i\epsilon_j\frac{4k_3^2}{|k_3+K_3|^2}+\frac{K}{2}f(\mathbf{k},2)(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}^*\times\boldsymbol{\varepsilon})_j\frac{4k_3^2}{|k_3+n_0^2K_3|^2}\right] \times\exp i[\mathbf{K}\cdot\mathbf{r}-\mathbf{K}^*\cdot\mathbf{r}'-K(t-t')]-\text{c.c.}$$

[22] For a treatment of continuous Fock space see, for example, J. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961), p. 159.
[23] Cf. R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[24] C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).
[25] J. R. Klauder, Phys. Rev. Letters **16**, 534 (1966).

$$+\frac{1}{(2\pi)^6\epsilon_0}\int_{K_3<0} d^3K\,\frac{K}{2}g(\mathbf{K},1)\epsilon_i\epsilon_j\Big\{\exp[i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')]+\left(\frac{K_3-k_3}{K_3+k_3}\right)^2\exp[i\mathbf{K}^{(R)}\cdot(\mathbf{r}-\mathbf{r}')]$$
$$+\left(\frac{K_3-k_3}{K_3+k_3}\right)[\exp i(\mathbf{K}\cdot\mathbf{r}-\mathbf{K}^{(R)}\cdot\mathbf{r}')+\exp i(\mathbf{K}^{(R)}\cdot\mathbf{r}-\mathbf{K}\cdot\mathbf{r}')]\Big\}\exp[-iK(t-t')]-\text{c.c.}$$
$$+\frac{1}{(2\pi)^6\epsilon_0}\int_{K_3<0} d^3K\,\frac{K}{2}g(\mathbf{K},2)\Big\{(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}\times\boldsymbol{\varepsilon})_j\exp i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')+(\mathbf{c}^{(R)}\times\boldsymbol{\varepsilon})_i(\mathbf{c}^{(R)}\times\boldsymbol{\varepsilon})_j\left(\frac{n^2K_3-k_3}{n^2K_3+k_3}\right)^2\exp i\mathbf{K}^{(R)}\cdot(\mathbf{r}-\mathbf{r}')$$
$$+\left(\frac{n^2K_3-k_3}{n^2K_3+k_3}\right)[(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}^{(R)}\times\boldsymbol{\varepsilon})_j\exp i(\mathbf{K}\cdot\mathbf{r}-\mathbf{K}^{(R)}\cdot\mathbf{r}')+(\mathbf{c}^{(R)}\times\boldsymbol{\varepsilon})_i(\mathbf{c}\times\boldsymbol{\varepsilon})_j\exp i(\mathbf{K}^{(R)}\cdot\mathbf{r}-\mathbf{K}\cdot\mathbf{r}')]\Big\}$$
$$\times\exp[-iK(t-t')]-\text{c.c.}\quad(76)$$

It is convenient to transform the integral over $\mathbf{k}$ in Eq. (76) to an integral over the vacuum wave vector $\mathbf{K}$, with the help of the relations (17), (18), and $n_0^2K_3^2=k_3^2-(n_0^2-1)(K_1^2+K_2^2)$. That part of the range of integration for which $k_3^2>(n_0^2-1)(K_1^2+K_2^2)$, or $K_3^2>0$, then includes only homogeneous plane waves, while the part of the range for which $k_3^2<(n_0^2-1)(K_1^2+K_2^2)$, or $(1/n_0^2-1)(K_1^2+K_2^2)<K_3^2<0$, includes only evanescent waves. It is therefore natural to decompose the integral into separate integrals. We can also introduce a slight simplification in some of the remaining terms in Eq. (76), by making the transformation $K_3\to -K_3$, which implies $k_3\to -k_3$, $\mathbf{K}\to\mathbf{K}^{(R)}$ and $\mathbf{c}\to\mathbf{c}^{(R)}$. With the help of these transformations, and on rearranging the order of the terms, we then obtain the equation

$$[\hat{E}_i(\mathbf{r},t),\hat{E}_j(\mathbf{r}',t')]=\frac{1}{\epsilon_0(2\pi)^6}\int_{K_3>0}d^3K\,\frac{K}{2}\left[f(\mathbf{k},1)\epsilon_i\epsilon_j\frac{4k_3K_3}{(k_3+K_3)^2}+f(\mathbf{k},2)(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}\times\boldsymbol{\varepsilon})_j\frac{4n_0^2k_3K_3}{(k_3+n_0^2K_3)^2}\right]$$
$$\times\exp i[\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-K(t-t')]-\text{c.c.}$$
$$+\frac{1}{\epsilon_0(2\pi)^6}\left\{\int_{K_3<0}d^3K\,g(\mathbf{K},1)+\int_{K_3>0}d^3K\,g(\mathbf{K}^{(R)},1)\frac{(K_3-k_3)^2}{(K_3+k_3)^2}\right\}\frac{K}{2}\epsilon_i\epsilon_j\exp i[\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-K(t-t')]-\text{c.c.}$$
$$+\frac{1}{\epsilon_0(2\pi)^6}\left\{\int_{K_3<0}d^3K\,g(\mathbf{K},2)+\int_{K_3>0}d^3K\,g(\mathbf{K}^{(R)},2)\left(\frac{n_0^2K_3-k_3}{n_0^2K_3+k_3}\right)^2\right\}\frac{K}{2}(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}\times\boldsymbol{\varepsilon})_j$$
$$\times\exp i[\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-\mathbf{K}(t-t')]-\text{c.c.}$$
$$+\frac{1}{\epsilon_0(2\pi)^6}\left\{\int_{K_3<0}d^3K\,g(\mathbf{K},1)+\int_{K_3>0}d^3K\,g(\mathbf{K}^{(R)},1)\right\}\frac{K}{2}\left(\frac{K_3-k_3}{K_3+k_3}\right)\epsilon_i\epsilon_j\exp i[\mathbf{K}\cdot\mathbf{r}-\mathbf{K}^{(R)}\cdot\mathbf{r}'-K(t-t')]-\text{c.c.}$$
$$+\frac{1}{\epsilon_0(2\pi)^6}\left\{\int_{K_3<0}d^3K\,g(\mathbf{K},2)+\int_{K_3>0}d^3K\,g(\mathbf{K}^{(R)},2)\right\}\frac{K}{2}\left(\frac{n_0^2K_3-k_3}{n_0^2K_3+k_3}\right)(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}^{(R)}\times\boldsymbol{\varepsilon})_j$$
$$\times\exp i[\mathbf{K}\cdot\mathbf{r}-\mathbf{K}^{(R)}\cdot\mathbf{r}'-K(t-t')]-\text{c.c.}$$
$$-\frac{1}{\epsilon_0(2\pi)^6}\int_{-\infty}^{\infty}d^2K\int_0^{[(1-1/n_0^2)(K_1^2+K_2^2)]^{1/2}}d|K_3|\left[f(\mathbf{k},1)\epsilon_i\epsilon_j\frac{2k_3|K_3|}{k_3^2+|K_3|^2}+f(\mathbf{k},2)(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}^*\times\boldsymbol{\varepsilon})_j\frac{2k_3n_0^2|K_3|}{k_3^2+n_0^2|K_3|^2}\right]$$
$$\times K\exp[-|K_3|(z+z')]\exp i[K_1(x-x')+K_2(y-y')-K(t-t')]-\text{c.c.}\quad(77)$$

