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Asymptotic Form of the Electron Propagator and the Self-Mass of the Electron*

M. BAKER
Physics Department, University of Washington, Seattle, Washington 98105

AND

K. JOHNSON
Laboratory for Nuclear Science and Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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The electron self-mass problem is discussed in the context of ordinary renormalized quantum electrodynamics. All perturbation contributions to the renormalized self-energy part $\Sigma(p)$, which diverge logarithmically or remain constant in the limit when $p \gg m$, are summed. The resulting $\Sigma(p)$ vanishes in the limit $p^2/m^2 \to \infty$ and yields a value for δm which is finite and equal to m. To obtain this result it is only assumed that the exact photon Green's function at small distances behaves like the bare propagator, which is the case if the eigenvalue equation for the bare coupling constant has a finite root. It is shown that in spite of the fact that the resulting mechanical mass m_0 vanishes identically, no conservation equation is obtained for any axial-vector current. Hence no Goldstone bosons appear in ordinary quantum electrodynamics when it is summed to all orders.

I. INTRODUCTION

IN an earlier paper[1] we studied the unrenormalized Schwinger-Dyson equations for the electron propagator $S(p)$ under the assumption that the photon propagator $D(k)$ is proportional to $1/k^2$ as $k^2 \to \infty$.[2] We found that if the electron bare mass m_0 was taken equal to be zero, we could obtain finite solutions for $S(p)$ in a certain approximation scheme. The resulting electron electromagnetic mass δm was finite and equal to the physical electron mass m, which of course was undetermined because the original $m_0=0$ equations contain no scale parameter.

The above work[1] was insufficient on two major counts:

(i) There was no explicit demonstration that these results would not be essentially modified as one went to higher orders in the approximation scheme.

(ii) The relation between the approximation scheme and the conventional renormalized perturbation expansion of quantum electrodynamics was not made clear.

The purpose of the present paper is to answer the questions raised by these points. We show [under the assumption that $D(k) \sim (1/k^2)$ as $k^2 \to \infty$] that the ordinary renormalized perturbation solution for $S(p)$ sums to a function which, in an appropriately chosen gauge, has the following behavior as $p \to \infty$:

$$S^{-1}(p) \sim C^{-1}[\gamma \cdot p + am(m^2/p^2)^\epsilon]. \qquad (1.1)$$

ϵ is a constant which is determined by the expansion of the renormalized Bethe-Salpeter kernel K for electron-positron scattering. C and a are also constants. We explicitly calculate ϵ to order α_0^2 and find[3]

$$\epsilon = \frac{3}{2}\frac{\alpha_0}{2\pi} + \frac{3}{8}\left(\frac{\alpha_0}{2\pi}\right)^2 + \cdots, \qquad (1.2)$$

where the unrenormalized fine-structure constant $\alpha_0 \equiv e_0^2/4\pi$ is precisely defined in terms of the renormalized theory in Sec. II, Eq. (2.1).

The proof of (1.1) will make essential use of those properties of the scattering kernel K which were proven to all orders of perturbation theory in our discussion of vacuum polarization.[2] We obtain result (1.1) if the asymptotic behavior of the exact K is the same as the asymptotic behavior of the individual terms of its perturbation expansion which is the same in each order. Equation (1.1) then gives the exact asymptotic expression for the renormalized electron propagator $S(p)$ of quantum electrodynamics, provided that the photon propagator $D(k) \sim 1/k^2$ as $k^2 \to \infty$.[2]

Furthermore, we show that the electromagnetic mass δm calculated in terms of the physical mass m is finite and equal to m for all values of the physical mass. The usual divergent expression for δm arises from using the perturbation-theory solution of $S(p)$ for high p^2 rather than the exact asymptotic solution given by (1.1).

In order to make the logic of the argument clear, we briefly summarize the basic outline of our approach, *ignoring* for clarity the technical difficulties associated

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[1] K. Johnson, M. Baker, and R. Willey, Phys. Rev. **136** B1111 (1964).

[2] In two later papers [K. Johnson, R. Willey, and M. Baker, Phys. Rev. **163**, 1699 (1967); M. Baker and K. Johnson, *ibid.* **183**, 1292 (1969)] it was shown that a sufficient condition for the validity of the assumption $D(k) \to 1/k^2$ as $k^2 \to \infty$ is the existence of a positive root of a certain equation $f(x)=0$. The function $f(x)$ is the coefficient of the logarithmic divergence in Z_3, calculated in the theory without photon self-energy insertions. See also M. Gell-Mann and F. E. Low, *ibid.* **95**, 1300 (1954).

[3] Equation (1.2) does not agree with the results obtained in Ref. 1. This is due to an incorrect treatment of electron self-energy insertions and Ward's identity in Ref. 1. See the end of Sec. IV of the present paper.

with gauge dependence and multiplicative renormalizations. Then, with these problems put aside, the function $S(p)$ is finite in perturbation theory, and

$$1/S(p)=\gamma\cdot p+m+\Sigma(p)-\delta m\,, \tag{1.3}$$

where m = the physical mass and the combination $\Sigma(p)-\delta m$ is finite. Formally $\Sigma(p)-\delta m=0$ when $\gamma\cdot p=-m$. However, this equation for δm will not be used. We first show that all the perturbation contributions to finite quantity $S^{-1}(p)-\gamma\cdot p$, which do not vanish in the limit when $p\gg m$, may be expressed in terms of $S^{-1}(p_0)-\gamma\cdot p_0$, where p_0 also is asymptotic. They will be related by an expression which does not involve the physical mass m. This is a nontrivial result and it is the reason that the analysis can be carried out. We next find that if we sum up all these nonvanishing perturbation contributions to $S^{-1}(p)-\gamma\cdot p$, the sum vanishes in the limit when $p/m\to\infty$. Thus, we shall obtain the nonperturbative result $S^{-1}(p)-\gamma\cdot p\to 0$ in the limit $p/m\to\infty$ or, because of (1.3),

$$m+\Sigma(p)-\delta m\to 0$$

in the limit when $p\gg m$. In this case the integrals which define the *unsubtracted* $\Sigma(p)$ when expressed in terms of the exact S converge and as a consequence we show that

$$\Sigma(p)\to 0 \quad \text{for} \quad p\gg m\,. \tag{1.4}$$

When we combine (1.3) and (1.4), we find

$$m-\delta m\equiv 0\,. \tag{1.5}$$

Although the above paragraph basically describes our approach, the technical questions referred to make the actual calculations somewhat more involved. Hence one should not apply Eqs. (1.3) and (1.4) above without the qualifications which are appended to them in the sections which follow.

II. BEHAVIOR OF $S(p)$ FOR LARGE p^2

We want to study the high-p^2 behavior of $S(p)$ under the assumption that the renormalized photon propagator $D(k^2)$ behaves like $1/k^2$ as $k^2\to\infty$. We group together all those Feynman graphs for $S(p)$ which differ from each other only by insertions in internal photon lines. This grouping affects no conservation law of the exact theory; that is, the graphs in each group respect Lorentz invariance and current conservation. The sum of all graphs in each group is then equal to an equivalent single graph. In this equivalent graph, the internal photon·lines stand for $D(k^2)$ and the coupling constant on the end of each line is the renormalized charge e. If e_0^2, the unrenormalized charge, is finite, this equivalent graph has the same behavior in the high-p^2 region as the graph in which $D(k)$ is replaced by $1/k^2$ and e^2 by e_0^2 defined in the renormalized theory

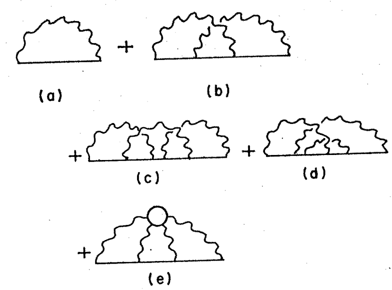

FIG. 1. Some typical diagrams for Σ^*.

by the equation

$$\lim_{k^2\to\infty} e^2k^2D(k)=e_0^2\,. \tag{2.1}$$

This is because the replacement of any single photon line in the graph by a contribution to $D(k)$ which vanishes more rapidly than $1/k^2$ makes all integrations converge. Because the graph is a function of only one external momentum p^2, this forces such a contribution to vanish as p^2 becomes large, as one can see from a simple scaling argument. Thus, if we assume that e_0^2 is finite,[4] we can calculate $S(p)$ in the uv region by omitting all graphs with photon self-energy insertions and by using the bare charge e_0 at the vertices.[5]

