used in the present work is much the same as in this review.

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[17]The argument given here is essentially that of Lee and Zumino (Ref. 13).

[18]It is an open question which of the currents j_μ^A or J_μ^A generate the electromagnetic and weak interactions. Only the j_μ^A lead to the appropriate single-particle poles in form factors when the Lagrangian is used in the tree approximation, but closed-loop diagrams must be considered in any event to treat the form factors for timelike momenta.

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[20]The data are analyzed in M. T. Vaughn, Lett. Nuovo Cimento 2, 851 (1969); see also Ref. 6.

[21]The colliding-beam reaction $e^+ + e^- \to A_1^\pm + \pi^\pm$ is a promising test [see Ref. 6 and also M. T. Vaughn and P. J. Polito, Lett. Nuovo Cimento 1, 74 (1971)], although it involves a rather long extrapolation off the mass shell of the ρ.

[22]*Note added in proof.* After the completion of this work, our attention was drawn to the work of V. Ogievetsky and B. M. Zupnik [Nucl. Phys. B24, 612 (1970)], who consider a general chiral Lagrangian for the nonlinear σ model and propose a restriction on derivative couplings which corresponds to the smoothness assumptions of the standard hard-pion method (Refs. 1 and 2). This restriction does not eliminate the new parameters which appear when the σ field is an independent field.

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K_{l3}-Decay Form Factors and Asymptotic $SU(3)$ Symmetry*

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A relation between the K_{l3}-decay form factors and the pion electromagnetic form factor is derived by using only the hypothesis of asymptotic $SU(3)$ symmetry. This relation enables us to predict the complete behavior of the K_{l3}-decay form factors, once the behavior of the pion electromagnetic form factor is given. For the choice of the dipole formula, $(1-t^2/m_\rho{}^2)^{-2}$, for the pion form factor, we predict the following for the parameters of the K_{l3}-decay form factors: $\xi(0) \simeq -0.53$, $\lambda_+ \simeq 0.023$, and $\lambda_- \simeq 0.0010$. If we extrapolate the result from the physical K_{l3}-decay region $0 \le t \le (m_K - m_\pi)^2$ to include the region $0 \le t \le (m_K + m_\pi)^2$, the presence of a dip in the K_{l3}-decay scalar form factor in this region is indicated. The extrapolated form factor satisfies approximately the soft-pion constraint given at $t = m_K{}^2$. The effect of particle mixing between the usual pseudoscalar nonet and the possible higher-lying pseudoscalar mesons is not considered.

I. INTRODUCTION

In the past years the K_{l3}-decay form factors have been intensively discussed.[1] We write the form factors with the obvious notation $[V_\mu^{K^-}(x) = V_\mu^4(x) - iV_\mu^5(x)]$

$$\langle \pi^0(\vec{p}) | V_\mu^{K^-}(0) | K^+(\vec{k}) \rangle = \frac{1}{(4p_0k_0)^{1/2}} \frac{1}{\sqrt{2}} [f_+(t)(k+p)_\mu + f_-(t)(k-p)_\mu]. \tag{1.1}$$

The ξ parameter is usually defined by

$$\xi(t) = f_-(t)/f_+(t),$$

where $t = -(k-p)^2$. The *scalar* form factor, $f(t)$, is defined by

$$f(t) = f_+(t) + \left(\frac{t}{m_K{}^2 - m_\pi{}^2}\right) f_-(t). \tag{1.2}$$

The usual parametrization of the $f_\pm(t)$ in the physical region is given by

$$f_\pm(t) = f_\pm(0)[1 + \lambda_\pm t/m_\pi{}^2].$$

In terms of this scalar form factor the well-known soft-pion prediction[2] is written as $f(m_K{}^2) \simeq f_K/f_\pi$. Here f_K and f_π denote the form factors of the $K \to \mu + \bar\nu$ and $\pi \to \mu + \bar\nu$ decays. This prediction may be subject to some error due to the soft-pion limit involved. The slope of $f(t)$ at $t = 0$ is given by

$$f'(0)=f_+'(0)+f_-(0)(m_K{}^2-m_\pi{}^2)^{-1}$$
$$=f_+(0)\left[\frac{\lambda_+}{m_\pi{}^2}+\frac{\xi(0)}{m_K{}^2-m_\pi{}^2}\right]. \quad (1.3)$$

The soft-pion constraint $f(m_K{}^2)\simeq f_K/f_\pi\simeq 1.28$ and the constraint $f(0)=f_+(0)\simeq 1$, imposed[3] by our asymptotic $SU(3)$ symmetry, predict a small positive value for $f'(0)$ if $f(t)$ is assumed to be a smooth *monotonic* function throughout the region $0\leqslant t\leqslant m_K{}^2$. If, in Eq. (1.3), we fix the value of λ_+ around $\lambda_+\simeq m_\pi{}^2/m_{K^*}{}^2\simeq 0.02$, which is given by the assumption of the K^* dominance of the $f_+(t)$, then $f'(0)=0$ gives a value $\xi(0)\simeq -0.28$. For the more realistic small positive value of $f'(0)$, $\xi(0)$ will then be closer to zero. Thus, the constraints mentioned above tend to predict a small value of $\xi(0)$ as long as the $f(t)$ is a *monotonic* function in the range considered. A typical example of such behavior of $f(t)$ is the curve 1 drawn in Fig. 1 (called case 1). Actually if the $f(t)$ is approximated by essentially a straight line passing through the points $f(0)=1$ and $f(m_K{}^2)\simeq(f_K/f_\pi)\simeq 1.28$ in the region $0\leqslant t\leqslant m_K{}^2$ (which may certainly be an oversimplification), then

$$f'(0)=\frac{f(m_K{}^2)-f(0)}{m_K{}^2}\simeq\frac{(f_K/f_\pi)-1}{m_K{}^2}. \quad (1.4)$$

Then from Eqs. (1.3) and (1.4), a relation,

$$\xi(0)\simeq(f_K/f_\pi-1)-(m_K{}^2-m_\pi{}^2)m_\pi{}^{-2}\lambda_+$$
$$\simeq\tfrac{1}{2}(f_K/f_\pi-f_\pi/f_K)-(m_K{}^2-m_\pi{}^2)m_\pi{}^{-2}\lambda_+, \quad (1.5)$$

follows. Actually Eq. (1.5) can also be derived from the theorem of Dashen and Weinstein which uses a perturbation argument.[4] Equation (1.5) explicitly demonstrates that for relatively small values of λ_+ (i.e., $\lambda_+\simeq 0.02$), the value of $\xi(0)$ is very close to zero. If Eq. (1.5) is taken seriously, only for unusually large values of λ_+ (i.e., $\lambda_+\gtrsim 0.08$) are large negative values of $\xi(0)$ $[\xi(0)\lesssim -0.5]$ possible.[5] However, there is mounting evidence[1] for a large negative value of $\xi(0)$.

