degenerate family. We now claim that

$$\psi e^{\sqrt{2}Q(1)\cdot k}|0\rangle_{p=0}=|0\rangle_{k^2=0}, \quad \text{(D5)}$$

that is, only the true (sector zero) vacuum contributes. For example, consider

$$_{k^2=0}\langle 0|K_l(p)e^{\sqrt{2}Q(1)\cdot k}|0\rangle_{p=0} \quad (l>0)$$
$$={}_{k^2=0}\langle 0|[K_l(p),e^{\sqrt{2}Q(1)\cdot k}]|0\rangle_{p=0}; \quad \text{(D6)}$$

then using the fact that $Q^\mu(1)$ satisfies the "stability conditions"

$$[K_l-K_0,Q^\mu(1)]=0, \quad \text{(D7)}$$

this reduces to

$$_{k^2=0}\langle 0|[K_0(p),e^{\sqrt{2}Q(1)\cdot k}]|0\rangle_{p=0}=0. \quad \text{(D8)}$$

Similarly, one shows that

$$_{k^2=0}\langle 0|(K_1)^l e^{\sqrt{2}Q(1)\cdot k}|0\rangle_{p=0}=0, \quad \text{(D9)}$$

etc. Q.E.D.

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Exact Consequences of Broken $O(4)$ Symmetry. III. Factorization and Mass Dependence*

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We use analyticity arguments to obtain the constraints imposed by factorization and broken $O(4)$ symmetry on Regge daughter sequences coupled to spinless external particles. Our results hold for arbitrary external masses.

I. INTRODUCTION

IN Papers I and II of this series, we derived the consequences of broken $O(4)$ symmetry for Regge-daughter sequences corresonding to Toller poles with $M=0$,[1] and integer $M\geqslant 1$.[2] Our results were not based on $O(4)$ symmetry directly, but rather on the requirement that scattering amplitudes be analytic at zero total energy ($t=0$ in our notation). Thus, our work is partially a derivation of new results, and partially a demonstration that the requirement of analyticity is interchangeable with the study of $O(4)$ symmetry.

In Papers I and II we emphasized the trajectory functions $\alpha(k,t)$, $k=0, 1, 2, \ldots$, which make up a daughter sequence. Here we wish to study the reduced residues $\gamma(k,t)$, and the many new complications which arise in a discussion of them. In the present paper we bypass the complications connected with spinning external particles. Among these are conspiracy relations and the requirement of factorization in the helicity indices. We do this by studying reactions involving spinless particles, as in Paper I. Accordingly, we can study only sequences with $M=0$, as coupled to spinless channels. However, even for this restricted case much remains to be shown beyond the results derived in Paper I. First, we want results which are valid for arbitrary external masses, and which show how the odd daughters decouple when either the initial or the final particle pair have equal masses. Second, we must impose the requirement of factorization, and verify that the daughter sequence constitutes a Toller pole when coupled to equal-mass initial and final particle pairs. Third, we want to find out if factorization is *necessary* to get a Toller pole, starting from analyticity requirements. It has been known for some time that analyticity and factorization are *sufficient* to get a Toller pole in equal-mass scattering,[3–5] but a Toller pole might also result from analyticity and continuity in the masses. We verify that factorization is necessary.

A modification of the procedures of Paper I must be made so that our results will be valid for arbitrary masses. We must recognize that a pseudothreshold (a point where the c.m. three-momentum vanishes) moves to $t=0$ when channel masses become equal. We must therefore deal only with functions which are analytic at pseudothresholds, as well as $t=0$, if we want our results to be valid in a neighborhood of $t=0$ for all mass configurations. For example, the function $R(k,t)$ defined in Eq. (I10), which differs from $\gamma(k,t)$ by a kinematic factor, has a square-root singularity at pseudothresholds for odd k because of the kinematic factor. Such kinematic singularities must be avoided. Our procedure for doing this is to use the expansion of $Q_{-\alpha(t)-1}(-z_t)$ in terms of powers of z_t^{-2} instead of the expansion in powers of $(1+z_t)^{-1}$ used in Paper I. The new choice entails one more summation, but is useful for all mass configurations.

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[2] J. B. Bronzan, Phys. Rev. **181**, 2111 (1969), Paper II.

[3] J. C. Taylor, Nucl. Phys. **B3**, 504 (1967).

[4] J. B. Bronzan and C. E. Jones, Phys. Rev. Letters **21**, 564 (1968).

[5] P. Di Vecchia and F. Drago, Phys. Letters **27B**, 387 (1968).