The sum of all such equivalent graphs yields an $S(p)$ which satisfies the functional equation

$$S^{-1}(p)=\gamma\cdot p+m_0+\Sigma^*(p\,;S(p'))\,, \tag{2.2}$$

where $m_0=m-\delta m$ is the bare mass of the electron. The functional $\Sigma^*(p\,;S(p'))$ is defined as the sum of all electron self-energy graphs which (a) cannot be broken by cutting a single electron line and (b) contain no insertions in either internal photon or electron lines. In each graph the internal electron lines stand for the full electron propagator $S(p)$, while the internal photon lines stand for the free photon propagator $D_{\mu\nu}{}^0(k)$, given by

$$D_{\mu\nu}{}^0(k)=\frac{1}{k^2}\left(g_{\mu\nu}+(b-1)\frac{k_\mu k_\nu}{k^2}\right), \tag{2.3}$$

where b is an arbitrary gauge parameter. Some of the typical low-order contributions to $\Sigma^*(p,S(p'))$ are depicted in Fig. 1.

The physical mass m of the electron is determined by

$$S^{-1}(\gamma\cdot p)=0 \quad \text{for} \quad \gamma\cdot p=-m \tag{2.4a}$$

or, equivalently,

$$\delta m=\Sigma^*[p\,;S(p')] \quad \text{for} \quad \gamma\cdot p=-m\,. \tag{2.4b}$$

[4] The value of e_0^2 is the first positive root of the equation $f(e_0^2/8\pi^2)=0$, where $f(x)$ is defined in Ref. 2. To order x^2, $f(x)=\frac{2}{3}+x-\frac{1}{4}x^2$. See J. L. Rosner, Phys. Rev. Letters **17**, 1190 (1966).

[5] It has been shown (M. Baker and K. Johnson, Ref. 2) that if e_0^2 is finite then the leading correction to the asymptotic limit (2.1) for $D(k)$ is of the form $(1/k^2)(m^2/k^2)^{K(e_0^2)}$.

The iterative solution of (2.2) generates the usual unrenormalized perturbation expansion for $S(p)$. The divergences of the integrals which appear in this expansion can be isolated in terms of two infinite constants: the electron self-mass $\delta m=m-m_0$ and the electron-wave-function renormalization constant Z_2. However, it has been shown[6] that the perturbation expression for Z_2 is finite if the gauge constant b is suitably chosen. In this gauge the only infinities in the perturbation expansion of (2.2) arise from the δm divergences. Hence, if we subtract (2.2) at $\gamma\cdot p=-m$ and use (2.4b), we obtain

$$S^{-1}(p)=\gamma\cdot p+m+\Sigma^*(p;S(p'))-[\Sigma^*(p;S(p'))]_{\gamma\cdot p=-m}\,. \qquad (2.5)$$

Because of the convenient choice of gauge, no second subtraction is necessary to render the perturbation solution of (2.5) finite. Thus from (2.5) we can obtain the usual renormalized perturbation expansion which expresses $S(p)$ in terms of m by a series of convergent integrals. The form for $S(p)$ in any other gauge can be obtained by a well-known transformation.[7]

The electromagnetic mass δm can be expressed in terms of $S(p)$ and hence m, using (2.4b). The resulting value of δm will be finite if $S(p)$ falls off sufficiently rapidly for $p^2\gg m^2$. Even if we ignore δm and the unrenormalized theory, and concern ourselves only with properties of the renormalized theory, the high-p^2 behavior is still of fundamental importance. For if $S^{-1}(p)$ contains terms which for large p^2 behave like $m[\ln(p^2/m^2)]$ as is indicated by renormalized perturbation theory, there then arises the possibility of inconsistencies in the renormalized theory when taken to all orders, i.e., the presence of "ghost" poles[8] in $S(p)$.

We will now show that the behavior of the solution of (2.5) at large p^2 is as indicated in (1.1). That is, we find

$$S(p)\to C(e_0^2)\left[\frac{1}{\gamma\cdot p}+\frac{m}{p^2}a_0(e_0^2)\left(\frac{m^2}{p^2}\right)^\epsilon\right], \qquad (2.6)$$

where $C(e_0^2)$, $a_0(e_0^2)$, and $\epsilon(e_0^2)$ are constants to be defined below.

If we expand the $(m^2/p^2)^\epsilon$ factor in (2.6), we obtain the asymptotic expression for the usual renormalized perturbation-theory expansion, namely,

$$S(p)\to C(e_0^2)\left\{\frac{1}{\gamma\cdot p}+\frac{ma_0(e_0^2)}{p^2}\times\left[1-\epsilon\ln\frac{p^2}{m^2}+\tfrac{1}{2}\epsilon^2\left(\ln\frac{p^2}{m^2}\right)^2+\cdots\right]\right\}. \qquad (2.7)$$

If expansion (2.7) for $S(p)$ is inserted in (2.4b), we obtain the usual perturbation series of logarithmically divergent integrals for δm. The logarithmic divergences arise from the terms proportional to ma_0/p^2 in (2.7). If instead we insert the correct exact asymptotic expression (2.6) for $S(p)$ in (2.4b), we obtain covergent integrals for δm because the factor $(m^2/p^2)^\epsilon$ with $\epsilon>0$ makes all logarithmically divergent integrals finite. Hence any discussion of self-mass integrals which involves the perturbation-theory estimate of the high-energy behavior of propagators is not relevant to the full theory.

Thus the basic problem is to show that the perturbation-theory logarithms sum to the form $(m^2/p^2)^\epsilon$ in (2.6). However, determination of the coefficient $C(e_0^2)$ of $1/\gamma\cdot p$ in (2.6) requires a bit of care and we will therefore make a few remarks about $C(e_0^2)$ before solving (2.5) for arbitrary m. If we set $m=0$, then the asymptotic solution (2.6) or (2.7) reduces to

$$S(p)^{m=0}=C(e_0^2)/\gamma\cdot p\,. \qquad (2.8)$$

That is, we can calculate $C(e_0^2)$ by looking at ordinary perturbation theory with physical mass, $m=0$. The resultant integrals for $C(e_0^2)$ are finite because of our choice of gauge but, because of their superficial linear divergence, the value of the constant C depends upon the way the external momentum is routed through the diagrams and, further, the order of doing subintegrations.

One can determine C a little less ambiguously by calculating the electron-photon vertex function

$$\Gamma_\mu{}^{m\to 0}(p,p+k)$$

and using Ward's identity,

$$\Gamma_\mu{}^{m=0}(p,p)=\frac{\partial}{\partial p^\mu}[S^{-1}(p)]^{m=0}=C^{-1}\gamma_\mu=\gamma_\mu+\frac{\partial}{\partial p^\mu}\left[\Sigma^*\left(p;\frac{C}{\gamma\cdot p'}\right)\right]. \qquad (2.9)$$

However, although the perturbation-theory integrals for Γ_μ may not be sensitive to the routing of the external momenta, they are not uniformly or absolutely covergent and still depend on the order of doing subintegrations. Different results for C can be obtained by introducing with different rules a cutoff Λ and then letting $\Lambda\to\infty$ for fixed p.[9] However, because the renormalized theory is free of ambiguities, the ambiguities associated with the different methods of defining

[6] K. Johnson, R. Willey, and M. Baker, Ref. 2.

[7] K. Johnson and B. Zumino, Phys. Rev. Letters 3, 351 (1959).

[8] These could arise in a manner similar to what might occur for the photon propagator if e_0^2 is infinite (see M. Baker and K. Johnson, Ref. 2).