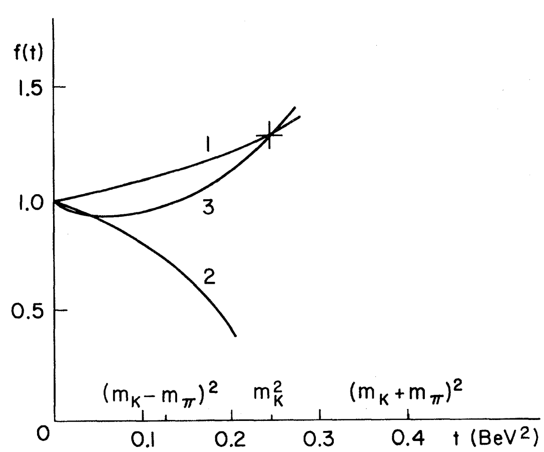

FIG. 1. Three typical possible behaviors of the scalar K_{l3}-decay form factor $f(t)$. The cross denotes the soft-pion constraint at $t=m_K{}^2$.

Recently some authors have given theoretical arguments in favor of a large negative ξ value: Kang[6] proposes a zero in the $f(t)$ below the $K\pi$ threshold. Brandt and Preparata[7] prefer $f(m_K{}^2)\simeq 0$ from their version of weak partially conserved axial-vector current. These cases (called case 2) are typified in Fig. 1 by the curve 2. They *drastically* violate the usual soft-pion constraint.

If both the soft-pion relation[2] and the relatively large negative value of $\xi(0)$[1] are to be accommodated, then the acceptable behavior of the $f(t)$ would be the one similar to the curve 3 given in Fig. 1 (called case 3). That is, the scalar form factor $f(t)$ will have a dip in the range $0<t<(m_K+m_\pi)^2$. The dip in the scalar form factor occurs in the theory of Hara,[8] which uses the Veneziano model and other assumptions. The experimental large negative value of $\xi(0)$ may thus force us to choose one of the two alternatives distinguished qualitatively by the curves 2 and 3 in Fig. 1.

In this paper we point out that the hypothesis of asymptotic $SU(3)$ symmetry[9] enables us to predict the complete behavior of the K_{l3} form factors in the physical region once the information on the pion electromagnetic form factor $f_{\rm em}^\pi(t)$ is given. A relatively large negative value of $\xi(0)$ is then obtained if the dipole formula is assumed for the $f_{\rm em}^\pi(t)$. The result for $f(t)$ is indicative[10] of the behavior of $f(t)$ similar to case 3 in Fig. 1. We deliberately avoid the use of the assumption of single-pole dominance for the form factors. It certainly failed in the case of nucleon form factors.

The hypothesis of asymptotic $SU(3)$ symmetry implies that for the system of particles with infinite momenta, the generator of $SU(3)$ group, V_i, acts as if it were the generator of exact $SU(3)$ symmetry. To be more explicit, consider, for example, the annihilation operator of the *physical* pseudoscalar meson, $a_\alpha(\vec{k})$, where α stands for $\pi^{\pm,0}$, $K^{+,0}$, $\overline{K}^{+,0}$, η^0, and η'^0, and $\vec{k}$ denotes their momenta. We write[9] (omitting the time-dependent factors)

$$[V_i,a_\alpha(\vec{k})]=i\sum_\beta u_{i\alpha\beta}(\vec{k})a_\beta(\vec{k})+\delta u_{i\alpha}(\vec{k}). \quad (1.6)$$

The first term on the right-hand side of (1.6) picks up all the terms linear in the a's (but not in the $a^\dagger$'s) and the remainder is denoted by the second term $\delta u_{i\alpha}$. The possible effect of $SU(3)$ particle mixing (such as the η-η' mixing) is contained in the first term. The $\delta u_{i\alpha}$ term is of the order of $SU(3)$ breaking.

The asymptotic $SU(3)$ symmetry is then expressed by

$$\delta u_{i\alpha}(\vec{k}) \to 0 \quad \text{when } |\vec{k}| \to \infty. \qquad (1.7)$$

Although the precise form of the asymptotic behavior of $\delta u_{i\alpha}(\vec{k})$ is not very well known (except for some soluble models), it has been shown[9] that the above hypothesis of asymptotic symmetry can be made consistently in the presence of the Gell-Mann–Okubo (GMO) hadron mass splittings. Conversely, exact validity (not as a lowest-order approximation) of the GMO mass formulas (including the important effect of particle mixing) is required if the asymptotic symmetry formulated above is correct.

From this hypothesis of asymptotic $SU(3)$ symmetry *alone*, we can derive a relation between the K_{l3}-decay form factors and the pion electromagnetic form factor. The derivation is given in Sec. II. It enables us to relate the K_{l3}-decay form factors in the region $t<(m_K-m_\pi)^2$ to the pion electromagnetic form factor in the spacelike momentum transfer region. Consequences of the relation are discussed in Sec. III. The analytic continuation of the relation to the whole complex t plane is examined in Sec. IV. When the analytic continuation of the K_{l3}-decay form factors is allowed up to the $K\pi$ threshold, the appearance of a dip in $f(t)$ is predicted and the behavior of the $f(t)$ in this range turns out to be similar to the case 3 shown in Fig. 1. Some relevant remarks are also added in Sec. V.

II. RELATION BETWEEN THE K_{l3}-DECAY FORM FACTORS AND THE PION ELECTROMAGNETIC FORM FACTOR

By using the assumption of asymptotic $SU(3)$ symmetry, we now derive a relation between the K_{l3}-decay form factors and the pion electromagnetic form factor. We follow the prescription[9] given by Eqs. (1.6) and (1.7). We of course neglect the effect of $SU(2)$ breaking. First we sandwich the commutator

$$[V_{\bar{K}^0}, V_\mu^{\pi^-}(0)] = V_\mu^{K^-}(0) \qquad (2.1)$$

between the states $\langle \pi^0(\vec{p})|$ and $|K^+(\vec{k})\rangle$.[11] Then we let $\vec{p}\to\infty$ and $\vec{k}\to\infty$, in which limit, according to the asymptotic symmetry, the generator $V_{\bar{K}^0}$ connects $\langle\pi^0(\vec{p})|$ to $|K^0(\vec{p})\rangle$ alone and $|K^+(\vec{k})\rangle$ to $\langle\pi^+(\vec{k})|$ alone. [If higher-lying pseudoscalar mesons exist, the effect of particle mixing between these mesons and the usual pseudoscalar mesons must be considered according to Eq. (1.6).] We then obtain

$$\langle \pi^0(\vec{p})|V_\mu^{K^-}(0)|K^+(\vec{k})\rangle = -(\tfrac{1}{2})^{1/2}\langle K^0(\vec{p})|V_\mu^{\pi^-}(0)|K^+(\vec{k})\rangle - \langle \pi^0(\vec{p})|V_\mu^{\pi^-}(0)|\pi^+(\vec{k})\rangle \quad (\vec{p}\to\infty,\ \vec{k}\to\infty). \qquad (2.2)$$