It will be seen that the terms under the first five integral signs in this rather complicated expansion differ from the others in that they depend on the separation of the space-time points $\mathbf{r}$, t and $\mathbf{r}'$, t' in the familiar way, via the factors $\exp i[\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-K(t-t')]$. The next four integrals, on the other hand, contain the factor

$$\exp i[K_1(x-x')+K_2(y-y')+K_3(z+z')-K(t-t')],$$

which is anomalous in its dependence on the z coordinates. The remaining terms depend on the space-time points via the factor $\exp[-|K_3|(z+z')]\exp i[K_1(x-x')+K_2(y-y')-K(t-t')]$, which is again anomalous in a different way with respect to the z coordinates.

However, the anomalous z dependence does have one benefit, which allows us to determine the functions

$f(\mathbf{k},s)$ and $g(\mathbf{K},s)$. For the contributions of these anomalous terms should become small compared with the others as $z\rightarrow\infty$ and $z'\rightarrow\infty$, when the difference $z-z'$ is kept constant. This means that only the first five integrals should contribute to the field commutator at a very great distance from the dielectric interface. But at a great distance from the interface, it may reasonably be argued that the commutator should reduce to the usual free-field commutator in empty space,[20] which is given by

$$[\hat{E}_i(\mathbf{r},t),\hat{E}_j(\mathbf{r}',t')]_{\text{vacuum}}$$

$$=\frac{\hbar}{2\epsilon_0(2\pi)^3}\int d^3K\, K(\delta_{ij}-c_ic_j)$$

$$\times\exp i[K\cdot(\mathbf{r}-\mathbf{r}')-\mathbf{K}(t-t')]-\text{c.c.}$$

$$=\frac{i\hbar}{\epsilon_0}\left[\delta_{ij}\frac{\partial^2}{\partial t\partial t'}-\frac{\partial^2}{\partial x_i\partial x_j'}\right]\Delta(\mathbf{r}-\mathbf{r}',t-t'),\tag{78}$$

where $\Delta(\mathbf{r},t)$ is the usual singular function defined by

$$\Delta(\mathbf{r},t)\equiv\frac{1}{(2\pi)^3}\int d^3K\frac{\sin Kt}{K}\exp i\mathbf{K}\cdot\mathbf{r}.\tag{79}$$

Comparison of the first five integrals in Eq. (77) with Eq. (78) now shows that equality requires

$$f(\mathbf{k},1)\frac{4k_3K_3}{(k_3+K_3)^2}+g(\mathbf{K}^{(R)},1)\frac{(K_3-k_3)^2}{(K_3+k_3)^2}=g(\mathbf{K},1)=(2\pi)^3\hbar,\tag{80}$$

$$f(\mathbf{k},2)\frac{4k_3n_0^2K_3}{k_3+n_0^2K_3}+g(\mathbf{K}^{(R)},2)\frac{(n_0^2K_3-k_3)^2}{(n_0^2K_3+k_3)^2}=g(\mathbf{K},2)=(2\pi)^3\hbar,\tag{81}$$

which is satisfied if

$$f(\mathbf{k},1)=f(\mathbf{k},2)=(2\pi)^3\hbar,\tag{82}$$

$$g(\mathbf{K},1)=g(\mathbf{K},2)=(2\pi)^3\hbar.\tag{83}$$

With this choice, the commutator (77) reduces to the somewhat more compact form

$$\begin{aligned}
[\hat{E}_i(\mathbf{r},t),\hat{E}_j(\mathbf{r}',t')]=&\frac{i\hbar}{\epsilon_0}\left[\delta_{ij}\frac{\partial^2}{\partial t\partial t'}-\frac{\partial^2}{\partial x_i\partial x_j'}\right]\Delta(\mathbf{r}-\mathbf{r}',t-t')+\frac{\hbar}{\epsilon_0}\frac{1}{(2\pi)^3}\int d^3K\frac{K}{2}\left[\left(\frac{K_3-k_3}{K_3+k_3}\right)\epsilon_i\epsilon_j\right.\\
&\left.+\left(\frac{n_0^2K_3-k_3}{n_0^2K_3+k_3}\right)(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}^{(R)}\times\boldsymbol{\varepsilon})_j\right]\exp i[K_1(x-x')+K_2(y-y')+K_3(z+z')-K(t-t')]-\text{c.c.}\\
&-\frac{\hbar}{\epsilon_0}\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}d^2K\int_0^{[(1-1/n_0^2)(K_1^2+K_2^2)]^{1/2}}d|K_3|\,K\left[\frac{2k_3|K_3|}{k_3^2+|K_3|^2}\epsilon_i\epsilon_j+\frac{2k_3n_0^2|K_3|}{k_3^2+n_0^2|K_3|^2}(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}^*\times\boldsymbol{\varepsilon})_j\right]\\
&\times\exp[-|K_3|(z+z')]\exp i[K_1(x-x')+K_2(y-y')-K(t-t')]-\text{c.c.}
\end{aligned}\tag{84}$$

The first term on the right-hand side is the usual free-field commutator, which vanishes off the light cone connecting the two events $\mathbf{r}$, t and $\mathbf{r}'$, t'. It is the dominant term at a great distance from the dielectric interface, and is obviously due to homogeneous light waves directly connecting the events. The next term contains the factor

$$\exp i[K_1(x-x')+K_2(y-y')+K_3(z+z')]$$

under the integral, and therefore connects one space-time point with the image of the other in the dielectric interface. This contribution is evidently due to homogeneous light waves connecting the space-time points $\mathbf{r}$, t and $\mathbf{r}'$, t' by reflection in the interface. Due to the dispersive nature of the reflection, this term has a different structure from the first, and is nonzero also off the light cone. The last term containing the factor $\exp[-|K_3|(z+z')]$ under the integral falls off exponentially with distance from the interface, and is evidently due to evanescent waves connecting the two events. Since the evanescent waves propagate parallel to the interface, only the x, y, and t coordinates appear in the oscillatory factor. This term is complicated and is also nonzero off the light cone connecting the two events.