[9] If we had introduced a cutoff Λ into the calculation of $S(p)$, then for p^2, $\Lambda^2\gg m^2$, we have $S(p)=(1/\gamma\cdot p)G(p^2/\Lambda^2,e_0^2)$. Now for fixed Λ^2, there are no uv divergences and the canonical commutation relations hold in their original form. This implies that as $p^2\to\infty$ for fixed Λ, $S(p)\to 1/\gamma\cdot p$, i.e., $G(\infty,e_0^2)=1$. However, since in our calculation no cutoff is introduced, we are effectively setting $\Lambda=\infty$ first. Hence we obtain $S(p)=(1/\gamma\cdot p)G(0,e_0^2)$, i.e., $C(e_0^2)=G(0,e_0^2)$. The fact that $G(0,e_0^2)$ is not necessarily equal to 1 and hence the fields have a modified canonical commutator reflects the sensitivity of the canonical commutation relations to the ambiguities of the perturbation theory integrals with no cutoff, even when there are no divergent quantities.

C cannot affect the value of any physical quantity. In particular the physically interesting gauge-invariant [see (5.17)] part of the asymptotic electron propagator, obtained by dividing out the factor C, will be independent of such ambiguities. For this reason, it is convenient to rewrite (2.2) in terms of a rescaled propagator $\tilde{S}(p)$ defined by the equation

$$\tilde{S}(p)=[1/C(e_0^2)]S(p)\,. \tag{2.10}$$

We could substitute (2.10) directly into the subtracted (2.5). However, since we are interested in the behavior of $S(p)$ for $p^2 \gg m^2$, and since (2.5) involves the value of Σ^* at $\gamma\cdot p=-m$ explicitly via the subtraction term, it is instead more convenient first to subtract (2.2) at a value of $p=p^0$, where $(p^0)^2 \gg m^2$. Then if we use the asymptotic solution of the resulting subtracted equation, we will be able to find the solution of (2.5) for $S(p)$ in terms of the physical mass m for $p^2 \gg m^2$.

For this reason, instead of studying (2.5) directly, we will first rewrite (2.2) in terms of $\tilde{S}$ and then perform a subtraction at $p=p_0$. If we define $\tilde{m}(p)$ by the equation

$$\tilde{S}^{-1}(p)=\gamma\cdot p+\tilde{m}(p)\,, \tag{2.11}$$

(2.2) becomes

$$\tilde{m}(p)=Cm_0+(C-1)\gamma\cdot p+C\Sigma^*(p;C\tilde{S}(p'))\,. \tag{2.12}$$

We then subtract (2.12) at $p=p^0$ to obtain

$$\tilde{m}(p)=\tilde{m}(p_0)+(C-1)\gamma\cdot(p-p_0) +C[\Sigma^*(p;C\tilde{S}(p'))-\Sigma^*(p_0;C\tilde{S}(p'))]\,. \tag{2.13}$$

Equation (2.13) is an integral equation for $\tilde{m}(p)$. It contains as parameters the subtraction point p_0 and the subtraction constant $\tilde{m}(p_0)$. We first note that in the limit

$$\tilde{m}(p_0)\to 0\,, \tag{2.14}$$

(2.13) is satisfied if at the same time

$$\tilde{S}\to 1/\gamma\cdot p\,. \tag{2.15}$$

This is just a consequence of our definition (2.9) of C, as can be explicitly verified by inserting (2.14) and (2.15) into (2.13). This yields

$$0=(C-1)\gamma\cdot(p-p_0)+C\{\Sigma^*(p;C/\gamma\cdot p') -\Sigma^*(p;C/\gamma\cdot p')\}\,. \tag{2.16}$$

Differentiating (2.16) with respect to p_μ yields

$$0=\left(1-\frac{1}{C}\right)\gamma_\mu+\frac{\partial}{\partial p^\mu}\Sigma^*\left(p;\frac{C}{\gamma\cdot p'}\right),$$

which coincides with (2.9). Thus the p and p_0 terms in (2.16) vanish independently.

For the general case $\tilde{m}(p_0)\neq 0$, we make explicit the dependence of $\tilde{m}(p)$ upon p_0 and $\tilde{m}(p_0)$ by writing

$$\tilde{m}(p)=\tilde{m}(p_0)H(p,p_0;\tilde{m}(p_0))\,. \tag{2.17}$$

In Sec. III, we show that

$$\lim_{\tilde{m}(p_0)\to 0} H(p,p_0;\tilde{m}(p_0))=\text{finite}=H^a(p,p_0)\,, \tag{2.18}$$

i.e., the ratio $\tilde{m}(p)/\tilde{m}(p_0)$ approaches a finite limit as $\tilde{m}(p_0)$ approaches zero, for fixed p, p_0. It should be emphasized that (2.18) is a nontrivial statement and its truth is the basic reason that one can carry out this analysis.[10]

We will now show that result (2.18) allows us to calculate the exact asymptotic form for $\tilde{m}(p)$ for $p^2 \gg m^2$. [$\tilde{m}(p)$ satisfies the rescaled version of the ordinary renormalized equation (2.5).] $\tilde{m}(p)$ has the form

$$\tilde{m}(p)=mF(p^2/m^2;e_0^2)+\gamma\cdot pG(p^2/m^2;e_0^2)\,, \tag{2.19}$$

where the functions F and G have the usual expansions in renormalized perturbation theory obtained by iterating (2.5) for $\tilde{S}$. For $p^2 \gg m^2$, these expansions take the form

$$F(p^2/m^2)=a_0(e_0^2)+a_1(e_0^2)\ln(p^2/m^2) +a_2(e_0^2)[\ln(p^2/m^2)]^2+\cdots +(m^2/p^2)[d_0+d_1\ln(p^2/m^2)+\cdots] +\cdots\,, \tag{2.20}$$

$$G(p^2/m^2)=(m^2/p^2)[b_0(e_0^2)+b_1(e_0^2) \times\ln(p^2/m^2)+\cdots]+\cdots\,. \tag{2.21}$$

Because of our choice of gauge and rescaling constant C, there are no terms in the asymptotic expansion (2.21) of G analogous to the $a_0+a_1\ln+a_2(\ln)^2+\cdots$ terms in expansion (2.20) of F.[11] Thus if we drop all those terms in the asymptotic expansion of the perturbation-series integrals for $\tilde{m}(p)$ which vanish as $p^2/m^2\to\infty$, we can write

$$\tilde{m}(p)\approx mF^a(p^2/m^2)\quad\text{for}\quad p^2\gg m^2\,, \tag{2.22}$$

where

$$F^a(p^2/m^2)=a_0(e_0^2)+a_1(e_0^2)\ln(p^2/m^2) +a_2(e_0^2)[\ln(p^2/m^2)]^2+\cdots\,. \tag{2.23}$$

Our problem is to sum the series (2.23), i.e., find a closed expression for the function $F^a(p^2/m^2)$. If we choose $p^2 \gg m^2$ and $p_0^2 \gg m^2$ and insert the asymptotic expression (2.22) for $\tilde{m}(p)$ and $\tilde{m}(p_0)$ into Eq. (2.17), we obtain the following functional equation for $F^a(p^2/m^2)$:

$$F^a(p^2/m^2)=F^a(p_0^2/m^2)H(p,p_0;mF^a(p_0^2/m^2))\,. \tag{2.24}$$

[10] The tautology (2.17) is of the same type that occurs in the so called "renormalization-group" analysis. However, that method has no content unless a zero-mass limit of the type (2.18) exists. This is made clear in the work of Gell-Mann and Low (Ref. 2), but it is not apparent in the work of many of the practioners of this method that an investigation of the sort carried out in (III) is required before one can believe its consequences.

[11] This is because one can verify that, to every order in perturbation theory, $m(p)=0$ when $m=0$. The argument is that used in the verification of (2.14) and (2.15). Thus G must vanish as $p^2/m^2\to\infty$, i.e., the asymptotic expansion of the perturbation-theory integrals for G must all have a factor m^2/p^2, as in (2.21).

Now let us assume that when $p^2/m^2 \to \infty$, F^a grows less rapidly than $(p^2/m^2)^{1/2}$, as indicated by perturbation theory; i.e., we assume

$$mF^a(p_0^2/m^2) \to 0 \tag{2.25}$$

as $m \to 0$. Then if we let $m \to 0$ in (2.24) and use the fundamental result (2.18), we obtain[12]

$$F^a(p^2/m^2) = F^a(p_0^2/m^2) H^a(p,p_0)\,, \tag{2.26}$$

which holds when $p^2/m^2 \gg 1$ and $p_0^2/m^2 \gg 1$.