By considering the matrix element

$$\langle K^0(\vec{p})|[V_\mu^{\pi^-}(0), V_{K^0}]|\pi^+(\vec{k})\rangle = 0 \quad (\vec{p}\to\infty,\ \vec{k}\to\infty) \qquad (2.3)$$

and again using the asymptotic symmetry, we obtain

$$\langle K^0(\vec{p})|V_\mu^{\pi^-}(0)|K^+(\vec{k})\rangle = -(\tfrac{1}{2})^{1/2}\langle \pi^0(\vec{p})|V_\mu^{\pi^-}(0)|\pi^+(\vec{k})\rangle = \langle \pi^+(\vec{p})|V_\mu^{\pi^0}|\pi^+(\vec{k})\rangle \quad (\vec{p}\to\infty,\ \vec{k}\to\infty). \qquad (2.4)$$

The last equality is due to $SU(2)$ symmetry. From Eqs. (2.2) and (2.4), therefore, we finally obtain

$$\langle \pi^0(\vec{p})|V_\mu^{K^-}(0)|K^+(\vec{k})\rangle = (\tfrac{1}{2})^{1/2}\langle \pi^+(\vec{p})|V_\mu^{\pi^0}(0)|\pi^+(\vec{k})\rangle \quad (\vec{p}\to\infty,\ \vec{k}\to\infty). \qquad (2.5)$$

This relation was first derived by Matsuda and Oneda.[12] Equation (2.5) leads to

$$(4p_0k_0)^{-1/2}[f_+(t)(k+p)_\mu + f_-(t)(k-p)_\mu] = (4p_0q_0)^{-1/2} f_{\text{em}}^\pi(t')(q+p)_\mu, \qquad (2.6)$$

where

$$t=-(p-k)^2, \quad t'=-(p-q)^2,$$

$$p_0=(\vec{p}^{\,2}+m_\pi^{\,2})^{1/2}, \quad q_0=(\vec{q}^{\,2}+m_\pi^{\,2})^{1/2},$$

and

$$k_0=(\vec{k}^2+m_K^{\,2})^{1/2},$$

with $\vec{q}=\vec{k}\to\infty$ and $\vec{p}\to\infty$.

When $\vec{p}=z\vec{k}$ $(z>0)$ and $\vec{k}\to\infty$, t and t' are found to be finite and are given by

$$t=(z-1)(m_\pi^{\,2}-zm_K^{\,2})/z \qquad (2.7)$$

and

$$t'=-(z-1)^2m_\pi^{\,2}/z. \qquad (2.8)$$

These relations are shown in Fig. 2. In this case Eq. (2.6) reduces to

$$(1+z)f_+(t)+(1-z)f_-(t)=(1+z)f_{\text{em}}^\pi(t'). \qquad (2.9)$$

For the case $\vec{p}=-z\vec{k}$ $(z>0)$ and the case where $\vec{p}$ and $\vec{k}$ are not parallel, t and t' are infinite and Eq. (2.6) reduces to

$$f_{\text{em}}^\pi(-\infty)=f_+(-\infty)+f_-(-\infty) \qquad (2.10\text{a})$$

and

$$f_{\text{em}}^\pi(-\infty)=f_+(-\infty)-f_-(-\infty). \qquad (2.10\text{b})$$

These equations can be satisfied only if

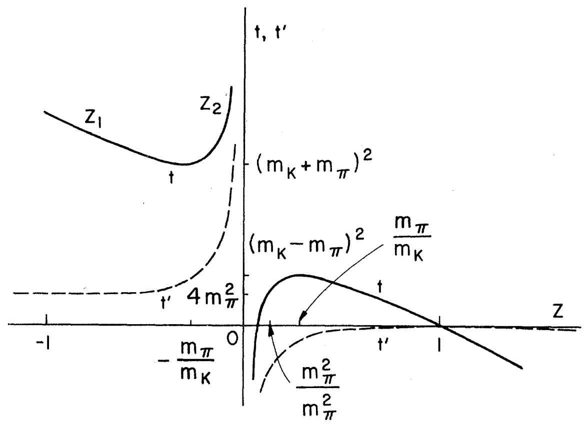

FIG. 2. Relations between the squared four-momentum transfers t and t' and the ratio z. The solid curve represents the relation t versus z and the broken curve that of t' versus z.

$$f_+(-\infty) = f_{\mathrm{em}}^{\pi}(-\infty) \tag{2.11a}$$

and

$$f_-(-\infty) = 0. \tag{2.11b}$$

[Here we note that Eqs.(2.11a) and (2.11b) are also inferred from Eq. (2.9) with (2.7) and (2.8). The limits $t \to -\infty$ and $t' \to -\infty$ can be realized in (2.7) and (2.8) by letting either $z \to +0$ or $z \to +\infty$. Equation (2.9) with $z \to +0$ and $z \to +\infty$ gives Eqs. (2.11a) and (2.11b).[13]] Equation (2.9) is the relation we wish to investigate in detail in the present paper. It is valid only for $z > 0$. The variables t and t' are the known functions of z given by Eqs. (2.7) and (2.8), respectively. For positive z the regions of t and t' are the following (see Fig. 2):

$$t \leq (m_K - m_\pi)^2, \tag{2.12a}$$

$$t' \leq 0. \tag{2.12b}$$

In Sec. III we examine consequences of the relation (2.9) for $z > 0$, namely, for t and t' in the range (2.11). In Sec. IV we study the possibility of continuing Eq. (2.9) over the whole complex z plane.