While causality would seem to require that the entire commutator vanish for two events having a spacelike separation, we must remember that an element of noncausality was introduced right at the beginning of our treatment, when we chose to ignore the high-frequency behavior of the refractive index, at frequencies in the region of anomalous dispersion and beyond. For this reason the commutator given by Eq. (84) cannot be expected to be strictly causal, although any noncausality might be expected to extend only over distances and time intervals of the order of the anomalous dispersion wavelength and period, which can be made as short as desired. It is clear that commutators for the magnetic fields, and for mixed electric and magnetic fields, may be derived in a strictly parallel manner.

Finally we note that, if we define photon number operators by integrating $\hat{u}^\dagger(\mathbf{k},s)\hat{u}(\mathbf{k},s)/\hbar$ or

$\hat{v}^\dagger(\mathbf{K},s)\hat{v}(\mathbf{K},s)/\hbar$ over finite domains $\mathbf{\Delta k}$ or $\mathbf{\Delta K}$,

$$\begin{aligned}\hat{N}(\mathbf{\Delta k},s)&\equiv\frac{1}{(2\pi)^3\hbar}\int_{\Delta\mathrm{k}}\hat{u}^\dagger(\mathbf{k},s)\hat{u}(\mathbf{k},s)d^3k\,,\\ \hat{N}(\mathbf{\Delta K},s)&\equiv\frac{1}{(2\pi)^3\hbar}\int_{\Delta\mathrm{K}}\hat{v}^\dagger(\mathbf{K},s)\hat{v}(\mathbf{K},s)d^3K\,,\end{aligned}\tag{85}$$

then it follows from the commutation relations (66) and (67) with (80) and (81) that

$$\begin{aligned}[\hat{u}(\mathbf{k}',s'),\hat{N}(\mathbf{\Delta k},s)]&=\hat{u}(\mathbf{k}',s')\delta_{ss'}U(\mathbf{k}'\subset\mathbf{\Delta k})\,,\\ [\hat{v}(\mathbf{K}',s'),\hat{N}(\mathbf{\Delta K},s)]&=\hat{v}(\mathbf{K}',s')\delta_{ss'}U(\mathbf{K}'\subset\mathbf{\Delta K})\,,\end{aligned}\tag{86}$$

where $U(\mathbf{k}'\subset\mathbf{\Delta k})=1$ or 0 according as $\mathbf{k}'\subset\mathbf{\Delta k}$ or $\mathbf{k}'\not\subset\mathbf{\Delta k}$. Moreover, from the form of $\mathcal{H}_{\mathrm{op}}$ given by Eq. (68), we have

$$\begin{aligned}[\hat{u}(\mathbf{k}',s'),\mathcal{H}_{\mathrm{op}}]&=\hbar K'\hat{u}(\mathbf{k}',s')\,,\\ [\hat{v}(\mathbf{K}',s'),\mathcal{H}_{\mathrm{op}}]&=\hbar K'\hat{v}(\mathbf{K}',s')\,.\end{aligned}\tag{87}$$

VII. CONFIGURATION-SPACE PHOTON ABSORPTION OPERATOR

In the usual treatment of the free electromagnetic field in vacuum, it has been found convenient to introduce a configuration-space photon absorption operator, which we call $\hat{V}_i(\mathbf{r},t)$, such that the integral over all space of $\hat{V}_i^\dagger(\mathbf{r},t)\hat{V}_i(\mathbf{r},t)$ gives the total number of photons.[26] $\hat{V}_i^\dagger(\mathbf{r},t)\hat{V}_i(\mathbf{r},t)$ is therefore the photon density per unit volume, and the operator $\hat{V}_i(\mathbf{r},t)$ plays a role in configuration space which is somewhat similar to the role played by the usual annihilation operator in the conjugate space. Moreover, it has been shown that the integral of $\hat{V}_i^\dagger(\mathbf{r},t)\hat{V}_i(\mathbf{r},t)$ over a finite volume also has a physical significance,[26] provided that the linear dimensions of the volume are large compared with the wavelengths of all modes contributing to $\hat{V}_i(\mathbf{r},t)$.

In the present situation we may again introduce a configuration-space absorption operator, although the somewhat more complicated mode structure limits the interpretation of $\hat{V}_i^\dagger(\mathbf{r},t)\hat{V}_i(\mathbf{r},t)$. Consider the operator defined by

$$\hat{V}_i(\mathbf{r},t)\equiv\sum_s\left(\frac{2}{\hbar}\right)^{1/2}\frac{1}{(2\pi)^3}\times\left[\int_{k_3>0}d^3k\,\hat{u}(\mathbf{k},s)\mathfrak{E}_{Li}(\mathbf{k},s,\mathbf{r})n(\mathbf{r})e^{-iKt}+\int_{K_3<0}d^3K\,\hat{v}(\mathbf{K},s)\mathfrak{E}_{Ri}(\mathbf{K},s,\mathbf{r})n(\mathbf{r})e^{-iKt}\right].\tag{88}$$

With the help of the orthogonality relations (50)–(52), we readily find that

$$\begin{aligned}\int\hat{V}_i^\dagger(\mathbf{r},t)\hat{V}_i(\mathbf{r},t)d^3x&=\sum_s\frac{1}{\hbar(2\pi)^3}\int_{k_3>0}d^3k\,\hat{u}^\dagger(\mathbf{k},s)\hat{u}(\mathbf{k},s)+\int_{K_3<0}d^3K\,v^\dagger(\mathbf{K},s)v(\mathbf{K},s)\\&=\hat{N}\,,\end{aligned}\tag{89}$$

where $\hat{N}$ is the total number of photons. We see that $\hat{V}_i^\dagger(\mathbf{r},t)\hat{V}_i(\mathbf{r},t)$ again has the dimensions of photon density, although its integral over a finite volume is less readily interpreted. In view of the relations (86), it follows at once that

$$[\hat{V}_i(\mathbf{r},t),\hat{N}]=\hat{V}_i(\mathbf{r},t)\,,\tag{90}$$

$$[\hat{V}_i^\dagger(\mathbf{r},t),\hat{N}]=-\hat{V}_i^\dagger(\mathbf{r},t)\,.\tag{91}$$

