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Simplified Equation for the Bare Charge in Renormalized Quantum Electrodynamics*

M. BAKER
Physics Department, University of Washington, Seattle, Washington 98105

AND

K. JOHNSON
Laboratory for Nuclear Science and Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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The bare coupling constant of quantum electrodynamics, if finite, is the zero of the Gell-Mann–Low function ψ. In the paper to which this note is being appended, it was shown that the zeros of a function f are also zeros of ψ. The quantity f is the sum of the coefficients of the single power of the logarithm of a cutoff present in every order of perturbation theory in the vacuum polarization, omitting all those diagrams which include insertions in single internal photon lines. In this note the converse is shown; the zeros of ψ are also zeros of f. It is further shown that the zeros of ψ are zeros of a still simpler function f_1, which is defined analogously to f, but with the exclusion of *all* diagrams which contain multiple closed electron loops.

LET us briefly review the results of our earlier paper.[1] We define the quantity

$$d(q^2)=e^2q^2D(q^2)\,, \tag{1}$$

where e is the renormalized charge and $D(q^2)$ is the exact renormalized photon propagator. Gell-Mann and Low[2] conjectured that when $d(q^2)$ is calculated as a power series in a suitably defined renormalized charge, the resulting perturbation-theory integrals remain finite when the physical election mass m is set equal to zero. This led to the result that

$$\lim_{q^2\to\infty} d(q^2)\to e_0{}^2\,, \tag{2}$$

where $e_0{}^2$ is the first positive root of the equation

$$\psi(x)=0\,, \tag{3}$$

provided such a root exists. The root e_0 is called the bare charge. $\psi(x)$ is a well-defined power series in x whose coefficients are mass-zero, perturbation-theory integrals which do not depend upon the magnitude of the renormalized charge e.

In Ref. 1 we proved the above conjecture of Gell-Mann and Low. We then showed that one can define a function $f(x)$ by a certain subset of the diagrams of the perturbation series for $\psi(x)$ such that

$$f(x)=0\Rightarrow\psi(x)=0\,. \tag{4}$$

Thus the existence of a positive root of the simpler equation

$$f(x)=0 \tag{5}$$

is a sufficient condition for the consistency of quantum electrodynamics with the asymptotic behavior of $d(q^2)$ given by (2). The function $f(x)$ was defined by the formula

$$-xf(x)\ln(q^2/\lambda^2)=\rho^*(q^2)-\rho^*(\lambda^2)\,, \tag{6}$$

where $(g^{\mu\nu}q^2-q^\mu q^\nu)\rho^*(q^2)=\pi^*{}_{\mu\nu}(q)$ is the sum of all $m=0$ photon self-energy diagrams, computed with coupling constant x, which do not contain insertions in single internal photon lines. It is this latter exclusion which makes $f(x)$ vastly simpler than $\psi(x)$.

In this addendum to Ref. 1 we will first show the converse of (4), namely,

$$\psi(x)=0\Rightarrow f(x)=0\,, \tag{7}$$

which means that the bare coupling constant $e_0{}^2$ is necessarily a zero of the function $f(x)$. We will then show that $e_0{}^2$ must also be a zero of the still simpler function $f_1(x)$ defined by the equation

$$-xf_1(x)\ln(q^2/\lambda^2)=\rho^{**}(q^2)-\rho^{**}(\lambda^2)\,, \tag{8}$$

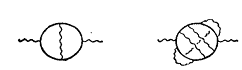
where $\pi^{**}{}_{\mu\nu}=(g_{\mu\nu}q^2-q_\mu q_\nu)\rho^{**}(q^2)$ is the sum of that subset of $m=0$ photon self-energy diagrams which not only do not contain insertions in single internal photon lines but also does not contain any closed electron-line loops except a single closed loop which couples directly to the two external photon lines. That is, π^{**} contains only those diagrams with a *single* closed electron line. Figure 1 contains examples of such diagrams for π^{**} while Fig. 2 gives diagrams contained in π^*, but not π^{**}. Figure 3 gives example of diagrams which are not included in the definition of π^*.

To prove (7), we first note that in quantum electrodynamics with electrons of zero physical mass, the *exact* photon propagator is $d(q^2)=e_0{}^2$. Thus

$$d(q^2)\,|_{m=0}=e_0{}^2\,. \tag{9}$$

Equation (9) follows from the fact that $d(q^2)=e_0{}^2$

FIG. 1. Some diagrams for $\pi^{**}(f_1)$.



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[1] M. Baker and K. Johnson, Phys. Rev. **183**, 1292 (1969).