III. PREDICTIONS FOR THE K_{l3}-DECAY FORM FACTORS

Now we investigate the implication of Eq. (2.9). When $z = 1$, we get, with our normalization $f_{\mathrm{em}}^{\pi}(0) = 1$, that

$$f(0) = f_+(0) = 1, \tag{3.1}$$

which is consistent with the assumption of asymptotic symmetry. For $z = m_\pi/m_K$ we obtain from Eq. (2.9)

$$f((m_K - m_\pi)^2) = f_{\mathrm{em}}^{\pi}(-m_\pi(m_K - m_\pi)^2/m_K), \tag{3.2}$$

which turns out to be 0.89 if we assume the following dipole formula for the $f_{\mathrm{em}}^{\pi}(t')$:

$$f_{\mathrm{em}}^{\pi}(t') = (1 - t'/m_\rho^{\,2})^{-2}, \tag{3.3}$$

and 0.94 for the simple-pole formula for the $f_{\mathrm{em}}^{\pi}(t')$:

$$f_{\mathrm{em}}^{\pi}(t') = (1 - t'/m_\rho^{\,2})^{-1}. \tag{3.4}$$

For the value $z = m_\pi^{\,2}/m_K^{\,2}$, we get

$$f_+(0) + f_-(0) \simeq f_{\mathrm{em}}^{\pi}(-m_K^{\,2}). \tag{3.5}$$

From Eqs. (3.5) and (3.1), $\xi(0)$ turns out to be approximately -0.5 and -0.3 for the choice of $f_{\mathrm{em}}^{\pi}(t')$ given by Eqs. (3.3) and (3.4), respectively. We thus see that the dipole formula (3.3) may be preferred to the simple-pole formula (3.4) since it predicts a larger negative value of $\xi(0)$. We already know that the nucleon form factors are better fitted by the dipole formula. In this connection the determination of the behavior of the pion electromagnetic form factor $f_{\mathrm{em}}^{\pi}(t)$ is very interesting.

We have drawn the relations given by Eqs. (2.7) and (2.8) in Fig. 2. From Fig. 2 we can see that for any given value of t in the range $t < (m_K - m_\pi)^2$, there correspond two z's (z_1 and z_2, $0 < z_1 < z_2$). Therefore, Eq. (2.9) actually gives two equations for each value of t in the range $t < (m_K - m_\pi)^2$; therefore both the $f_+(t)$ and $f_-(t)$ can be expressed in terms of the corresponding $f_{\mathrm{em}}^{\pi}(t')$. The values of t' corresponding to the z_1 and z_2 are given by

$$t_i' = -(z_i - 1)^2 m_\pi^{\,2}/z_i \quad (i = 1, 2), \tag{3.6}$$

where

$$z_1 = (1/2m_K^{\,2})\{m_K^{\,2} + m_\pi^{\,2} - t - [((m_K + m_\pi)^2 - t)((m_K - m_\pi)^2 - t)]^{1/2}\} \tag{3.7}$$

and

$$z_2 = (1/2m_K^{\,2})\{m_K^{\,2} + m_\pi^{\,2} - t + [((m_K + m_\pi)^2 - t)((m_K - m_\pi)^2 - t)]^{1/2}\}. \tag{3.8}$$

Then

$$f_+(t) = \left[\frac{1+z_1}{1-z_1} f_{\mathrm{em}}^{\pi}(t_1') - \frac{1+z_2}{1-z_2} f_{\mathrm{em}}^{\pi}(t_2')\right] \times \frac{t}{2m_K^{\,2}(z_1 - z_2)} \tag{3.9}$$

and

$$f_-(t) = [f_{\mathrm{em}}^{\pi}(t_1') - f_{\mathrm{em}}^{\pi}(t_2')]\,\frac{2(m_K^{\,2} + m_\pi^{\,2}) - t}{2m_K^{\,2}(z_2 - z_1)}. \tag{3.10}$$

Once $f_{\mathrm{em}}^{\pi}(t')$ is given for $t' < 0$, we can determine $f_+(t)$ and $f_-(t)$ for $t < (m_K - m_\pi)^2$ by Eqs. (3.9) and (3.10).

From Eqs. (3.9) and (3.10) we predict also that

$f_+(t) \sim t^{-\alpha}$ and $f_-(t) \sim t^{-\alpha}$ as $t \to -\infty$, if $f^{\pi}_{\rm em}(t') \sim (t')^{-\alpha}$ as $t' \to -\infty$.

The parameters $\xi(0)$, λ_+, and λ_- can also be determined if $f^{\pi}_{\rm em}(t')$ is given. Since the dipole formula for $f^{\pi}_{\rm em}(t')$ seems to be preferred, we choose the following parametric form for $f^{\pi}_{\rm em}(t')$:

$$f^{\pi}_{\rm em}(t') = (1 - t/M^2)^{-2}. \qquad (3.11)$$

By expanding (3.9) and (3.10) around $t = 0$, we then get

$$\xi(0) = -(m_K{}^2 + m_\pi{}^2)(1 - A^{-2})/(m_K{}^2 - m_\pi{}^2), \qquad (3.12)$$

$$\lambda_+ = \tfrac{1}{2}\, m_\pi{}^2(m_K{}^2 + m_\pi{}^2)(1 - A^{-2})/(m_K{}^2 - m_\pi{}^2)^2, \qquad (3.13)$$

and

$$\lambda_- = m_\pi{}^2 \left\{ \frac{m_K{}^2 + m_\pi{}^2}{(m_K{}^2 - m_\pi{}^2)^2}[1 - 2A^{-1}(1 + A)^{-1}] - \frac{1}{2(m_K{}^2 + m_\pi{}^2)} \right\}, \qquad (3.14)$$

where

$$A \equiv 1 + (m_K{}^2 - m_\pi{}^2)^2/m_K{}^2M^2. \qquad (3.15)$$

For various choices of the parameter M^2 we have calculated $\xi(0)$, λ_+, and λ_- from Eqs. (3.12), (3.13), and (3.14), respectively. The result is given in Table I. ($M^2 = 0.710$ BeV^2 corresponds to the best fit for the nucleon electromagnetic form factor.) Numerical calculation of Eqs. (3.9) and (3.10) in the case of the dipole formula Eq. (3.11) with $M = m_\rho$ has been carried out. It shows that $f(t)$ has a small curvature $[f''(t) > 0]$ for $0 < t < (m_K - m_\pi)^2$. The $f_-(t)$ is almost constant in the physical decay region. The result is depicted in Fig. 3. It is worthwhile to note that the value of λ_+ turns out to be around 0.02 for the choice $M^2 \simeq m_\rho{}^2$ without using any assumption about the K^* contribution to the $f_+(t)$. Our result suggests that $\xi(0)$ could take a large negative value, even if the value of λ_+ is relatively *small*. Compare with the remark given in Ref. 5.

In order to obtain more insight from our sum rule, Eq. (2.9), we try to continue it analytically to the whole complex z plane. This will be discussed in Sec. IV.