By making use of the relations (22)–(24), (28)–(30), and (32)–(49), together with their Hermitian adjoints, and the commutation relations (65)–(67) with (82) and (83), we can evaluate the commutator of $\hat{V}_i(\mathbf{r},t)$ with $\hat{V}_j^\dagger(\mathbf{r}',t')$. For two points $\mathbf{r}$ and $\mathbf{r}'$ in the right half-space we find, after introducing a transformation of variables and proceeding as in the derivation of Eq. (84),

$$\begin{aligned}[\hat{V}_i(\mathbf{r},t),\hat{V}_j^\dagger(\mathbf{r}',t')]&=\frac{1}{(2\pi)^3}\int d^3K(\delta_{ij}-c_ic_j)\exp i[\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-K(t-t')]\\&+\frac{1}{(2\pi)^3}\int d^3K\left[\epsilon_i\epsilon_j\left(\frac{K_3-k_3}{K_3+k_3}\right)+(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}^{(R)}\times\boldsymbol{\varepsilon})_j\left(\frac{n_0^2K_3-k_3}{n_0^2K_3+k_3}\right)\right]\\&\qquad\times\exp i[K_1(x-x')+K_2(y-y')+K_3(z+z')-K(t-t')]\\&-\frac{2}{(2\pi)^3}\int_{-\infty}^{\infty}d^2K\int_0^{[(1-1/n_0^2)(K_1^2+K_2)]^{1/2}}d|K_3|\left[\epsilon_i\epsilon_j\left(\frac{2k_3|K_3|}{k_3^2+|K_3|^2}\right)+(\mathbf{c}\times\boldsymbol{\varepsilon})_i(\mathbf{c}^*\times\boldsymbol{\varepsilon})_j\left(\frac{2k_3n_0^2|K_3|}{k_3^2+n_0^4|K_3|^2}\right)\right]\\&\qquad\times\exp[-|K_3|(z+z')]\exp i[K_1(x-x')+K_2(y-y')-K(t-t')]\,,\end{aligned}\tag{92}$$

in which the three terms can be identified as before as due to direct waves, reflected waves, and evanescent waves, respectively.

[26] L. Mandel, Phys. Rev. **144**, 1071 (1966).

VIII. PHOTOELECTRIC EMISSION IN EVANESCENT WAVE FIELD

So far we have been treating the electromagnetic field effectively as a free field, despite the presence of the dielectric in the left half-space. But the questions which are of most physical significance obviously relate to the interaction of this field with atoms and charges.

Let us therefore consider the problem of a bound electron, located in the right half-space, interacting with an electromagnetic field having evanescent wave components in the right half-space. We shall calculate the probability, to the first order in perturbation theory, that the electron makes an upward transition to the continuum of positive-energy states, after a short time T following the turn-on of the interaction. Since the electron is then free, we refer to this as the problem of photoelectric emission. However, the calculation is substantially similar for upward transitions to a broad band of bound states, which is the situation often encountered in radiation-induced fluorescence.

If $\hat{\rho}(t)$ is the density operator of the combined system at time t in the interaction picture, and $\hat{H}_1(t)$ is the interaction at time t, which is turned on at time t_0, then we have from the usual perturbation expansion,[20] up to the second order in $\hat{H}_1$,

$$\hat{\rho}(t_0+T)=\hat{\rho}(t_0)+\frac{1}{i\hbar}\int_{t_0}^{t_0+T}[\hat{H}_1(t),\hat{\rho}(t_0)]dt + \frac{1}{(i\hbar)^2}\int_{t_0}^{t_0+T}dt_1\int_{t_0}^{t_1}dt_2[\hat{H}_1(t_1),[\hat{H}_1(t_2),\hat{\rho}(t_0)]]. \quad (93)$$

We suppose that the electron is initially in some bound state $|\psi_0\rangle$ with energy $E_0<0$ and $-E_0/\hbar$ in the range of optical frequencies, and that the field is in a coherent state $|\{u(\mathbf{k},s)\},0\rangle$ [cf. Eq. (71)], in which the modes labelled by left-going waves are all unoccupied. Moreover, it is convenient to suppose that the field is quasimonochromatic, with frequencies centered on some optical frequency. Then

$$\hat{\rho}(t_0)=|\psi_0\rangle\langle\psi_0|\,|\{u(\mathbf{k},s)\},0\rangle\langle 0,\{u(\mathbf{k},s)\}|\,. \quad (94)$$

We take the interaction to be of the usual form

$$\hat{H}_1(t)=-(e/m)\exp[i\hat{H}_0(t-t_0)/\hbar]\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}(\mathbf{r}_0,t_0) \times\exp[-iH_0(t-t_0)/\hbar], \quad (95)$$

in which $\hat{\mathbf{p}}$ is the electron momentum, $\mathbf{r}_0$ is some fixed point within the potential well wherein the electron is bound, $\hat{\mathbf{A}}$ is the vector potential in the Coulomb gauge, and $\hat{H}_0$ is the noninteracting part of the Hamiltonian. In taking $\hat{\mathbf{A}}$ in the interaction $\hat{H}_1$ at a fixed point $\mathbf{r}_0$, we are making the usual assumption that the dimensions of the well are small compared with the wavelengths of all occupied modes of the field.

From the expansion (59) and the relation

$$\hat{E}_i(\mathbf{r},t)=-\frac{\partial}{\partial t}\hat{A}_i(\mathbf{r},t)$$

between the electric field and the vector potential, together with the commutation relations (87), it follows at once that

$$\exp[i\hat{H}_0(t-t_0)/\hbar]\hat{\mathbf{A}}(\mathbf{r}_0,t_0)\exp[-i\hat{H}_0(t-t_0)/\hbar]\equiv\hat{\mathbf{A}}(r_0,t) = \frac{-i}{(2\pi)^3}\int_{k_3>0}d^3k\sum_s\frac{1}{(K\epsilon_0)^{1/2}}[\hat{u}(\mathbf{k},s)\mathfrak{E}_L(\mathbf{k},s,\mathbf{r}_0)e^{-iK(t-t_0)}-\text{H.c.}] \times\frac{-i}{(2\pi)^2}\int_{K_3<0}d^3K\sum_s\frac{1}{(K\epsilon_0)^{1/2}}[\hat{v}(\mathbf{K},s)\mathfrak{E}_R(\mathbf{K},s,\mathbf{r}_0)e^{-iK(t-t_0)}-\text{H.c.}]. \quad (96)$$