Differentiating (2.26) with respect to p^2 and setting p_0^2 equal to p^2, we obtain

$$\frac{F^{a\prime}(p^2/m^2)}{F^a(p^2/m^2)} = -\epsilon \frac{m^2}{p^2}\,, \tag{2.27}$$

where

$$-\epsilon = p^2 \frac{d}{dp^2}[H^a(p,p_0)]_{p^2=p_0^2} \tag{2.28}$$

(H^a depends only on the ratio p^2/p_0^2). Hence

$$F^a(p^2/m^2) = A(m^2/p^2)^\epsilon\,. \tag{2.29}$$

Thus if $\epsilon > -\frac{1}{2}$, our assumption (2.25) is justified and (2.29) gives the exact asymptotic behavior of $\bar{m}(p)$ for $p^2 \gg m^2$. In Sec. III, when we study the equation for $H^a(p,p_0)$, we will calculate ϵ to order e_0^4. The result is given by (1.2). The positivity of the first two terms in the power-series expansion then guarantees the validity of (2.29), at least for small e_0^2.

The constant A in (2.29) is clearly not determined from (2.26). However, (2.29) determines the values of all the constants a_0, a_1, a_2, a_3, $\ldots$ in expansion (2.23) in terms of any one of them [say, $a_0(e_0^2)$] and the constant ϵ; i.e., if we expand (2.29) in a power series in ϵ, we find

$$F^a(p^2/m^2) = A\{1 - \epsilon \ln(p^2/m^2) + \tfrac{1}{2}\epsilon^2[\ln(p^2/m^2)]^2 + \cdots\}\,. \tag{2.30}$$

Comparing (2.30) with (2.23), we obtain

$$A = a_0(e_0^2)\,. \tag{2.31}$$

Thus from (2.31), (2.29), and (2.22), we can write

$$\bar{m}(p) \to m a_0(e_0^2)(m^2/p^2)^\epsilon\,, \quad p^2 \gg m^2\,. \tag{2.32}$$

Then using (2.10), (2.11), and (2.32), we obtain result (2.6) for the sum of all nonvanishing terms in the asymptotic expansion of the usual renormalized perturbation integrals for the electron propagator $S(p)$. (2.32) and (2.6) are valid in the gauge where Z_2 is finite, and include the contributions of all Feynman diagrams in which there are no photon self-energy insertions or, equivalently, of *all* diagrams if the photon self-energy insertions sum to the form $e^2 D_F(k) \to e_0^2/k^2$ with e_0^2 finite.

[12] Equation (2.26) implies that, for large p^2 and p_0^2, H^a depends only upon p^2 and p_0^2. This will be seen explicitly when we solve the equation for H^a.

It may seem that we have obtained the powerful result (2.32) without having made use of any properties of higher-order Feynman diagrams except for the general structure of their high-energy behavior [(2.20) and (2.21)]. However, the crucial ingredient for (2.32) was the assertion (2.18) that the ratio $\bar{m}(p)/\bar{m}(p_0)$ determined from (2.13) approached a finite limit as $\bar{m}(p_0) \to 0$. The proof of this assertion requires that certain detailed and nontrivial properties of Feynman diagrams remain valid to every order in perturbation theory, as we shall see in Sec. III, where we derive (2.18).

III. DERIVATION OF EQ. (2.18)

When $\bar{m}(p_0)=0$, the solution of (2.13) is $\bar{m}(p)=0$. We can obtain an expression for $\bar{m}(p)/\bar{m}(p_0)$ by differentiating (2.13) with respect to $\bar{m}(p_0)$ and setting $\bar{m}(p_0)=0$. This gives

$$\lim_{\bar{m}(p_0)\to 0} \frac{\bar{m}(p)}{\bar{m}(p_0)} = \frac{\partial \bar{m}(p)}{\partial \bar{m}(p_0)}\bigg|_{\bar{m}(p_0)=0} = 1 + C\frac{\partial}{\partial \bar{m}(p_0)}[\Sigma^*(p;C\tilde{S}(p')) - \Sigma^*(p_0,C\tilde{S}(p'))]_{\bar{m}(p_0)=0}\,. \tag{3.1}$$

We now show that (3.1) yields a finite solution for

$$H^a = \lim_{\bar{m}(p_0)\to 0} \frac{\bar{m}(p)}{\bar{m}(p_0)} \tag{3.2}$$

and hence we will establish our basic assertion (2.18). Now

$$\frac{\partial}{\partial \bar{m}(p_0)}\{C\Sigma^*[p;C\tilde{S}(p'')]\}_{\bar{m}(p_0)=0} = \int d^4p' C^2 \left\{\frac{\delta\Sigma^*[p;\tilde{S}(p'')]}{\delta\tilde{S}(p')}\right\}_{\bar{m}(p_0)=0} \times \left[\frac{\partial \tilde{S}(p')}{\partial \bar{m}(p_0)}\right]_{\bar{m}(p_0)=0}\,. \tag{3.3}$$

But

$$-(2\pi)^4 \frac{\delta\Sigma^*(p,S(p''))}{\delta S(p')} \equiv K(p,p')\,, \tag{3.4}$$

where $K(p,p')$ is the Bethe-Salpeter kernel for electron position scattering.[13] We can understand (3.4) graphic-

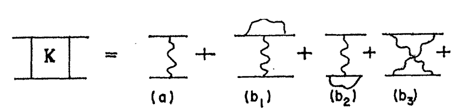

FIG. 2. Diagrams for K corresponding to diagrams (a) and (b) of Fig. 1.

[13] For a formal proof of (3.3), see for example, M. Baker, K. Johnson, and B. W. Lee, Phys. Rev. **133**, B209 (1964).

ally by differentiating the contributions to Σ^* depicted in diagrams (a) and (b) of Fig. 1. The resulting diagrams for K are depicted in Fig. 2.

$K(p,p')$ can be expressed as a functional of S and the full vertex Γ_μ according to the expansion depicted in Fig. 3. From Fig. 3 it is clear that $K(S,\Gamma)$ satisfies the following scaling property:

$$\tilde{K}=C^2K(S,\Gamma)=K(\tilde{S},\tilde{\Gamma})\,, \tag{3.5}$$

where

$$\tilde{\Gamma}_\mu=C\Gamma_\mu\,. \tag{3.6}$$

Thus $\tilde{K}\equiv C^2K$ can also be represented by the expansion depicted in Fig. 3, provided we interpret that intermediate electron lines and vertex blobs in that diagram as representing $\tilde{S}$ and $\tilde{\Gamma}$, respectively.

In (3.3) we need $\tilde{K}$ evaluated for $\tilde{m}(p_0)=0$. Let us call

$$\tilde{K}|_{\tilde{m}(p_0)=0}\equiv\tilde{K}^a(p,p')\,.$$

From (3.5), (2.14), and (2.15), we see that

$$\tilde{K}^a(p,p')=K(p,p';1/\gamma\cdot p'',\tilde{\Gamma}^a)\,, \tag{3.7}$$

where $\tilde{\Gamma}_\mu{}^a$ is the value of $\tilde{\Gamma}_\mu$ in (3.6) at $\tilde{m}(p_0)=0$.