TABLE I. The parameters $\xi(0)$, λ_+, and λ_- of the K_{l3}-decay form factors calculated by the formulas (3.12)–(3.15) for several values of the mass parameter M. The physical ρ-meson mass squared is 0.585 BeV^2 and the best-fit value of M^2 in the case of the nucleon electromagnetic form factor is 0.710 BeV^2.

M^2 (BeV^2)	0.460	0.585	0.710	1.170
$\xi(0)$	−0.62	−0.53	−0.47	−0.33
λ_+	0.027	0.023	0.021	0.014
λ_-	0.0077	0.0010	−0.0038	−0.015

IV. ANALYTIC CONTINUATION OF Eq. (2.9)

Before making the analytic continuation we wish to emphasize the fact that Eq. (2.9) is not an *exact* equation. If higher-lying pseudoscalar mesons exist, they certainly give a contribution through mixing even in our asymptotic limit. The many-particle contribution to Eq. (2.9), which we denote as $f(z, |\vec{k}|)$, is a function of z and $|\vec{k}|$ and vanishes according to our asymptotic symmetry, in the limit $|\vec{k}| \to \infty$, but only for $z > 0$. Therefore, it is not very surprising even if a sizable discrepancy is encountered for the region of *large negative z* which is far from the original region $z > 0$.

For small positive (negative) z, t and t' become large negative (positive), and Eq. (2.9) becomes $f_+(\mp\infty) + f_-(\mp\infty) = f^{\pi}_{\rm em}(\mp\infty)$. This may represent a relation $0 = 0$. The case where the form factors $f_\pm(t)$ and $f^{\pi}_{\rm em}(t')$ satisfy unsubtracted dispersion relations seems to be preferable, both from the theoretical and experimental viewpoints. Therefore, the result of asymptotic symmetry, Eq. (2.9), seems to be reasonable also around $z = 0$.

Let us first consider the left-hand side of Eq. (2.9). We write it as $F_L(z)$:

$$F_L(z) = (1 + z)f_+(t) + (1 - z)f_-(t). \qquad (4.1)$$

In order to see the analytic structure of $F_L(z)$, we examine the mapping from the t plane to the z plane by means of Eq. (2.7). In Fig. 4(a) (t plane), cuts are drawn from $(m_K + m_\pi)^2$ to $+\infty$ and from $-\infty$ to $(m_K - m_\pi)^2$. The upper (lower) edge of the right-hand cut in the t plane maps onto the segment $-m_\pi/m_K < z < 0$ $(-\infty < z < -m_\pi/m_K)$ of the real nega-

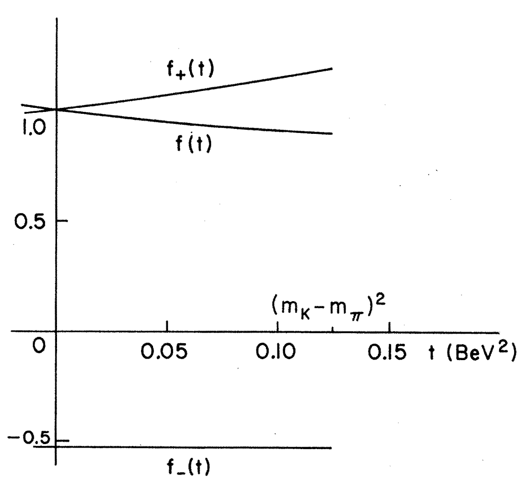

FIG. 3. Predicted form factors of the K_{l3} decay when the dipole formula is used for the pion electromagnetic form factor.

tive z axis; see Fig. 4(b). The region $(m_K - m_\pi)^2 < t < (m_K + m_\pi)^2$ maps onto the circle (with the origin at $z=0$ and the radius m_π/m_K) in the z plane. The whole physical sheet of the t plane maps onto the upper half z plane; the upper (lower) half t plane maps inside (outside) of the half-circle of the upper half z plane. The value of the function $F_L(z)$ in the lower half z plane is determined by means of the reality relation because $F_L(z)$ is real for real positive z:

$$F_L(z^*) = [F_L(z)]^*. \tag{4.2}$$

In the z plane the elastic cut extends from 0 to $-\infty$. The analytic structure of the right-hand side of Eq. (2.9) [denoted by $F_R(z)$] is the same as above if one replaces m_K by m_π.

Now, we examine the consequences of the equality $F_L(z) = F_R(z)$ for negative z. Let us first set $z \simeq (-m_\pi^2/m_\rho^2) + i\epsilon$ ($\epsilon > 0$). Then the corresponding t' and t are given by $t' \simeq m_\rho^2 + i\epsilon$ and $t \simeq m_\rho^2 + m_K^2 + i\epsilon$. The function $F_R(z)$ has a singularity at this point, t', due to the presence of the ρ meson.

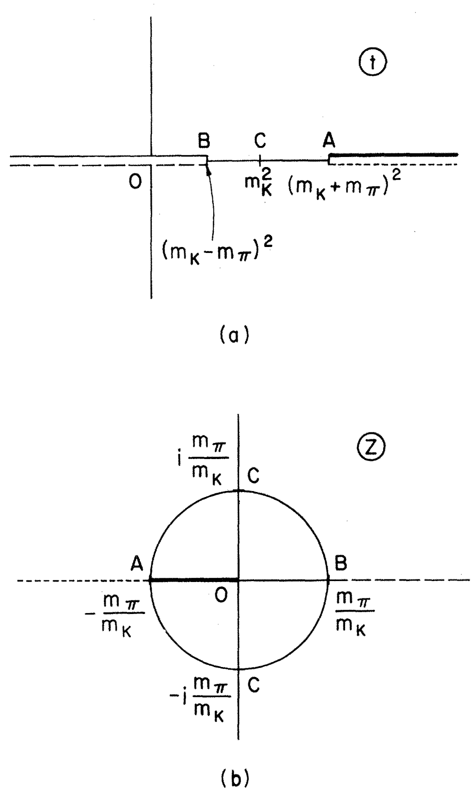

FIG. 4. Mapping from the t plane (a) to the z plane (b) according to Eq. (2.7). The images of the points A, B, and C in the t-plane are denoted by the same letters in the z plane. The thick (thin) solid line in the t plane maps onto the thick (thin) solid line in the z plane. The two kinds of broken lines in the t plane map onto the corresponding lines in the z plane.