It is convenient to denote the positive- and negative-frequency parts of $\hat{\mathbf{A}}(\mathbf{r},t)$, which are Hermitian conjugates of each other, by $\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)$, and $\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)$, respectively. If we denote $\exp[i\hat{H}_0(t-t_0)/\hbar]\hat{\mathbf{p}}\exp[-i\hat{H}_0(t-t_0)/\hbar]$ by $\hat{\mathbf{p}}(t)$, we may write

$$[\hat{H}_1(t),\hat{\rho}(t_0)]=-(e/m)[\hat{p}_i(t)|\psi_0\rangle\langle\psi_0|\hat{A}_i(\mathbf{r}_0,t)|\{u(\mathbf{k},s)\},0\rangle\langle 0,\{u(\mathbf{k},s)\}| -|\psi_0\rangle\langle\psi_0|\hat{p}_i(t)|\{u(\mathbf{k},s)\},0\rangle\langle 0,\{u(\mathbf{k},s)\}|\hat{\mathbf{A}}(\mathbf{r}_0,t)] \quad (97)$$

and

$$[\hat{H}_1(t_1),[\hat{H}_1(t_2),\hat{\rho}(t_0)]]=(e/m)^2\{\hat{p}_j(t_1)\hat{p}_i(t_2)|\psi_0\rangle\langle\psi_0|\hat{A}_j(\mathbf{r}_0,t_1)\hat{A}_i(\mathbf{r}_0,t_2)|\{u(\mathbf{k},s)\},0\rangle\langle\{u(\mathbf{k},s)\},0| -\hat{p}_j(t_1)|\psi_0\rangle\langle\psi_0|\hat{p}_i(t_2)\hat{A}_j(\mathbf{r}_0,t_1)|\{u(\mathbf{k},s)\},0\rangle\langle 0,\{u(\mathbf{k},s)\}|\hat{A}_i(\mathbf{r}_0,t_2) -\hat{p}_i(t_2)|\psi_0\rangle\langle\psi_0|\hat{p}_j(t_1)\hat{A}_i(\mathbf{r}_0,t_2)|\{u(\mathbf{k},s)\},0\rangle\langle 0,\{u(\mathbf{k},s)\}|\hat{A}_j(\mathbf{r}_0,t_1) +|\psi_0\rangle\langle\psi_0|\hat{p}_i(t_2)\hat{p}_j(t_1)|\{u(\mathbf{k},s)\},0\rangle\langle 0,\{u(\mathbf{k},s)\}|\hat{A}_i(\mathbf{r}_0,t_2)\hat{A}_j(\mathbf{r}_0,t_1)\}. \quad (98)$$

In order to evaluate the probability that the electron ends up in some unbound energy eigenstate $|\psi_\alpha\rangle$ at a time t_0+T following the turn-on of the interaction, we take the expectation value of $\hat{\rho}(t_0+T)$ with respect to $|\psi_\alpha\rangle$, and trace over all field variables. When these operations are performed on the commutators given by Eqs. (97) and (98), we find

$$\text{Tr}_F\langle\psi_\alpha|[\hat{H}_1(t),\hat{\rho}(t_0)]|\psi_\alpha\rangle=0 \quad (99)$$

and

$$\mathrm{Tr}_F\langle\psi_\alpha|[\hat{H}_1(t_1),[\hat{H}_1(t_2),\hat{\rho}(t_0)]]|\psi_\alpha\rangle=-(e/m)^2\{\exp[i(E_\alpha-E_0)(t_1-t_2)/\hbar]\langle\psi_\alpha|\hat{p}_j|\psi_0\rangle\langle\psi_0|\hat{p}_i|\psi_\alpha\rangle \times\mathrm{Tr}_F[\hat{A}_j(\mathbf{r}_0,t_1)|\{u(\mathbf{k},s)\},0\rangle\langle 0,\{u(\mathbf{k},s)\}|\hat{A}_i(\mathbf{r}_0,t_2)]+\exp[i(E_\alpha-E_0)(t_2-t_1)/\hbar] \times\langle\psi_\alpha|\hat{p}_i|\psi_0\rangle\langle\psi_0|\hat{p}_j|\psi_\alpha\rangle\,\mathrm{Tr}_F[\hat{A}_i(\mathbf{r}_0,t_2)|\{u(\mathbf{k},s)\},0\rangle\langle 0,\{u(\mathbf{k},s)\}|\hat{A}_j(\mathbf{r}_0,t_1)]\}. \quad (100)$$

Now the initial electron state is a bound state, with $-E_0/\hbar$ somewhere in the range of optical frequencies, and the final electron state is free, so that $(E_\alpha-E_0)/\hbar$ is always a positive frequency. If first-order photoelectric emission is to take place, it is necessary for the electromagnetic field to have frequency components which are at least as high as $-E_0/\hbar$. If we decompose the $\hat{\mathbf{A}}(\mathbf{r}_0,t)$ operators into their positive- and negative-frequency parts, and substitute in Eq. (100), we find that most of the terms are highly oscillatory in t_1 or t_2 or both, so that their contributions under the integral in Eq. (93) are very small for any T which is great compared with the optical period. If we eliminate these terms and retain only the contributions which, at least for some $E_\alpha\geq 0$, are more slowly varying, we obtain

$$\mathrm{Tr}_F\langle\psi_\alpha|[\hat{H}_1(t_1),[\hat{H}_1(t_2),\hat{\rho}(t_0)]]|\psi_\alpha\rangle =-(e/m)^2\{\exp[i(E_\alpha-E_0)(t_1-t_2)/\hbar]\langle\psi_\alpha|\hat{p}_j|\psi_0\rangle\langle\psi_0|\hat{p}_i|\psi_\alpha\rangle\,\mathrm{Tr}_F[\hat{A}_j^{(+)}(\mathbf{r}_0,t_1)|\{u(\mathbf{k},s)\},0\rangle \times\langle 0,\{u(\mathbf{k},s)\}|\hat{A}_i^{(-)}(\mathbf{r}_0,t_2)]+\exp[i(E_\alpha-E_0)(t_2-t_1)/\hbar]\langle\psi_\alpha|\hat{p}_i|\psi_0\rangle\langle\psi_0|\hat{p}_j|\psi_\alpha\rangle \times\mathrm{Tr}_F[\hat{A}_i^{(+)}(\mathbf{r}_0,t_2)|\{u(\mathbf{k},s)\},0\rangle\langle 0,\{u(\mathbf{k},s)\}|\hat{A}_j^{(-)}(\mathbf{r}_0,t_1)]\}, \quad (101)$$

in which the second term is the complex conjugate of the first.