If any of the integrals in the perturbation expansion for $\tilde{K}^a$ diverged, then (3.1) would make no sense and $\lim\tilde{m}(p)/\tilde{m}(p_0)$ as $\tilde{m}(p_0)\to 0$ would not exist. However, we have already investigated the properties of $\tilde{K}^a$ in our previous discussion of the Bethe-Salpeter equation for the vertex function $\tilde{\Gamma}$,[14]

$$\tilde{\Gamma}_\mu=C\gamma_\mu+\tilde{K}\tilde{S}\tilde{\Gamma}_\mu\tilde{S}\,. \tag{3.8}$$

We showed that $\tilde{K}^a(p,p')$ is finite to all orders in perturbation theory[15] and *furthermore* no infrared divergences arise when we set $p=0$; i.e., $\tilde{K}^a(0,p')$ is also finite. The latter property allowed us to choose the gauge constant b so that (3.8) has a finite iteration solution or, equivalently, so that Z_2 is finite to all orders in perturbation theory.[16] We now see that the finiteness of $\tilde{K}^a(0,p)$, which was essential to our previous discussion of Z_2 and Z_3, also guarantees the finiteness of the solution H^a of (3.1). We can put this equation in its final form by noting

$$\left.\frac{\partial\tilde{S}(p')}{\partial\tilde{m}(p_0)}\right|_{\tilde{m}(p_0)=0}=-\tilde{S}(p')\left.\frac{\partial\tilde{m}(p')}{\partial\tilde{m}(p_0)}\tilde{S}(p')\right|_{\tilde{m}(p_0)=0}=-\frac{1}{\gamma\cdot p'}\left.\frac{\partial\tilde{m}(p')}{\partial\tilde{m}(p_0)}\right|_{\tilde{m}(p_0)=0}\frac{1}{\gamma\cdot p'}. \tag{3.9}$$

Combining (3.1)–(3.3) and (3.9), we obtain

$$\left.\frac{\tilde{m}(p)}{\tilde{m}(p_0)}\right|_{\tilde{m}(p_0)\to 0}=H^a(p,p_0)=1+\int\frac{d^4p'}{(2\pi)^4}[\tilde{K}^a(p,p')-\tilde{K}^a(p_0,p')]\times\frac{1}{\gamma\cdot p'}H^a(p',p_0)\frac{1}{\gamma\cdot p'}. \tag{3.10}$$

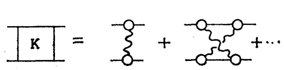

FIG. 3. Some diagrams for expansion of K in terms of the full vertex Γ.

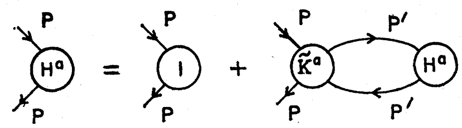

FIG. 4. Graphical representation of (3.10). The blob with K_s stands for difference $\tilde{K}^a(p,p')-\tilde{K}^a(\mathbf{p}_0,\mathbf{p}')$.

Equation (3.10) is depicted graphically in Fig. 4. The kernel $\tilde{K}^a$ is defined by integrals with zero-mass internal electron lines, (3.7), and those zero-mass integrals might have diverged in the infrared region. The demonstration that such infrared divergences do not arise in any order of perturbation theory was the essential part of our previous proof[14] that $\tilde{K}^a(p,p')$ is finite.

We now see how our previous analysis of zero-mass Feynman integrals not only guarantees that the kernel in (3.10) exists, but also guarantees the existence of a solution of (3.10) to all orders in perturbation theory. The first iteration of (3.10) yields the integral[17]

$$I(p,p_0)=-\int\frac{d^4p'}{(2\pi)^4}[\tilde{K}^a(p,p')-\tilde{K}^a(p_0,p')]\frac{1}{(p')^2}. \tag{3.11}$$

We can choose the vectors p and p_0 in (3.10) and (3.11) to be spacelike so that we can rotate the contour in the p'^0 integration. We then write

$$\int\frac{d^4p'}{(2\pi)^4}=\frac{i}{16\pi^2}\int(p')^2d(p')^2\int\frac{d\Omega'}{2\pi^2}, \tag{3.12}$$

---

[14] K. Johnson, R. Willey, and M. Baker, Ref. 2, Sec. IV and Appendix.

[15] Analogous properties of $\tilde{K}^a$ play an essential role in our discussion of Z_3. See K. Johnson, M. Baker, and R. Willey and M. Baker and K. Johnson (Ref. 2).

[16] The gauge constant b is determined by the following condition:

$$\int d\Omega'\tilde{K}^a(0,p')\frac{1}{\gamma\cdot p'}\gamma_\mu\frac{1}{\gamma\cdot p'}=0.$$

See Ref. 14.

[17] We have suppressed the indices of the Dirac matrices appearing in (3.10) and (3.11); e.g., the 1 in (3.10) stands for the Dirac matrix unity. With indices included, (3.10) becomes

$$I_{\alpha\beta}(p,p_0)=-\sum_\gamma\int\frac{d^4p'}{(2\pi)^4}[\tilde{K}_{\alpha\beta,\gamma\gamma}{}^a(p,p')-\tilde{K}_{\alpha\beta,\gamma\gamma}{}^a(p_0,p')]\frac{1}{p'^2}.$$

Since $\tilde{K}^a$ contains an even number of γ matrices, $I_{\alpha\beta}$ cannot contain a term like $(\gamma\cdot p)_{\alpha\beta}$ or $(\gamma\cdot p_0)_{\alpha\beta}$ and hence must be proportional to the unit Dirac matrix.

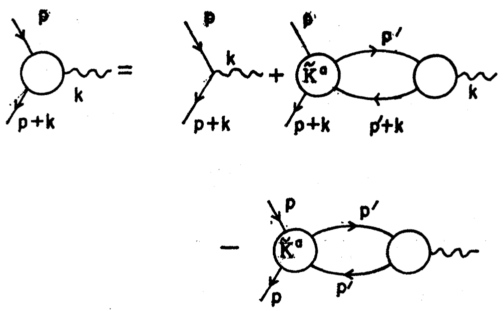

FIG. 5. Graphical representation of (4.1). Each vertex stands for $\tilde{\Gamma}^a$.

where the solid-angle integration in (3.12) is carried out over a sphere in four-dimensional Euclidean space. Since $\tilde{K}^a$ has dimensions $1/p^2$, we can write

$$\frac{i}{16\pi^2}\int\frac{d\Omega}{2\pi^2}\tilde{K}^a=\frac{1}{p'^2}k\left(\frac{p^2}{p'^2}\right), \tag{3.13}$$

where $k(p^2/p'^2)$ is a dimensionless function of p^2/p'^2. Equation (3.11) can then be written

$$I=-\int_0^\infty\frac{dp'^2}{p'^2}\left[k\left(\frac{p^2}{p'^2}\right)-k\left(\frac{p_0^2}{p'^2}\right)\right]. \tag{3.14}$$

Now if $k(p^2/p'^2)$ contained terms which behaved like $\ln(p'^2/p^2)$ as $p'^2\to\infty$, the subtraction in Eq. (3.14) would not make the integral I converge in the high-p'^2 region. However, from definition (3.13) and our result that $\tilde{K}^a(0,p)$ is finite, it follows that

$$\lim_{p'^2\to\infty}\frac{1}{p'^2}k\left(\frac{p^2}{p'^2}\right)\to\frac{k(0)}{p'^2}, \tag{3.15}$$

where $k(0)$ is finite constant. Thus the subtraction in (3.14) produces convergence in the high-p'^2 region.

From (3.15) and the symmetry property

$$\frac{1}{p'^2}k\left(\frac{p^2}{p'^2}\right)=\frac{1}{p^2}k\left(\frac{p'^2}{p^2}\right), \tag{3.16}$$

it also follows that the integrals in (3.14) converge in the low-p'^2 region even though we have set $m=0$.[18]

Thus we conclude that the first iteration of (3.10) yields a convergent integral I, which is a function $I(p_0^2/p^2)$ of the ratio p_0^2/p^2. Since $\tilde{K}^a(0,p')$ is finite,[19] $I(p_0^2/p^2)$ diverges logarithmically at $p^2\to 0$ and thus behaves (to within logarithms) like the inhomogeneous term 1 as $p^2\to 0$. Hence the higher-order iterations of (3.10) will also yield convergent integrals for $H^a(p,p_0)=H^a(p^2/p_0^2)$.[20]

We have therefore shown that our previous analysis[14] of the zero-mass integrals for K^a is sufficient to guarantee existence of the function $H^a(p^2/p_0^2)$ to every order in perturbation theory. Thus, without any new analysis of Feynman graphs, we have derived the basic equation (2.18) from which the asymptotic behavior of $\tilde{m}(p)$, (2.32), follows. The constant ϵ can be calculated from (2.28) by using the solution of (3.10) for $H(p^2/p_0^2)$ as a power series in e_0^2. However, it is more convenient and illuminating to calculate ϵ directly from the exact solution of (3.10) instead of using the iterative solution. With (3.12) and (3.13) and the fact that H^a depends only upon the ratio of p^2 and p_0^2, we can write (3.10) in the form

$$H^a\left(\frac{p^2}{p_0^2}\right)=1-\int_0^\infty\frac{dp'^2}{p'^2}H^a\left(\frac{p'^2}{p_0^2}\right)\times\left[k\left(\frac{p^2}{p'^2}\right)-k\left(\frac{p_0^2}{p'^2}\right)\right]. \tag{3.17}$$