(For simplicity we neglect the widths of resonances.) Then, for the same z, $F_L(z)$ must also develop a singularity if Eq. (2.9) were to be valid for *small* negative value of z. We note that the value of t corresponding to $t' = m_\rho^2$ is

$$t = m_\rho^2 + m_K^2 \simeq m_{K^*}^2. \tag{4.3}$$

Therefore, we see that $F_L(z)$ develops a singularity at the correct place, i.e., $t \simeq m_{K^*}^2$. [For the possible singularity of $F_L(z)$ due to the existence of the κ meson, see Appendix A.] This relation, Eq. (4.3), is the well-known nonet formula and is well satisfied experimentally. (m_π^2 is small compared with the other masses squared.) The same relation holds also for the recurrences of the K^* and the ρ mesons. These results seem to indicate strongly that Eq. (2.9) may be valid at least in the neighborhood of $z=0$, i.e., $-m_\pi^2/m_\rho^2 \lesssim z \lesssim 0$.

Next, let us take $z = -m_\pi/m_K + i\epsilon$ ($\epsilon > 0$). Then Eq. (2.9) gives

$$f((m_K + m_\pi)^2) = f_{\mathrm{em}}^\pi(m_\pi(m_K + m_\pi)^2/m_K), \tag{4.4}$$

which turns out to be 1.54 (and 1.24) for the dipole formula of $f_{\mathrm{em}}^\pi(t')$, Eq. (3.3) [and for the simple-pole formula of $f_{\mathrm{em}}^\pi(t')$, Eq. (3.4)]. In Fig. 5 we have plotted $f(t)$ for the case of the dipole formula of $f_{\mathrm{em}}^\pi(t')$ including the information obtained from Eq. (4.4). We see now that our result essentially reproduces the case 3 in Fig. 1 mentioned in the Introduction. The soft-pion theorem is approximately satisfied in contrast[14] to the theories discussed in Refs. 6 and 7 and the deviation (in the dipole fit case) is around 20%. A deviation of this order of magnitude is not very serious, since the

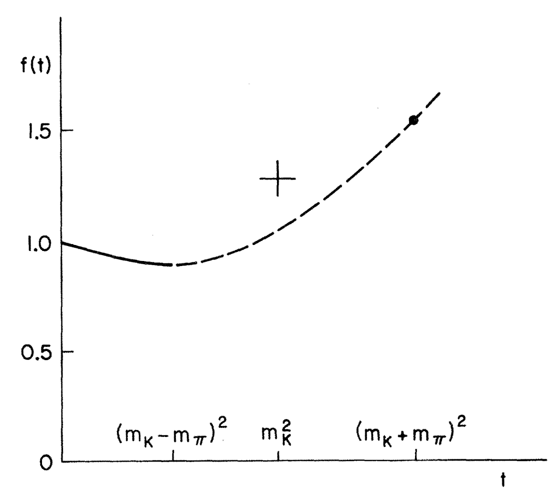

FIG. 5. A tentative behavior of the scalar K_{l3}-decay form factor $f(t)$. The broken line denotes the extrapolated form factor by using Eq. (4.4).

soft-pion prediction itself involves off-mass-shell extrapolation.

When we continue Eq. (2.9) to the region $z \lesssim -2$, there appear difficulties. This can be seen as follows: For $z = z_0 + i\epsilon$ ($\epsilon > 0$),

$$z_0 = \frac{1}{2m_K^2}\{m_K^2 + m_\pi^2 - m_{K^*}^2 - [(m_{K^*}^2 - m_\pi^2 - m_K^2)^2 - 4m_\pi^2 m_K^2]^{1/2}\} \simeq -2.1,$$

we find $t = m_{K^*}^2 - i\epsilon$ and $t' = m_{K^*}^2 + (m_K^2 - m_\pi^2) \times (z_0 - 1) - i\epsilon \simeq 0.09\ \text{BeV}^2 - i\epsilon$. Therefore, if Eq. (2.9) were still valid for $z \lesssim -2$, there should be another ρ meson around 300 MeV. Thus we see that Eq. (2.9) cannot be extended, at least, beyond $z \lesssim -2$. It is possible that even the result, Eq. (4.4), may already contain an appreciable error because of the extrapolation.

In passing we note that the form factors $f_+(t)$ and $f_-(t)$, as given by Eqs. (3.9) and (3.10), do not have branch points at $t = (m_K - m_\pi)^2$ in spite of the appearance of Eqs. (3.7) and (3.8). In fact, for $(m_K - m_\pi)^2 \leq t \leq (m_K + m_\pi)^2$, z_1 and z_2 are complex conjugates and therefore t_1' and t_2' are also complex conjugates. Then Eqs. (3.9) and (3.10) tell us that $f_+(t)$ and $f_-(t)$ are real for $(m_K - m_\pi)^2 \leq t \leq (m_K + m_\pi)^2$, which implies the absence of the branch point at $t = (m_K - m_\pi)^2$ in $f_+(t)$ and $f_-(t)$. Strictly speaking, the threshold behavior of the form factors $f_\pm(t)$ in the extrapolated region is not correct. In fact, in Eq. (4.4), $m_\pi(m_K + m_\pi)^2/m_K > 4m_\pi^2$ and therefore the right-hand side has an imaginary part, while the left-hand side should not have an imaginary part. In the numerical calculation of the right-hand side of Eq. (4.4), we simply neglected the imaginary part, which is expected to be very small.

In Sec. V we discuss further the problem of extrapolation.

V. DISCUSSION

If we take our asymptotic symmetry [Eq. (1.7)] strictly, corrections to Eq. (2.9) in the physical region ($z > 0$) come from the possible higher-lying pseudoscalar meson states which will contribute when we sandwich the commutator [Eq. (2.1)] between the states $\langle \pi^0|$ and $|K^+\rangle$. Then, on the right-hand side of Eq. (2.9), we expect, for example, a term $f_{\text{em}}^{\pi\pi'}(t'')$ which denotes the pion electromagnetic transition form factor, where π' is a heavy pion. The momentum transfer t'' is given by Eq. (2.7) with m_K replaced by $m_{\pi'}$. The analytic structure of the form factor $f_{\text{em}}^{\pi\pi'}(z)$ is, however, different from those of the $f_+(z)$, $f_-(z)$, and $f_{\text{em}}^{\pi}(z)$. In general, the analytic structure of each correction term coming from heavy pseudoscalar mesons is different for each intermediate state. Therefore the correction terms coming from particle mixing do not seem to remove the difficulty of the analytic continuation encountered in Sec. IV for large negative values of z. That is, the cancellation of the unwanted singularities may not be achieved by including such correction terms.

As remarked before, there is another argument which may invalidate the analytic continuation of Eq. (2.9). Before we take the limit of infinite momenta, there exists a term $f(z, |\vec{k}|)$ which vanishes only when $\vec{k}$ tends to infinity for positive z. This term $f(z, |\vec{k}|)$, however, may not vanish when analytic continuation with respect to z is first carried out before the limit of infinite momentum is taken. In this respect, the fact that the extrapolation achieves an attractive result only for small z and fails for large z seems quite reasonable.