Now, in view of Eq. (71), the coherent state $|\{u(\mathbf{k},s)\},0\rangle$ is a right eigenstate of $\hat{\mathbf{A}}^{(+)}(\mathbf{r}_0,t)$ with eigenvalue

$$\mathbf{W}(\mathbf{r}_0,t)=\frac{-i}{(2\pi)^3}\int_{k_3>0}d^3k\sum_s\frac{1}{(K\epsilon_0)^{1/2}}u(\mathbf{k},s)\mathfrak{E}_L(\mathbf{k},s,\mathbf{r})\exp[-iK(t-t_0)], \quad (102)$$

and a left eigenstate of $\hat{\mathbf{A}}^{(-)}(\mathbf{r}_0,t)$ with eigenvalue $\mathbf{W}^*(\mathbf{r}_0,t)$. On making use of these properties in Eq. (101), and taking the trace over the field variables, we obtain

$$\mathrm{Tr}_F\langle\psi_\alpha|[\hat{H}_1(t_1),[\hat{H}_1(t_2),\hat{\rho}(t_0)]]|\psi_\alpha\rangle=-(e/m)^2\exp[i(E_\alpha-E_0)(t_1-t_2)/\hbar] \times\langle\psi_\alpha|\hat{p}_j|\psi_0\rangle\langle\psi_0|\hat{p}_i|\psi_\alpha\rangle W_j(\mathbf{r}_0,t_1)W_i^*(\mathbf{r}_0,t_2)+\text{c.c.}, \quad (103)$$

so that, from Eqs. (93) and (102),

$$\begin{aligned}\mathrm{Tr}_F\langle\psi_\alpha|\hat{\rho}(t_0+T)|\psi_\alpha\rangle &=-\left(\frac{e}{m}\right)^2\langle\psi_\alpha|\hat{p}_j|\psi_0\rangle\langle\psi_0|\hat{p}_i|\psi_\alpha\rangle\frac{1}{(2\pi)^6}\int\int d^3kd^3k'\sum_s\sum_{s'}\frac{1}{(K\epsilon_0K'\epsilon_0)^{1/2}}u(\mathbf{k},s)u^*(\mathbf{k}',s')\mathfrak{E}_{Lj}(\mathbf{k},s,r_0)\mathfrak{E}_{Li}^*(\mathbf{k}',s',r_0)\\&\quad\times\frac{1}{(i\hbar^2)}\int_{t_0}^{t_0+T}dt_1\int_{t_0}^{t_1}dt_2\,\exp i[(E_\alpha-E_0)(t_1-t_2)/\hbar-Kt_1+K't_2+(K-K')t_0]+\text{c.c.}\\&=\left|\frac{e}{\hbar m}\langle\psi_\alpha|\hat{p}_j|\psi_0\rangle\frac{1}{(2\pi)^3}\int d^3k\sum_s\frac{1}{(K\epsilon_0)^{1/2}}u(\mathbf{k},s)\mathfrak{E}_{Lj}(\mathbf{k},s,\mathbf{r}_0)\int_0^T dt_1\exp[i(E_\alpha-E_0-K)t_1]\right|^2\\&=\left|\frac{e}{\hbar m}\langle\psi_\alpha|\hat{p}_j|\psi_0\rangle\frac{1}{(2\pi)^3}\int d^3k\sum_s\frac{1}{(K\epsilon_0)^{1/2}}u(\mathbf{k},s)\mathfrak{E}_{Lj}(\mathbf{k},s,r_0)\exp(-\tfrac{1}{2}iKT)\frac{\sin[\frac{1}{2}(E_\alpha-E_0-K)T]}{\frac{1}{2}(E_\alpha-E_0-K)}\right|^2. \quad (104)\end{aligned}$$

The second line follows from the first when the changes of variables $\mathbf{K}\leftrightarrow\mathbf{K}'$, $s\leftrightarrow s'$, and $t_1\leftrightarrow t_2$ are introduced in the complex-conjugate term. If the time interval T is short compared with the reciprocal frequency spread of the incident electromagnetic field, we may replace the factor

$$\{\sin[\tfrac{1}{2}(E_\alpha-E_0-K)T]\}/[\tfrac{1}{2}(E_\alpha-E_0-K)]$$

by

$$\{\sin[\tfrac{1}{2}(E_\alpha-E_0-K_0)T]\}/[\tfrac{1}{2}(E_\alpha-E_0-K_0)]$$

to a good approximation, where K_0 is some midfrequency. From Eq. (102) we then see that Eq. (104) reduces to

$$\mathrm{Tr}_F\langle\psi_\alpha|\hat{\rho}(t_0+T)|\psi_\alpha\rangle =\left(\frac{e}{\hbar m}\right)^2|\langle\psi_\alpha|\hat{p}_j|\psi_0\rangle W_j(r_0,\,t_0+\tfrac{1}{2}T)|^2 \times\left[\frac{\sin\frac{1}{2}(E_\alpha-E_0-K_0)T}{\frac{1}{2}(E_\alpha-E_0-K_0)}\right]^2. \quad (105)$$

Finally, in order to arrive at the probability of photoelectric emission $P(t_0+T)$, we sum over free electron states, or integrate over positive electron energies E_α, with the density of states $\sigma(E_\alpha)$ as a weight function. If $\sigma(E_\alpha)$ and the matrix element $\langle\psi_\alpha|\hat{p}_j|\psi_0\rangle$ vary slowly with E_α in the neighborhood of the peak of the function $\{\sin^2[\frac{1}{2}(E_\alpha-E_0-K_0)T]\}/[\frac{1}{2}(E_\alpha-E_0-K_0)]^2$, then we have the usual result given by the golden rule,

$$P(t_0+T)=\int_0^\infty \mathrm{Tr}_F\langle\psi_\alpha|\hat{\rho}(t_0+T)|\psi_\alpha\rangle\sigma(E_\alpha)dE_\alpha$$
$$=2\pi T(e/\hbar m)^2|\langle\psi_\alpha(E_\alpha=E_0+\hbar K_0)|\hat{p}_j|\psi_0\rangle \times W_j(\mathbf{r}_0,\, t_0+\tfrac{1}{2}T)|^2\sigma(E_0+\hbar K_0). \quad (106)$$

If $\mathbf{W}(\mathbf{r}_0,t)$ is proportional to a real unit vector $\mathbf{l}$, as for a TE wave, we can write

$$P(t_0+T)=2\pi T(e/\hbar m)^2\sigma(E_0+\hbar K_0) \times|\langle\psi_\alpha(E_\alpha=E_0+\hbar K_0)|\hat{\mathbf{p}}\cdot\mathbf{l}|\psi_0\rangle|^2 \times|W(\mathbf{r}_0,\, t_0+\tfrac{1}{2}T)|^2, \quad (107)$$

which shows that the emission probability is proportional to the intensity $|W(\mathbf{r}_0,\, t_0+\frac{1}{2}T)|^2$ of the field, at the midtime $t_0+\frac{1}{2}T$ within the interval.