[2] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

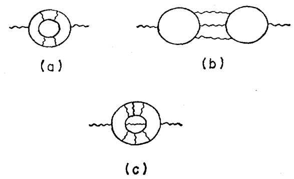

FIG. 2. Some diagrams for $\pi^*(f)$ which are not included in definition of $\pi^{**}(f_1)$.

solves the Gell-Mann–Low equation

$$q^2\frac{\partial}{\partial q^2}d(q^2)=d^2\psi(d) \tag{10}$$

if $\psi(e_0^2)=0$. But for mass-zero electrons, the Gell-Mann–Low equation, (10), for the propagator is *exact.*

Thus the field theory of zero-physical-mass, spin-$\frac{1}{2}$ charged particles[3] will have a solution in which the exact photon propagator is equal to the free-photon propagator if the coupling constant x is chosen to be a zero, e_0^2, of $\psi(x)$. The fact that the exact photon propagator is proportional to $1/q^2$ implies that its imaginary part which is proportional to the imaginary part of ρ vanishes. But this means that the sum over all photon self-energy diagrams which differ from each other only by insertions in a single internal photon line must vanish. For such a sum contains an internal $\rho(q')$ which vanishes when $\psi=0$. For example, in Fig. 3 the sum of diagrams (a)–(c) and all the other higher-order diagrams of that type vanish. Thus those graphs which do not contain an internal $\rho(q')$ must by themselves sum to zero. However, these are just the graphs which contribute to f.

In order to show that

$$\psi(x)=0\Rightarrow f_1(x)=0\,, \tag{11}$$

we make use of the fact that absorptive part of $D(q^2)$ is given by $\langle 0|j^\mu(x)j^\nu(y)|0\rangle$, where $j^\mu(x)$ is the electro-

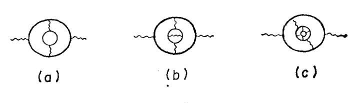

FIG. 3. Diagrams for the complete photon self-energy part π which are not included in the definition of $\pi^*(f)$.

[3] We remark that although $d=e_0^2$, in the theory of zero-physical-mass electrons which is the limit of ordinary finite-physical-mass electrodynamics as $m\to 0$, a more general renormalizable zero-mass theory exists. For, we may also solve (10), with the boundary condition $d=e_\lambda^2$, when $q^2=\lambda^2$. In such a theory d will approach e_0^2 when $q^2/\lambda^2\to\infty$. This sort of zero-mass theory is the same as that discussed by T. D. Lee and M. Nauenberg, Phys. Rev. **133**, B1549 (1964).

magnetic current operator. Thus $\psi(e_0^2)=0$ implies

$$\langle 0|j^\mu(x)j^\nu(y)|0\rangle=0\,. \tag{12}$$

But now we apply the theorem[4]

$$\langle 0|j^\mu(x)j^\nu(y)|0\rangle=0$$
$$\Rightarrow\langle 0|j^{\mu_1}(x_1)j^{\mu_2}(x_2)\cdots j^{\mu_n}(x_n)|0\rangle=0\,. \tag{13}$$

Thus when the coupling constant is e_0^2, and the physical mass is zero, all the multiple-current correlation functions of the form (13) also vanish. This kind of multiple correlation function is represented in perturbation theory by a sum of all closed-loop diagrams with n external photon lines. Thus the sum of all contributions to π^* (or f) which differ only by insertions in a given external closed loop must by themselves vanish when $x=e_0^2$. For example, in Fig. 2 the contribution to $f(x)$ of graphs (a)–(c), and all other higher-order corrections to the external photon-photon scattering must vanish when $x=e_0^2$. We are thus left with diagrams of the type depicted in Fig. 1 which contain no internal closed loops. The sum of these diagrams, which define f_1, must then vanish by themselves, when $x=e_0^2$. This is the assertion of (11).

To summarize: We now know that a zero of ψ will also be a zero of f and a zero of f_1. Further, a zero of f will be a zero of ψ.[1,5] We cannot yet establish that a zero of f_1 will be a zero of f. Thus the existence of a positive root of the equation

$$f_1(x)=0 \tag{14}$$

is a necessary condition for consistent quantum electrodynamics with a finite bare coupling constant. It may or may not be a sufficient condition. The calculation of f_1 is of course vastly simpler than that of f since multiple-loop diagrams are not included in f_1. This fact means that all of the diagrams for $f_1(x)$ are generated by a single functional equation,

$$\gamma^\mu\left(\frac{1}{i}\frac{\partial}{\partial z^\mu}-A_\mu(z)+ix\int D_{\mu\nu}(z-z'')\frac{\delta}{\delta A^\nu(z'')}\right)G(z,z') =\delta^{(4)}(z-z')\,, \tag{15}$$

where $D_{\mu\nu}$ is the free photon propagator. The relative simplicity of (15) may hopefully lead to a method for computing $f_1(x)$ to all orders.

[4] P. G. Federbush and K. Johnson, Phys. Rev. **120**, 1296 (1960).

[5] This also completely settles the questions raised by R. Jackiw, Nucl. Phys. **B5**, 158 (1968).