As a final remark we add a comment on the apparent difficulty of the expression (2.5) when the four-divergence of both sides of Eq. (2.5) is taken blindly:

$$\frac{-i}{(4p_0k_0)^{1/2}}(m_K^2 - m_\pi^2)f(t) = \frac{-i}{(4p_0q_0)^{1/2}} f_{\text{em}}^{\pi}(t')(m_\pi^2 - m_\pi^2), \quad |\vec{k}| \to \infty. \qquad (5.1)$$

Equation (5.1) appears to lead to $f(t) = 0$. However, the procedure taken to derive Eq. (5.1) is not justified. In fact, the four-divergence of the commutator (2.1) reads

$$[\dot{V}_{\bar{K}^0}, V_0^{\pi^-}(0)] = \partial_\mu V_\mu^{K^-}(0), \qquad (5.2)$$

where $\dot{V}_{\bar{K}^0}$ denotes the time derivative of $V_{\bar{K}^0}$. When we take a matrix element of Eq. (5.2) and use asymptotic symmetry, we may no longer retain only the terms coming from the one-particle states selected by the charge $V_{\bar{K}^0}$, as was done in deriving Eq. (5.1). This is because we acquire an extra power of $|\vec{k}|$ from the presence of the time derivative of $V_{\bar{K}^0}$. To see this explicitly, consider a solvable model of a pseudoscalar nonet whose members have different physical masses but no other interactions. In the case of charge-current commutators involving the divergence of the current and the time derivative of the charge, such as Eq. (5.2), the so-called Z diagrams cannot be neglected, and in fact, they give contributions of the same order of magnitude as the terms coming from the diagonal terms selected by the charge $\dot{V}_{\bar{K}^0}$ in the asymptotic limit. In the case of a charge-current commutator without time derivative of the charge, the model shows that the contribution from the Z diagrams vanishes as $1/|\vec{k}|$ when $\vec{k} \to \infty$, compared with the diagonal terms retained in the asymptotic limit. (This model provides some illustration of the real mechanism of asymptotic symmetry.) Therefore, greater care is necessary when we treat the charge-current commutator involving the time derivative of $V_{\bar{K}^0}$.

Finally, we emphasize the fact that we did not make any assumption as to the mechanism of $SU(3)$ breaking, except that it should be compatible with the asymptotic $SU(3)$ symmetry. We have demonstrated that it is not impossible to derive a relatively large negative value of ξ. In this connection, further experimental study of the pion electromagnetic form factor is very interesting and important.

ACKNOWLEDGMENTS

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APPENDIX A

If the κ meson exists, its singularity in $F_L(x)$ may require the existence of another ρ meson [Eq. (2.9)]. This can be avoided if $m_{K^*} = m_\kappa$ or if the residue of the κ-meson pole becomes vanishingly small.

Actually, one can show that if $m_{K^*} = m_\kappa$ and if we, furthermore, assume simple-pole formulas for the form factors $f_+(t)$ and $f^\pi_{\rm em}(t')$,[15]

$$f_+(t) = m_{K^*}^2/(m_{K^*}^2 - t)\,, \qquad (A1)$$

$$f^\pi_{\rm em}(t') = m_\rho^2/(m_\rho^2 - t')\,, \qquad (A2)$$

then the scalar form factor $f(t)$ has a vanishing residue at $t = m_{K^*}^2$. In this case Eq. (2.9) reads

$$\frac{1}{m_{K^*}^2 - t}\left\{m_{K^*}^2 + \frac{1-z}{1+z}\,\frac{m_K^2 - m_\pi^2}{t}\,[(m_{K^*}^2 - t)f(t) - m_{K^*}^2]\right\} = \frac{m_\rho^2}{m_\rho^2 - t'}\,. \qquad (A3)$$

The momentum transfers t and t' are given by Eqs. (2.7) and (2.8) in terms of z. We write

$$m_{K^*}^2 = (z_* - 1)(m_\pi^2 - z_* m_K^2)/z_* \qquad (A4)$$

and

$$m_\rho^2 = -(z_* - 1)^2 m_\pi^2/z_*\,, \qquad (A5)$$

where

$$z_* = z_2(t = m_{K^*}^2)\,. \qquad (A6)$$

The function $z_2(t)$ is defined in Eq. (3.8). From (2.7) and (2.8) we have a relation among t, t', and z,

$$t' = t + (m_K^2 - m_\pi^2)(z-1)\,. \qquad (A7)$$

Now we calculate the following quantity:

$$B \equiv \lim_{z\to z_*} \frac{m_\rho^2 - t'}{m_{K^*}^2 - t}\left[m_{K^*}^2 + \frac{1-z}{1+z}\,\frac{m_K^2 - m_\pi^2}{t}\,(-m_{K^*}^2)\right]. \qquad (A8)$$

Using (A4), (A5), (A7), (2.7), and (2.8), we get

$$B = \frac{1}{(dt/dt')}\left[m_{K^*}^2 - \frac{1-z_*}{1+z_*}(m_K^2 - m_\pi^2)\right] = m_\rho^2\,. \qquad (A9)$$

We write

$$g_\kappa \equiv \lim_{t\to m_{K^*}^2 = m_\kappa^2} (m_{K^*}^2 - t)f(t). \qquad (A10)$$

From Eqs. (A3) and (A9) we conclude that

$$g_\kappa = 0. \qquad (A11)$$

This result is actually in line with the usual result of current algebra where single spin-zero- and spin-one-meson dominance is assumed for the matrix elements of the currents. There, usually, the relations $f_K = f_\pi$ and $f_\kappa = 0$ hold.[16] In a realistic case, where $f_+(t)$ and $f^\pi_{\rm em}(t')$ are no longer given by the simple-pole formulas, one cannot conclude Eq. (A11).

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[1]See, for example, the review article by M. K. Gaillard and L. M. Chounet, CERN Report No. 70-14, 1970 (unpublished). A summary of this report is published in Phys. Letters 32B, 505 (1970). Their values for $\xi(0)$, λ_+, and λ_- are $\xi(0) = -0.85 \pm 0.20$, $\lambda_+ = 0.045 \pm 0.012$, and $\lambda_- \simeq 0.05 \pm 0.10$.

[2]C. G. Callan and S. B. Treiman, Phys. Rev. Letters 16, 153 (1966); M. Suzuki, *ibid.* 16, 312 (1966); V. S. Mathur, S. Okubo, and L. K. Pandit, *ibid.* 16, 371 (1966).