We emphasize that the results embodied in Eqs. (106) and (107) were obtained for an electromagnetic field which is in a pure coherent state. But since an arbitrary state of the field can be given a diagonal representation in terms of coherent states,[23–25] with a certain weighting functional, we can derive the emission probability for the general case from Eq. (106) by "averaging" with the same weighting functional. It is interesting to note that the same result would also be obtained from a semiclassical treatment, in which the electromagnetic field is represented by a c-number analytic signal $\mathbf{W}(\mathbf{r}_0,t)$.[27] We may also point out that, for a quasimonochromatic field, $\mathbf{W}(\mathbf{r}_0,t)$ is proportional to $\mathbf{V}(\mathbf{r}_0,t)$, where $V_i(\mathbf{r}_0,t)$ is the eigenvalue of the operator $\hat{V}_i(\mathbf{r}_0,t)$, defined in Eq. (88), belonging to the coherent state $|\{u(\mathbf{k},s)\},0\rangle$. $|W(\mathbf{r}_0,\, t_0+\frac{1}{2}T)|^2$ is therefore proportional to $|V(\mathbf{r}_0,\, t_0+\frac{1}{2}T)|^2$, which is the photon number density.

The formulas (106) and (107) hold regardless whether the occupied modes of the field in the right half-space are homogeneous plane waves, evanescent waves, or both. In particular, Eq. (107) therefore holds for a single evanescent wave produced by a TE wave of complex amplitude $u(\mathbf{k},1)$ which is incident from the left at an angle greater than the critical angle, for which, according to Eqs. (24) and (102),

$$|W(\mathbf{r}_0,\, t_0+\tfrac{1}{2}T)|^2\propto|u(\mathbf{k},1)|^2\frac{k_3^2}{K(k_3^2+|K_3|^2)} \times\exp(-2|K_3|z_0). \quad (108)$$

[27] See for example, L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).

The photoemission probability therefore falls off exponentially with distance z_0 from the dielectric interface, in a manner characteristic of evanescent waves. This result is to be compared with the expression

$$|W(\mathbf{r}_0,\, t_0+\tfrac{1}{2}T)|^2\propto|u(\mathbf{k},1)|^2\frac{k_3^2}{K(k_3+K_3)^2}, \quad (109)$$

for a TE wave of the same complex amplitude incident from the left below the critical angle. The two expressions of course coincide at the critical angle. The corresponding photoemission probabilities for TM waves are a little more complicated, but can be obtained from Eqs. (37) and (106).

Although we have treated this as a problem of photoelectric emission, the results are substantially similar for upward transitions of the electron to a band of bound states, as in some situations of radiation-induced fluorescence.

IX. CONCLUSION

We have developed a description of a quantized electromagnetic field, which includes evanescent waves and allows the properties of the field and its interactions with atomic systems to be studied. The essence of the treatment is the introduction of a set of modes, labelled by a continuous wave vector index, each of which consists of three waves, including evanescent waves. With the help of these modes the development closely parallels the usual one for the electromagnetic field in empty space, and we can define Fock states, coherent states, etc., in an analogous manner. Although the field commutators in configuration space are considerably more complicated in the present case, they have the usual form in the conjugate space.

Since evanescent waves extend only over a half-physical space, and only over a limited range of modes, there are no excitations of the field corresponding to completely evanescent photons. A one-photon state like $|\mathbf{k},s\rangle$, with $k_3>+[(n_0^2-1)(k_1^2+k_2)]^{1/2}$, is a state with only an evanescent wave in the right half-space, but a homogeneous plane wave in the left. If we attempt to form a more localized one-photon state in the right half-space we need the nonevanescent components also.

In the treatment of the photoelectric emission of a bound charge under the influence of an evanescent wave, we find, as in other photoemission problems,[28] that the result is identical with that obtained by treating the electromagnetic field as a classical, c-number field. In this respect the evanescent waves are no different from homogeneous waves.

[28] L. Mandel, E. C. G. Sudarshan, and E. Wolf, Proc. Phys. Soc. (London) **84**, 435 (1964).

APPENDIX: PROOF OF ORTHOGONALITY OF TRIPLET MODES

By expanding $\mathfrak{E}_L(\mathbf{k},s,\mathbf{r})$ in terms of the three components defined by Eqs. (22)–(24) and (35)–(37), we see that every term under the integral on the left-hand side of Eq. (50) contains the factor

$$\int\int_{-\infty}^{\infty} dxdy \exp-i[(k_1-k_1')x+(k_2-k_2')y] = (2\pi)^2\delta(k_1-k_1')\delta(k_2-k_2'), \quad \text{(A1)}$$

which implies that $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{K}$, and $\mathbf{K}'$ differ only in their z components. From this fact and the definitions of $\boldsymbol{\varepsilon}$ following Eq. (24), it follows immediately that $\boldsymbol{\varepsilon}(\mathbf{k}) = \boldsymbol{\varepsilon}(\mathbf{k}')$, and therefore $\boldsymbol{\varepsilon}(\mathbf{k})\cdot\boldsymbol{\varepsilon}(\mathbf{k}')=1$, and that

$$\begin{aligned}\boldsymbol{\varepsilon}(\mathbf{k})\cdot[\boldsymbol{\kappa}'\times\boldsymbol{\varepsilon}(\mathbf{k}')]&=0,\\ \boldsymbol{\varepsilon}(\mathbf{k})\cdot[\boldsymbol{\kappa}^{(R)\prime}\times\boldsymbol{\varepsilon}(\mathbf{k}')]&=0,\\ \boldsymbol{\varepsilon}(\mathbf{k})\cdot[\mathbf{c}'\times\boldsymbol{\varepsilon}(\mathbf{k}')]&=0.\end{aligned} \quad \text{(A2)}$$

The orthogonality of the mode functions for the two polarizations TE and TM follows immediately from this. Thus each scalar product on the left-hand side of Eq. (50) gives rise to the factor $\delta_{ss'}$.

As an example, let us now evaluate the integrals in Eq. (50) for the TE ($s=1$) case. Using Eqs. (22)–(24) to expand the $\mathfrak{E}_L(\mathbf{k},1,\mathbf{r})$ function, we have

$$\begin{aligned}\int d^3x\, &\mathfrak{E}_L^*(\mathbf{k},1,\mathbf{r})\cdot\mathfrak{E}_L(\mathbf{k}',1,\mathbf{r})n^2(\mathbf{r})\\ &=(2\pi)^2\frac{\delta(k_1-k_1')\delta(k_2-k_2')}{2n_0^2(k_3+K_3^*)(k_3'+K_3')}\\ &\times[(k_3+K_3^*)(k_3'+K_3')n_0^2I^*I+(k_3-K_3^*)\\ &\times(k_3'-K_3')n_0^2R^*R+(k_3+K_3^*)(k_3'-K_3')n_0^2I^*R\\ &+(k_3-K_3^*)(k_3'+K_3')n_0^2R^*I+4k_3k_3'T^*T],\end{aligned} \quad \text{(A3)}$$

where

$$I^*I=(R^*R)^*=\int_{-\infty}^{0}\exp[-i(k_3-k_3')z]dz = \pi\delta(k_3-k_3')+iP\left(\frac{1}{k_3-k_3'}\right), \quad \text{(A4)}$$

$$\begin{aligned}I^*R=RI^*&=\int_{-\infty}^{0}\exp[-i(k_3+k_3')z]dz\\ &=\pi\delta(k_3+k_3')+iP\left(\frac{1}{k_3+k_3'}\right)\\ &=iP\left(\frac{1}{k_3+k_3'}\right).\end{aligned} \quad \text{(A5)}$$

