we refer to Chap. 4 of Eden *et al.*[5] Its relevance to this discussion can be seen from Fig. 1, where we show the 2π, 3π, and $N\bar{N}$ thresholds. For convenience we have separated their associated branch cuts. Physical unitarity relates the amplitudes at points A and B (both on the real axis), i.e.,

$$\mathbf{f}(J,s_A)-\mathbf{f}(J,s_B)=\mathbf{f}(J,s_A)\boldsymbol{\rho}\mathbf{f}(J,s_B)\,,$$

where $\mathbf{f}$ describes the scattering to all open coupled channels, and $\boldsymbol{\rho}$ is a diagonal phase-space matrix. Generalized unitarity, however, allows us to consider separately the $N\bar{N}$ channels,[6] relating the appropriate submatrix $\mathbf{f}_{N\bar{N}}(J,s)$ at the points A and C, i.e.,

$$\mathbf{f}_{N\bar{N}}(J,s_A)-\mathbf{f}_{N\bar{N}}(J,s_C)=\mathbf{f}_{N\bar{N}}(J,s_A)\boldsymbol{\rho}\mathbf{f}_{N\bar{N}}(J,s_C)\,.$$

This relation rules out fixed poles except in the region $-1<J<0$, i.e., at all integer values of J. We can similarly rule out fixed poles at interesting points in the other cases discussed by Finkler. We are left only with the possibility of fixed poles in the region $-\lambda<J<\lambda-1$ as noted above.

I am grateful to Dr. Collins, who drew my attention to the paper of Finkler.

[5] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge U. P., London, 1966).

[6] See, for example, E. J. Squires, in *Strong Interactions and High Energy Physics* (Oliver and Boyd, Edinburgh, 1964), pp. 55, 56.

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Remarks on a Test of the Padé-Approximant Approach in the Coulomb Bound-State Problem*

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An integral occurring in a paper by Crater with the same title is calculated exactly by numerical methods in order to further test the conclusions of that paper.

Equation (13) of Crater's paper[1] is

$$T(s,t,u)=\frac{4\pi}{\mu^2}(4M^2s-s^2)^{1/2}\sum_{n=0}^{\infty}\frac{1}{[(2n)!!]^2}\left(-\frac{t}{\mu^2}\right)^n\int_0^\infty dx\,x^{2n+1}(e^{zK_0(x)}-1)\,. \tag{1}$$

Table I. A_m^n: Coefficients used in calculating the Padé approximants through the [4,4] for $n=1,\ldots,8$.

n	$m=1$	$m=2$	$m=3$	$m=4$
0	1.0	0.25	$9.766\,269\,4\times10^{-2}$	$4.382\,439\,3\times10^{-2}$
1	4.0	0.166 666 67	$1.910\,593\,8\times10^{-2}$	$3.143\,745\,5\times10^{-3}$
2	64.0	0.533 333 33	$2.168\,256\,3\times10^{-2}$	$1.595\,307\,4\times10^{-3}$
3	2304.0	4.114 277 9	$6.365\,400\,8\times10^{-2}$	$2.252\,061\,9\times10^{-3}$
4	$1.474\,55\times10^{5}$	58.514 17	$3.574\,635\,9\times10^{-1}$	$6.314\,857\,5\times10^{-3}$
5	$1.474\,55\times10^{6}$	$1.329\,867\times10^{3}$	$3.281\,515\,4\times10^{0}$	$2.962\,041\,8\times10^{-2}$
6	$2.123\,356\,9\times10^{9}$	$4.419\,251\,0\times10^{4}$	$4.472\,900\,8\times10^{1}$	$2.095\,308\,7\times10^{-1}$
7	$4.161\,779\,0\times10^{11}$	$2.021\,070\,6\times10^{6}$	$8.484\,425\,1\times10^{2}$	$2.085\,899\,3\times10^{0}$
8	$1.065\,415\,3\times10^{14}$	$1.217\,397\,7\times10^{8}$	$2.137\,589\,0\times10^{4}$	$2.781\,537\,3\times10^{1}$
n	$m=5$	$m=6$	$m=7$	$m=8$
0	$2.080\,493\,9\times10^{-2}$	$1.013\,484\,5\times10^{-2}$	$4.999\,701\,2\times10^{-3}$	$2.482\,224\,8\times10^{-3}$
1	$6.139\,498\,7\times10^{-4}$	$1.315\,852\,4\times10^{-4}$	$2.978\,567\,3\times10^{-5}$	$6.973\,824\,5\times10^{-6}$
2	$1.577\,943\,7\times10^{-4}$	$1.849\,441\,5\times10^{-5}$	$2.049\,447\,0\times10^{-6}$	$3.365\,319\,9\times10^{-7}$
3	$1.216\,291\,2\times10^{-4}$	$8.426\,960\,6\times10^{-6}$	$6.849\,102\,2\times10^{-7}$	$6.203\,097\,4\times10^{-8}$
4	$1.935\,493\,8\times10^{-4}$	$8.247\,421\,5\times10^{-6}$	$4.355\,643\,2\times10^{-7}$	$2.667\,316\,4\times10^{-8}$
5	$5.274\,534\,4\times10^{-4}$	$1.415\,818\,6\times10^{-5}$	$4.979\,030\,6\times10^{-7}$	$2.113\,909\,7\times10^{-8}$
6	$2.202\,219\,2\times10^{-3}$	$3.783\,973\,7\times10^{-5}$	$9.006\,964\,0\times10^{-7}$	$2.695\,633\,5\times10^{-8}$
7	$1.308\,736\,2\times10^{-2}$	$1.456\,116\,2\times10^{-4}$	$2.373\,473\,0\times10^{-6}$	$5.066\,994\,6\times10^{-8}$
8	$1.050\,757\,5\times10^{-1}$	$7.635\,829\,2\times10^{-4}$	$8.598\,083\,7\times10^{-6}$	$1.320\,992\,6\times10^{-7}$