From (2.26) and (2.29), we see that a solution has the form

$$H^a(p^2/p_0^2)=(p_0^2/p^2)^\epsilon. \tag{3.18}$$

The form (3.18) of the solution can also be directly obtained from (3.17). The integrals on the right-hand side of (3.17) will converge near $p'^2=0$ provided $\epsilon<1$. There will be convergence in the high-p^2 region provided $\epsilon>-1$. However, (2.26), from which we determined $\tilde{m}(p)$ [(2.32)], will be valid only if ϵ turns out to be $>-\frac{1}{2}$. If we substitute (3.18) into (3.17) and use the symmetry property (3.16), we obtain the following equation determining ϵ in terms of k:

$$-1=\frac{k(0)}{\epsilon}+\int_0^1du\,k(u)\frac{1}{u^\epsilon}+\int_0^1du\frac{k(u)-k(0)}{u^{1-\epsilon}}, \tag{3.19}$$

which is valid for $-1<\epsilon<1$. If $\epsilon>0$ then (3.19) may be put in the form

$$-1=\int_0^1du\,k(u)(u^{-\epsilon}+u^{\epsilon-1}), \tag{3.20}$$

and in this case H^a then satisfies the *homogeneous* equation

$$H^a\left(\frac{p^2}{p_0^2}\right)=-\int_0^\infty\frac{dp'^2}{p'^2}H^a\left(\frac{p'^2}{p_0^2}\right)k\left(\frac{p^2}{p'^2}\right). \tag{3.21}$$

The $u^{-\epsilon}$ term in (3.20) arises from the p'^2 integration in the region $p'^2<p^2$ in (3.21) while the $u^{\epsilon-1}$ term is the contribution of the region $p'^2>p^2$.

[18] Equation (3.14) would of course converge at low p'^2 even if $k(p'^2/p^2)\sim\ln(p^2/p'^2)$, as $p'^2\to 0$. That is, the absence of such logarithms is only necessary for the high-p'^2 behavior of the integral.

[19] Setting $p=0$ in the integrals for $K^a(p,p')$ enhances the possibility of infrared divergences. In fact, our previous analysis of these integrals showed that the general diagram had just enough factors to guarantee convergence in the infrared region.

[20] We will give the explicit form of the higher-order interactions below when we calculate the exact solution of (3.10).

Thus we have shown that (3.10) has the finite solution (3.18), where ϵ is determined by (3.19) or (3.20). If the resulting value of ϵ lies between $-\frac{1}{2}$ and 1, then the assumptions (2.18) and (2.25) are justified and the proof of our basic result (2.32) is completed. In Sec. IV we calculate the first two terms in the power-series expansion of $\tilde{K}^a$ so that we can determine ϵ to order e_0^4 from (3.20).

IV. CALCULATION OF ε TO ORDER e_0^4

We first calculate k using its definition (3.13) in terms of $\tilde{K}^a$. From (3.7), $\tilde{K}^a(p,p')=K(1/\gamma\cdot p,\tilde{\Gamma}_\mu{}^a)$, where the functional K is represented by the series of diagrams depicted in Fig. 3. Thus to calculate $\tilde{K}^a$, we simply replace each internal photon line by $D_{\mu\nu}{}^0(k)$ [(2.3)], each internal electron line by $1/\gamma\cdot p$, and each vertex blob $\tilde{\Gamma}_\mu{}^a$. $\tilde{\Gamma}_\mu{}^a$ is determined by (3.8) with m set equal to zero. Since by choice of C, $\tilde{\Gamma}_\mu{}^a(pp)=\gamma_\mu$, we can determine C by setting the photon momentum $k=0$ in (3.8). If we insert the resulting expression for C in (3.8), we obtain the following equation for $\tilde{\Gamma}_\mu{}^a(p,\,p+k)$:

$$\tilde{\Gamma}_\mu{}^a(p,\,p+k)=\gamma_\mu+\int\frac{d^4p'}{(2\pi)^4}\tilde{K}^a(p,\,p+k;\,p',\,p'+k)\times\frac{1}{\gamma\cdot p'}\tilde{\Gamma}_\mu(p',\,p'+k)\frac{1}{\gamma\cdot(p'+k)}-\int\frac{d^4p'}{(2\pi)^4}\tilde{K}^a(p,p;\,p',p')\frac{1}{\gamma\cdot p'}\tilde{\Gamma}_\mu(p',p')\frac{1}{\gamma\cdot p'}. \quad (4.1)$$

Equation (4.1) is depicted graphically in Fig. 5. To lowest order,

$$\tilde{K}^{a(2)}=-ie_0{}^2\gamma^a\left(g^{ab}-\frac{(p-p')^a(p-p')^b}{(p-p')^2}\right)\gamma^b\frac{1}{(p-p')^2};\quad (4.2)$$

the gauge constant b in (4.2) was chosen so that the condition in Ref. 16 is satisfied. To this order this gives $b=0$; thus, as is well known that the second-order vertex is finite in the Landau gauge. From (4.1) and (4.2), we then obtain the e_0^2 contribution to $\tilde{\Gamma}_\mu{}^a(p,\,p+k)$,

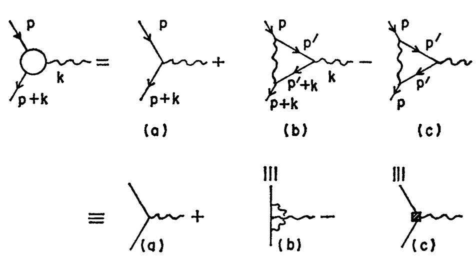

FIG. 6. Graphs for $\tilde{\Gamma}_\mu{}^a$ to order e_0^2. The second representation is just a convenient abbreviated notation.

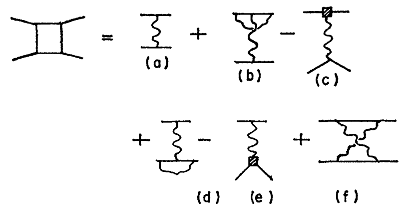

FIG. 7. Graphs for $\tilde{K}^a$ to order e_0^4.

which is depicted graphically in Fig. 6. The e_0^2 contribution to $\tilde{K}^a$ is then obtained by inserting the diagrams of Fig. 6 in the vertex blobs of Fig. 3. The resulting diagrams for the e_0^4 contribution to $\tilde{K}^a$ are shown in Fig. 7. Graph (a) of Fig. (7) includes a contribution of order e_0^4 arising from the e_0^2 term in the gauge constant b given by[1]

$$B^{(2)}=3\alpha_0/8\pi\,. \quad (4.3)$$

To order e_0^2 we find, using (4.2) and (3.12),

$$k(u)=-3\alpha_0/4\pi\,,\quad u\leqslant 1\,. \quad (4.4)$$

If we insert (4.4) in (3.19) or (3.20), we find

$$1=\frac{3\alpha_0}{4\pi}\left(\frac{1}{1-\epsilon}+\frac{1}{\epsilon}\right)+\cdots, \quad (4.5)$$

where the $1/\epsilon$ term arises from the second term in (3.20), which in turn comes from the large-P'^2 integration in (3.17). From (4.5), we find

$$\epsilon=3\alpha_0/4\pi+\cdots\,. \quad (4.6)$$

The higher-order terms in the expansion of ϵ in a power series in α_0 arise both from the contribution of the $1/(1-\epsilon)$ term in (4.5) and from the higher-order corrections to $\tilde{K}^a$ depicted in Fig. 7. However, in order to calculate ϵ to order α_0^2 we need only evaluate the α_0^2 contribution to $k(u)$ at $u=0$. This is easily seen if we write (3.19) in the form

$$-\epsilon=k(0)+\epsilon\int_0^1 du\,k(u)u^{-\epsilon}+\epsilon\int_0^1 du\,u^{\epsilon-1}[k(u)-k(0)]. \quad (4.7)$$

Since the integrals on the right-hand side of (4.7) are finite as $\epsilon\to 0$, they give a contribution which is of $\alpha_0\times$(first-order term). To order α_0^2, (4.7) becomes

$$\epsilon^{(4)}=-k^{(4)}(0)+(3\alpha_0/4\pi)^2\,. \quad (4.8)$$

We can calculate $k^{(4)}(0)$ by setting $p=0$ in the integrals represented by Fig. 7. The calculation is straightforward and the contributions of the various diagrams of Fig. 7 to $k^{(4)}(0)$ are displayed in Table I. Using (4.3), we sum all the contributions listed in Table I. This

TABLE I. Contributions of diagrams of Fig. 7 to $k^{(4)}(0)$.