[3]Present experiment indicates $f_+(0) = 1$ to a good accuracy. See, for example, Ref. 9 below.

[4]R. Dashen and M. Weinstein, Phys. Rev. Letters 22, 1337 (1969). See also Fayyazuddin and Riazuddin, Phys. Rev. D 1, 361 (1970), and V. S. Mathur and S. Okubo, *ibid.* 2, 619 (1970). When we take $f_K/f_\pi \simeq 1.28$, we get $\xi(0) \simeq -0.03$, -0.31, and -0.59 for $\lambda_+ = m_\pi^2/m_{K^*}^2$, $2m_\pi^2/m_{K^*}^2$, and $3m_\pi^2/m_{K^*}^2$, respectively.

[5]Equation (1.5) can only be satisfied with a very large value of λ_+ ($\lambda_+ \gtrsim 0.08$) if $\xi(0)$ is negative and large [$\xi(0)$

$\lesssim -0.5$]. However, off-hand, it is rather hard to expect such a large value of λ_+. See R. Acharya and L. Brink, Phys. Rev. D 3, 1579 (1971).

[6]K. Kang, Phys. Rev. Letters 25, 414 (1970).

[7]R. A. Brandt and G. Preparata, Lett. Nuovo Cimento 4, 80 (1970).

[8]Y. Hara, Phys. Rev. D 1, 874 (1970), and report (unpublished). The fact that a dip occurs in the model of Hara was pointed out in Ref. 1. See also H. Banerjee, Phys. Letters 32B, 691 (1970).

[9]S. Matsuda and S. Oneda, Phys. Rev. 174, 1992 (1968); Nucl. Phys. B9, 55 (1969); S. Oneda, H. Umezawa, and S. Matsuda, Phys. Rev. Letters 25, 71 (1970). For a recent review, see S. Oneda and S. Matsuda, Nucl. Phys. (to be published).

[10]For this, we need another assumption in addition to the hypothesis of asymptotic $SU(3)$ symmetry (see Sec. IV).

[11]$V_\mu^{\pi^\pm}(x) = V_\mu^1(x) \pm iV_\mu^2(x)$, $V_\mu^{K^0}(x) = V_\mu^6(x) + iV_\mu^7(x)$, $V_\mu^{\bar{K}^0}(x) = V_\mu^6(x) - iV_\mu^7(x)$.

[12]S. Matsuda and S. Oneda (unpublished). Assuming that the $f_+(t)$ and $f_-(t)$ satisfy unsubtracted dispersion relations which are dominated by the K^* and the κ mesons, Matsuda and Oneda were able to fix all the parameters using the information given by asymptotic symmetry. Extrapolation of their result also reproduces the soft-pion result. However, the value of $\xi(0)$ is negative but small [S. Matsuda and S. Oneda, Phys. Rev. 169, 1172 (1968)]. This indicates that single-pole dominance over the dispersion integrals does not work. See also, for example, H. T. Nieh and H. S. Tsao, Phys. Rev. D 1, 2663 (1970), and N. G. Deshpande, *ibid.* 2, 569 (1970).

[13]There is another limiting procedure which was not considered in the text. Now we do not keep fixed the directions of the momenta $\vec{p}$ and $\vec{k}$ ($|\vec{p}| = z|\vec{k}|$). The angle θ between the two momenta $\vec{p}$ and $\vec{k}$ is varied as $\theta = c/|\vec{k}|$ ($c \neq 0$). Then t and t' are given by $t = (z-1)(m_\pi^2 - zm_K^2)/z - zc^2$, and $t' = -(z-1)^2 m_\pi^2/z - zc^2$. In this case, if we insist on using all the sum rules coming from Eq. (2.6) ($\mu = 0, 1, 2, 3$) on the same footing, we obtain $f_+(t) = f^\pi_{\text{em}}(t')$ and $f_-(t) = 0$. [Both the $\mu = 0$ and $\mu = 3$ ($\vec{k}$ is in this direction) components of Eq. (2.6) give $(1+z)f_+(t) + (1-z)f_-(t) = (1+z)f^\pi_{\text{em}}(t')$. The $\mu = 1$ and $\mu = 2$ components lead to $f_+(t) - f_-(t) = f^\pi_{\text{em}}(t')$. In the case treated in the text, the latter components give only an identity, $0 = 0$. Usually, the $\mu = 0$ sum rule is the most trustworthy, since it follows from the so-called good commutator. In fact, the leading terms of the $\mu = 1$ and $\mu = 2$ sum rules are proportional to $1/|\vec{k}|$.] Therefore, we see that to the approximation adopted in this paper, the hypothesis of asymptotic $SU(3)$ symmetry gives rise to the two solutions: One is the above solution giving $f_-(t) = 0$, which is unphysical, and the other is the solution of broken $SU(3)$ symmetry given in Eq. (2.9). The limiting procedure taken in the text seems to be the best one under our approximation, and the gross features of the problem may be revealed by this procedure. The limiting procedure mentioned in this footnote may be the one sensitive to the approximation, i.e., the neglect of particle mixing.

We thank O. W. Greenberg, Seisaku Matsuda, J. Shapiro, J. Sucher, and C. H. Woo for discussion on this point.

[14]We note that this conclusion depends seriously on the validity of our continuation of z for $z < 0$. In Fig. 5, strictly speaking, asymptotic symmetry gives information only for $0 \leq t \leq (m_K - m_\pi)^2$.

[15]If we use Eqs. (3.9) and (3.10) for the whole negative-z region, we are not allowed to use simple-pole formulas for the form factors, because Eqs. (3.9) and (3.10) require infinitely many poles for the form factors. This fact results from the nonlinear relations, given by Eqs. (2.7) and (2.8), between the momentum transfers (t and t') and the variable z (see Fig. 2). As was discussed in Sec. IV, the relation (2.9) may be valid only for small negative values of z, that is to say, for the branch of z_2 [Eq. (3.8)] (see again Fig. 2). The relation (2.9) is surely wrong for the other branch, z_1 [Eq. (3.7)]. Therefore, in Appendix A, we do not use Eqs. (3.9) and (3.10) but we use Eq. (2.9) for the branch of z_2 alone. Then we are allowed to assume simple-pole formulas for the $f^\pi_{\text{em}}(t')$ and $f_+(t)$.

[16]Seisaku Matsuda and S. Oneda, Phys. Rev. 185, 1887 (1969), and Refs. 11–13 cited in that paper. For the possible relation, $f_K = f_\pi - f_\kappa$, see S. Okubo and V. S. Mathur, Phys. Rev. D 1, 2046 (1970), and L. Bessler, T. Muta, H. Umezawa, and D. Welling, *ibid.* 2, 349 (1970).