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[1] Horace W. Crater, Phys. Rev. D 2, 1060 (1970).

TABLE II. Lowest roots of the denominator of the [N,N] Padé approximants.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
[1,1]	4.0	24.0	120.0	560.0	2520
[2,2]	2.149 142 4	5.737 312 8	12.883 670	27.109 186	54.947 812
[3,3]	2.009 546 5	4.316 562 4	7.614 823 5	12.656 273	20.428 003
[4,4]	2.000 560 7	4.047 446 0	6.430 220 6	9.538 028 2	13.745 971
[5,5]		4.003 937 0	6.095 702 8	8.507 987 6	11.482 297
[6,6]			6.014 093 3	8.142 910 6	10.563 179
[7,7]			6.000 832 2	8.029 475 9	10.185 489
[8,8]				8.003 301 2	10.049 905
[9,9]				7.999 969 6	10.018 969
[10,10]					10.002 271
	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
[1,1]	11 088	48 048	205 920	875 158	3 695 101.9
[2,2]	108.594 27	210.743 50	403.498 76	764.740 52	1438.142 1
[3,3]	32.377 664	50.672 261	78.564 161	120.928 61	185.057 96
[4,4]	19.495 942	27.363 908	38.120 552	52.803 108	72.808 277
[5,5]	15.242 327	20.035 221	26.160 318	33.991 139	43.997 984
[6,6]	13.438 377	16.917 009	21.156 328	26.337 278	32.674 975
[7,7]	12.603 760	15.402 036	18.686 176	22.565 189	27.159 344
[8,8]	12.223 333	14.635 178	17.370 641	20.511 684	24.137 331
[9,9]	12.068 032	14.260 017	16.659 258	19.342 925	22.374 801
[10,10]	12.028 456	14.122 384	16.263 918	18.693 349	21.324 402

(The $(2n!)^2$ appearing in Crater's Eq. (11) *et seq.* is either a misprint or an error: It should be $[(2n)!!]^2$; also, the x^{2n-1} appearing in the first term of the right-hand side of Eq. (13) is a misprint for x^{2n+1}. These points are inessential to Crater's discussion.) Crater further expands the coefficient of each power of $-t/\mu^2$ in a power series in z:

$$T(s,t,u)=\sum_{n=0}^{\infty} T_n\left(-\frac{t}{\mu^2}\right)^n,$$

$$T_n=\frac{4\pi}{\mu^2}(4M^2s-s^2)^{1/2}\frac{1}{[(2n)!!]^2}\sum_{m=1}^{\infty} A_m{}^n z^m, \quad (2)$$

$$A_m{}^n=\int_0^\infty x^{2n+1}[K_0(x)]^m\,dx.$$

It should be observed that for each n, the first two terms in power-series expansion in z agree with Eq. (3) of Crater's paper exactly, as they must (the eikonal approximation sums the ladder graphs and crossed ladder graphs exactly in Crater's approximation $t<\mu^2\ll 4M^2-s$).

According to the known relativistic Balmer series, the lowest roots of the successive Padé approximants to T_n must converge to $2(n+1)$.

In Table I we list the $A_m{}^n$, and in Table II we list the lowest roots of the various Padé approximants for each n. The lowest roots do approach $2(n+1)$ for each n, but increasingly slowly as n increases.

The higher roots suggest that $2(n+1)$ is a branch point: The higher roots are greater than $2(n+1)$, increase in density as the order of the Padé approximant increases, and are interspersed with zeros.