(a)	(b)	(c)	(d)	(e)	(f)
$-3(\alpha_0/4\pi)-b^{(2)}(\alpha_0/4\pi)$	$\frac{3}{2}(\alpha_0/4\pi)^2$	$\frac{9}{2}(\alpha_0/4\pi)^2$	$\frac{3}{2}(\alpha_0/4\pi)$	$\frac{9}{2}(\alpha_0/4\pi)^2$	$-3(\alpha_0/4\pi)^2$

yields

$$k^{(4)}(0)=-\frac{3\alpha_0}{4\pi}+\frac{15}{2}\left(\frac{\alpha_0}{4\pi}\right)^2. \tag{4.9}$$

Equations (4.4) and (4.9) then give

$$\epsilon^{(4)}=\frac{3\alpha_0}{4\pi}+\frac{3}{2}\left(\frac{\alpha_0}{4\pi}\right)^2 \tag{4.10}$$

as stated in Sec. I. Thus we see that at least for small α_0, ϵ is positive, and $S(p)$ behaves as indicated in (1.1) for large p^2. In Ref. 1, we overlooked the contribution of diagrams (c) and (e) of Fig. 7 to $\tilde{K}^a$. These diagrams arise from the vertex subtraction in (4.1) for $\tilde{\Gamma}_\mu{}^a$ [diagram (c) of Fig. 6]. The omission was due to an incorrect treatment of Ward's identity given there and thus our previous result[1] for ϵ differed from (4.10) in the order α_0^2.

V. SHORT-DISTANCE BEHAVIOR OF ELECTRON PROPAGATOR IN ARBITRARY GAUGE

Our result (2.6) for the large-p behavior of $S(p)$ is valid in the gauge in which Z_2 is finite. To order α_0 this gauge is determined by (4.3). In any other gauge, the unrenormalized $S(p)$ is infinite and hence one must introduce a cutoff Λ in order to define it. We can do this by replacing the photon propagator $D_{\mu\nu}{}^0(k)$ by $D_{\mu\nu}{}^0(k,\Lambda)$, where

$$D_{\mu\nu}{}^0(k,\Lambda)=\frac{\Lambda^2}{\Lambda^2+k^2}D_{\mu\nu}{}^0(k). \tag{5.1}$$

This makes the electron propagator $S_b(p,\Lambda)$, calculated using the photon propagator (5.1) for internal photon lines, finite in any gauge. We have introduced the subscript b to make explicit the dependence upon the gauge parameter. Then one can relate the *coordinate space* electron propagagor in the gauges b_1 and b_2 according to the formula[7]

$$S_{b_2}(x-x',\Lambda)=S_{b_1}(x-x',\Lambda)\times\exp\{ie_0^2(b_2-b_1)[D^0(x-x',\Lambda)-D^0(0,\Lambda)]\}, \tag{5.2}$$

where

$$D^0(x,\Lambda)=-\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{(k^2-i\epsilon)^2}\frac{\Lambda^2}{\Lambda^2+k^2} \tag{5.3}$$

and

$$S_b(x-x',\Lambda)\equiv\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-x')}S_b(p,\Lambda). \tag{5.4}$$

We will choose b_1 so that $S_{b_1}(x-x',\Lambda)$ is finite as $\Lambda\to\infty$, i.e.,

$$\lim_{\Lambda\to\infty}S_{b_1}(p,\Lambda)=S(p), \tag{5.5}$$

where for large p, $S(p)$ is given by (2.6). This means that

$$b_1=3\alpha_0/8\pi+\cdots. \tag{5.6}$$

We let b_2 be arbitrary and define

$$\delta=b_2-b_1. \tag{5.7}$$

Then in the limit of large Λ, (5.2) becomes

$$S_{b_2}(x-x',\Lambda)=S(x-x')\exp[ie_0^2\delta I(x-x',\Lambda)], \tag{5.8}$$

where

$$I(x-x',\Lambda)=\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-x')}-1}{(k^2-i\epsilon)^2}\left(\frac{\Lambda^2}{\Lambda^2+k^2}\right) \tag{5.9}$$

and $S(x-x')$ is the Fourier transform (5.4) of $S(p)$ [(5.5)]. It is clear since $S(x-x')$ remains finite as $\Lambda\to\infty$, that in order to define a finite S_{b_2} an infinite renormalization is required. The resulting finite renormalized S_{b_2} will be related to $S(x-x')$ by a *factor* which is a finite function of $x-x'$, and which is so smooth that no divergences in the Fourier transform are produced by the confluence of the light cone singularities of $S(x-x')$ and the factor. We notice that since this factor is independent of spin that the coefficients of the Dirac matrix $\gamma\cdot(x-x')$ and 1 in S is the same, that is the ratio of these coefficients is gauge invariant, and is furthermore independent of the way the multiplicative renormalization is carried out. Therefore, we write

$$S(x-x')=\frac{\gamma\cdot(x-x')}{(x-x')^4}A(x-x')+m\frac{B(x-x')}{(x-x')^2}$$

and in an arbitrary gauge b,

$$S_b{}^{\text{ren}}(x-x')=\frac{\gamma\cdot(x-x')}{(x-x')^4}A_b(x-x')+m\frac{B_b(x-x')}{(x-x')^2}.$$

We can calculate S at short distances by using the large-p behavior of $S(p)$, i.e.,

$$S(x-x')|_{(x-x')^2\to0}\to C(e_0^2)\int\frac{dp}{(2\pi)^4}e^{ip\cdot(x-x')}\times\left[\frac{1}{\gamma\cdot p}+\frac{m}{p^2}a_0\left(\frac{m^2}{p^2}\right)^\epsilon\right]=\frac{C(e_0^2)}{2\pi^2}\left\{\frac{\gamma\cdot(x-x')}{(x-x')^4}+i\frac{\Gamma(1-\epsilon)}{2^{1+2\epsilon}\Gamma(1+\epsilon)}a_0\frac{m}{(x-x')^2}[m^2(x-x')^2]^\epsilon\right\};$$

the ratio

$$R(x-x')=\frac{mB(x-x')}{A(x-x')}=m\frac{B_b(x-x')}{A_b(x-x')}$$

is gauge invariant and independent of the method of carrying out the multiplicative renormalization of S_b. At small distances, R takes the form

$$R\to mi\frac{\Gamma(1-\epsilon)}{2^{1+2\epsilon}\Gamma(1+\epsilon)}a_0(e_0^2)[m^2(x-x')^2]^\epsilon.$$

R is a gauge-invariant finite quantity which at small distances contains functions of the charge: $a_0(e_0^2)$ and $\epsilon(e_0^2)$, which are expressed in terms of unambiguous integrals and which occur in renormalized perturbation theory. Note that the constant $C(e_0^2)$, which contains ambiguities, does not occur in $R(x-x')$.