P denotes the principal part. $\delta(k_3+k_3')$ drops out in Eq. (A5) because both wave components are incident from the left, and k_3 and k_3' are both positive. The term T^*T is given by

$$\begin{aligned}T^*T&=\int_0^{\infty}\exp[-i(K_3^*-K_3')z]dz\\ &=\left(\frac{K_3+K_3^*}{2K_3}\right)\left(\frac{K_3'+K_3'^*}{2K_3'}\right)\pi\delta(K_3-K_3')\\ &\quad -iP\left(\frac{1}{K_3^*-K_3'}\right).\end{aligned} \quad \text{(A6)}$$

Notice that in Eqs. (A3)–(A6), k_3 and k_3' are taken to be real, but K_3 and K_3' are given the option of being real or imaginary, according as the transmitted waves are homogeneous or evanescent.

With the condition that $k_1=k_1'$ and $k_2=k_2'$, we have from Eqs. (20) and (21)

$$K_3^2-K_3'^2=(k_3^2-k_3'^2)/n_0^2, \quad \text{(A7)}$$

and on replacing K_3 by K_3^* in (A7) we obtain the relation

$$1/(K_3^*-K_3')=n_0^2(K_3^*+K_3')/(k_3^2-k_3'^2). \quad \text{(A8)}$$

If in Eq. (A6) K_3 and K_3' are both real, then, since they have the same sign, we have

$$\begin{aligned}\delta(K_3-K_3')&=\delta(K_3-K_3')+\delta(K_3+K_3')\\ &=2|K_3|\,\delta(K_3^2-K_3'^2),\end{aligned}$$

and from Eq. (A7)

$$\begin{aligned}&=2|K_3|\,\delta[(k_3^2-k_3'^2)/n_0^2]\\ &=n_0^2(K_3/k_3)\delta(k_3-k_3').\end{aligned} \quad \text{(A9)}$$

With the help of Eqs. (A8) and (A9), Eq. (A6) now becomes

$$\begin{aligned}T^*T=&[(K_3+K_3^*)/2K_3]n_0^2\pi\delta(k_3-k_3')\\ &-in_0^2P(K_3^*+K_3')/(k_3^2-k_3'^2),\end{aligned} \quad \text{(A10)}$$

where the factor $(K_3'+K_3'^*)/2K_3'$ has been dropped since it does not affect the value of the result.

Substituting from Eqs. (A4), (A5), and (A10) into Eq. (A3), we see that there are two kinds of terms: those containing the $\delta(k_3-k_3')$ function and those containing the principal parts; we readily find

$$\begin{aligned}\int d^3x\, &\mathfrak{E}_L^*(\mathbf{k},1,\mathbf{r})\cdot\mathfrak{E}_L(\mathbf{k}',1,\mathbf{r})\\ &=\tfrac{1}{2}(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')+\text{principal-part terms}.\end{aligned} \quad \text{(A11)}$$

To show that the sum of the principal-part terms vanishes, we group them as follows:

$$\text{principal-part terms} = \frac{(2\pi)^2\delta(k_1-k_1')\delta(k_2-k_2')iP}{2(k_3+K_3^*)(k_3'+K_3')(k_3^2-k_3'^2)}$$
$$\times\{[(k_3+K_3^*)(k_3'+K_3')-(k_3-K_3')(k_3'-K_3')]$$
$$\times(k_3+k_3')+[(k_3+K_3^*)(k_3'-K_3')$$
$$-(k_3-K_3^*)(k_3'+K_3')](k_3-k_3')$$
$$-4k_3k_3'(K_3^*+K_3')\}\,, \quad (A12)$$

when the term in the curly parentheses is readily seen to vanish. Thus Eq. (50) has been verified for the TE case. The TM case is treated in essentially the same way.

The derivation of Eq. (51) proceeds exactly as above, except that the roles of $\mathbf{k}$ and $\mathbf{K}$ are interchanged and there are no evanescent waves involved. The algebra for the corresponding expressions for the magnetic fields is almost identical to that for the electric fields, except that the TE magnetic field behaves as the TM electric field and vice versa. Addition of the electric and magnetic field contributions results in Eqs. (53)–(55).

Turning our attention to Eqs. (56)–(58), we note that the integrals over x and y yield δ functions implying $k_1=-k_1'$, $k_2=-k_2'$, and thus

$$\boldsymbol{\varepsilon}(\mathbf{k})\cdot\boldsymbol{\varepsilon}(\mathbf{k}')=-1\,, \quad (A13)$$

$$\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}'=-k_1^2-k_2+k_3k_3'\,, \quad (A14)$$

$$\boldsymbol{\kappa}^{(R)}\cdot\boldsymbol{\kappa}'=-k_1^2-k_2^2-k_3k_3'\,, \quad (A15)$$

$$\mathbf{c}^*\cdot\mathbf{c}'=-k_1^2-k_2^2+K_3^*K_3'\,. \quad (A16)$$

Proceeding as above for both waves incident from the left and $s=1$, we obtain the relations

$$\int \mathfrak{E}_L(\mathbf{k},1,\mathbf{r})\cdot\mathfrak{E}_L(\mathbf{k}',1,\mathbf{r})n^2(\mathbf{r})d^3x$$
$$=-\tfrac{1}{2}(2\pi)^3\delta(k_1+k_1')\delta(k_2+k_2')$$
$$\times\delta(k_3-k_3')(k_3-K_3)/(k_3+K_3) \quad (A17)$$

and

$$\int \mathfrak{B}_L(\mathbf{k},1,\mathbf{r})\cdot\mathfrak{B}_L(\mathbf{k}',1,\mathbf{r})d^3x$$
$$=\tfrac{1}{2}(2\pi)^3\delta(k_1+k_1')\delta(k_2+k_2')$$
$$\times\delta(k_3-k_3')(k_3-K_3)/(k_3+K_3)\,, \quad (A18)$$

which add to give zero. The results for $s=2$ are similar and the terms also add to give zero, as they do in the remaining cases covered by Eqs. (56)–(58).