VI. CALCULATION OF δm

We now insert our result for $S(p')$, expressed in terms of the physical mass m, in the original unrenormalized (2.2). In the appropriate gauge (5.6), all the integrals defining $\Sigma^*(p,S(p'))$ now converge since $S(p')$ possesses the large-p behavior given by (2.6). Hence $\Sigma^*(p,S(p'))$ can be expressed in terms of finite functions of the ratio (p^2/m^2). We can then determine the high-p behavior of Σ^* by letting m approach zero. The high-p limit of (2.2) yields the following equation for δm, or $m_0=m-\delta m$:

$$C^{-1}\gamma\cdot p=\gamma\cdot p+m_0+\Sigma^*(p,C/\gamma\cdot p') -m\int\frac{d^4p'}{(2\pi)^4}K(p,p')\big|_{m\sim0}\frac{\partial S(p')}{\partial m}\bigg|_{m\sim0} \tag{6.1}$$

as $p/m\to\infty$. In arriving at (6.1) we have used the definition (3.4) of K and the fact that $S(p')|_{m=0}=C/\gamma\cdot p'$. By Ward's identity (2.9) the γp terms in (6.1) are canceled by the $\Sigma^*(p, C/\gamma\cdot p')$ term. Equation (6.1) then becomes

$$\delta m=m-m\int\frac{d^4p'}{(2\pi)^4}K(p,p')\big|_{m=0}\frac{\partial S(p')}{\partial m}\bigg|_{m\sim0}. \tag{6.2}$$

We know that $K(p,0)|_{m=0}$ is finite and we see from (2.6) that

$$m\frac{\partial S(p')}{\partial m}\bigg|_{m\sim0}=C(2\epsilon+1)\frac{m}{p^2}\left(\frac{m^2}{p^2}\right)^\epsilon a_0 \tag{6.3}$$

when $\epsilon>0$ [and as $m\to0$, no divergent integral which multiplies m appears in (6.2)]. Thus (6.2) becomes

$$\delta m=m\,, \tag{6.4}$$

i.e., the electromagnetic mass δm, when expressed in terms of the physical mass m, is identically equal to it for all values of m.[21]

We can gain some insight into (6.4) by looking at the subtracted equation (2.13) for $\bar{m}(p)$. In (2.13) the quantity p_0 acts as a cutoff for the integrals generated by the perturbation expansion of $\Sigma^*(p)-\Sigma^*(p_0)$. Equation (2.13) can then be interpreted as the equation for the electron propagator in a theory which contains a cutoff p_0 and in which the mechanical mass is $\bar{m}(p_0)$. For values of the cutoff, $p_0\gg m$, the mechanical mass $\bar{m}(p_0)$ is then determined in terms of the physical mass m and the cutoff p_0 by (2.32); i.e.,

$$\bar{m}(p_0)=a_0(e_0^2)m(m^2/p_0^2)^\epsilon\,, \tag{6.5}$$

which is the sum of the usual perturbation-theory logarithms in the expansion of the bare mass in terms of the physical mass and the cutoff. From (6.5) we see that as the cutoff becomes larger it requires less and less mechanical mass to generate the same physical mass and in the limit when the cutoff p_0 becomes infinite, it takes only a vanishingly small mechanical mass to generate a finite physical mass.[21] Accordingly, in this limit δm becomes equal to m.

VII. CONSERVATION LAWS

The axial-vector current $j_5^\mu(x)=-i\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$ obeys the formal equation of motion,

$$\partial_\mu j_5^\mu=2m_0\bar{\psi}\gamma_5\psi=2m_0j_5\,, \tag{7.1}$$

where m_0 is the mechanical mass.[22] We wish to discuss (7.1) in the context of the renormalized theory. The unrenormalized operator $j_5^\mu(x)$ exists because the divergent part of every diagram for the proper vertex is the same as that of the corresponding proper vertex of the unrenormalized vector current $\bar{\psi}\gamma^\mu\psi$ which does exist. The same cannot be said for the unrenormalized operator $j_5(x)$. Hence, the argument that $m_0=0$ implies a conservation law for the axial current must be regarded with some caution, since when (7.1) is made precise with the use of a cutoff, both m_0 and $j_5(x)$ diverge in perturbation theory in the limit as the cutoff is removed, whereas the left-hand side of (7.1) remains finite. This difficulty was first pointed out by Maris and Jacob.[23]

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[21] In deriving (6.4) we used (2.2), in which all internal photon propagators $D(k)$ in Σ^* were replaced by their asymptotic value e_0^2/k^2. It is easy to see, using scaling arguments, that the corrections (Ref. 5) to the asymptotic limit for $D(k)$ yield a contribution to Σ^* of the form $\gamma\cdot p(m^2/p^2)^{K(e_0^2)}+m(m^2/p^2)^{\epsilon+K(e_0^2)}$. If we then look at the high-p limit of the exact version of (2.2), we again find $\delta m=m$. Thus (6.4) is valid in the complete quantum electrodynamics including photon self-energy insertions provided $D(k^2)\to e_0^2/k^2$ as $k^2\to\infty$.

[22] We may remark also that anomalies in the divergence of the axial current of the sort discussed by S. Adler, Phys. Rev. **177**, 2426 (1969), R. Jackiw and K. Johnson, *ibid.* **182**, 1459 (1969), and others play no role because we can use a non-gauge-invariant axial-vector current whose divergence is consistent with (7.1) to discuss the Goldstone phenomena formally.

[23] Th. A. J. Maris and G. Jacob, Phys. Rev. Letters **17**, 1300 (1966).

To discuss this limit we shall merely paraphrase an argument already given.[24] A general matrix element of (7.1) can be obtained by the symmetrical insertion into single lines of the off-shell version of (7.1). Therefore, consider the equation for the proper vertex,

$$q^\mu\Gamma_\mu{}^5(p+q,p)=S^{-1}(p+q)\gamma_5+\gamma_5S^{-1}(p)+2m_0\Gamma^5(p+q,p)\,. \quad (7.2)$$

Here $\Gamma_\mu{}^5$ is the proper vertex corresponding to the axial-vector current and Γ^5 is the proper vertex corresponding to the pseudoscalar density $j_5(x)$.

We may re-express (7.2) in terms of renormalized quantities in the form

$$(Z_2/Z_1{}^A)q^\mu\bar{\Gamma}_\mu{}^S(p+q,p)=\bar{S}^{-1}(p+q)\gamma_5+\gamma_5\bar{S}^{-1}(p)+2m(Z_1{}^S/Z_1{}^{PS})\bar{\Gamma}^5(p+q,p)\,, \quad (7.3)$$

where

$$m_0=mZ_1{}^S/Z_2\,, \quad (7.4)$$

$Z_1{}^A$ is a renormalization constant of the axial-vector vertex, $Z_1{}^S$ is a suitably defined renormalization constant of the proper scalar vertex, and $Z_1{}^{PS}$ is the corresponding quantity for the pseudoscalar vertex. In the limit as $\Lambda\to\infty$, $\bar{\Gamma}_\mu{}^5$, $\bar{S}$, $\bar{\Gamma}^5$, and $Z_2/Z_1{}^A$ exist order by order in perturbation theory, so that

$$\frac{Z_1{}^S}{Z_1{}^{PS}}=\frac{Z_1{}^S/Z_2}{Z_1{}^{PS}/Z_2}=\frac{Z^S}{Z^{PS}}$$

approaches a finite nonvanishing limit, order by order, which is the ratio of the renormalization constants of the scalar and pseudoscalar currents. Since when we sum up the perturbation-theory contributions, $m_0=0$, and Z_2 is finite, Z^S vanishes as the cutoff tends to infinity. Since the ratio Z^S/Z^{PS} is finite (in our gauge), $Z_1{}^{PS}$ also vanishes in this limit [which we could have demonstrated by treating this vertex in exactly the same fashion as we discussed $m(p_0)$ in the earlier sections of this paper]. We can see the reason intuitively by observing that the divergent parts of the scalar and pseudoscalar vertices are independent of mass terms, and hence are the same since the interaction vertex is chirally invariant.

In conclusion, we find that in spite of a vanishing mechanical mass, there is no conservation of the unrenormalized axial-vector current and hence no chiral symmetry which is broken by the finite physical mass (at least in the sense of the sort of symmetry which when broken is accompanied by a Goldstone boson).

[24] G. Preparata and W. I. Weisberger, Phys. Rev. **175**, 1965 (1968).

VIII. CONCLUSION

We have shown that if the photon propagator is set equal to $1/k^2$, then the high-p^2 behavior of the sum of all the terms in the perturbation expansion of $S(p)$ is given by (2.6). The coefficient in the power-series expansion of ϵ are determined by the renormalized perturbation expansion of the $m=0$, electron, positron, Bethe-Salpeter kernel K. This exact $(m^2/p^2)^\epsilon$ behavior leads to finite self-energy integrals, giving $\delta m\equiv m$ for all values of m.

To complete this study of the short-distance behavior of quantum electrodynamics, we must calculate the function $f(\alpha_0)$,[2] in order to determine whether the basic assumption, $e^2D(k)\to e_0{}^2/k^2$, $e_0{}^2<\infty$, is justified.