The main results we derive can be summarized in a few equations. First, we rederive the expression for $\alpha(k,t)$ originally given in Paper I:

$$\alpha(k,t)=\alpha_0-k+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{\partial^n}{\partial\alpha_0{}^n}[\mathcal{A}(k,t,\alpha_0)]^{n+1},$$
$$\mathcal{A}(k,t,\alpha_0)=\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}A_i{}^q\frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}. \quad (1)$$

The $A_i{}^q$ are arbitrary constants. We show in Appendix C that they may be taken to be independent of α_0 without loss of generality. From Eq. (1) one can derive the relations among the derivatives of $\alpha(k,t)$ discussed in Paper I. Second, there are new results for the factorized residue functions:

$$R(k,t)=F(k,t,\sigma,\delta)F(k,t,\sigma',\delta'),$$
$$F(k,t,\sigma,\delta)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}\left\{\mathfrak{F}(k,t,\alpha_0,\sigma,\delta)\times\left[1-\frac{\partial\mathcal{A}(k,t,\alpha_0)}{\partial\alpha_0}\right]^{1/2}[\mathcal{A}(k,t,\alpha_0)]^n\right\}, \quad (2)$$
$$\mathfrak{F}(k,t,\alpha_0,\sigma,\delta)=[(-1)^kM^{-1}{}_{k0}(\alpha_0)2^{-k}]^{-1/2}\times\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}\frac{b_i{}^q(\sigma,\delta)}{(\sigma\delta)^i}N^{-1}{}_{ki}(\alpha_0,\sqrt{\Delta}).$$

Here

$$\sigma=m_1+m_2,\quad \delta=m_1-m_2,$$
$$\sqrt{\Delta}=(\sigma^2-t)(\delta^2-t)/\sigma^2\delta^2, \quad (3)$$

and m_1 and m_2 are the masses of the initial pair of particles in the t channel. $R(k,t)$ differs from the standard reduced residue by kinematic factors given in Eq. (7), and M^{-1} and N^{-1} are matrices given in Eqs. (20) and (15), respectively. The coupling parameters $b_i{}^q(\sigma,\delta)$ are arbitrary analytic functions in the masses of the external particles, and according to Appendix C may be taken to be independent of α_0. In Appendix E we show that Eq. (2) is equivalent to the results of Durand *et al.*,[6] which were presented without proof. Equations (1) and (2), taken together, are necessary and sufficient for the contribution of the factorizing daughter sequence to be analytic at $t=0$.

II. ANALYTICITY CONDITIONS

A. Step I

The constraints imposed by analyticity, summarized by Eqs. (22) and (23), are derived in three steps paralleling the development in Paper I. Here we stress only the modifications which arise because the masses are arbitrary, and refer the reader to Paper I for many of the details.

We consider the spinless scattering process $m_1+m_2\rightarrow m_3+m_4$ in the t channel. The cosine of the t-channel scattering angle is

$$z_t=(\sigma\delta\sigma'\delta'+2xt)/4qq't, \quad (4)$$

where

$$x=u+\tfrac{1}{2}t-\tfrac{1}{2}\Sigma,$$
$$\sigma=m_1+m_2,\quad \sigma'=m_3+m_4,$$
$$\delta=m_1-m_2,\quad \delta'=m_3-m_4, \quad (5)$$
$$\Sigma=m_1{}^2+m_2{}^2+m_3{}^2+m_4{}^2=\tfrac{1}{2}(\sigma^2+\delta^2+\sigma'^2+\delta'^2),$$
$$q^2=(\sigma^2-t)(\delta^2-t)/4t,\quad q'^2=(\sigma'^2-t)(\delta'^2-t)/4t.$$

Here q and q' are the initial and final c.m. three-momenta. We use x rather than s or u as the Mandelstam variable to go with t. The contribution of a Regge pole at $\alpha(t)$ to the full amplitude has the form $r(t)Q_{-\alpha(t)-1}(-z_t)$. As explained in the Introduction, we expand Q in powers of $z_t{}^{-2}$ to avoid spurious square-root singularities at pseudothresholds, where $q=0$ or $q'=0$. These pseudothresholds move to $t=0$ when either $\delta=0$ or $\delta'=0$, that is, when the initial or final channel is an equal-mass channel. Using the specified expansion, we eliminate pseudothreshold singularities and derive constraints from analyticity which are valid for all mass configurations. A glance at Eq. (4) shows that a further binomial expansion is required to convert the expansion in powers of $z_t{}^{-2}$ into a series in powers of x^{-1}. This complication introduces the index n in Eq. (6), below, and is the price one must pay to generalize Paper I to general mass scattering.

A daughter trajectory sequence, $\alpha(k,t)$, $k=0$, 1, 2, ..., is required to make the full amplitude analytic at $t=0$. To restore analyticity, we require $\alpha(k,0)=\alpha_0-k$, and the reduced residue of the kth daughter must behave like t^{-k} at $t=0$. We sum the contributions of these daughters, expanding Q and z_t as outlined above. We also make use of Eq. (I8), and obtain the full amplitude in the form

$$T(t,x)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{x^{\alpha_0-r}(\ln x)^s}{s!t^r}\left(\frac{\sigma\delta\sigma'\delta'}{2}\right)^r\left\{\frac{[\alpha(k,t)-\alpha_0+k]^sR(k,t)\Delta^n}{\Gamma(-\alpha(k,t)+n+\frac{1}{2})\Gamma(r-k-2n+1)\Gamma(\alpha(k,t)-r+k+1)n!2^{2n}}\right\}, \quad (6)$$

where

$$R(k,t)=\frac{-\pi^{3/2}2^ke^{-i\pi\alpha(k,t)}[2\alpha(k,t)+1]\gamma(k,t)}{\sin\pi\alpha(k,t)\,\cos\pi\alpha(k,t)(\sigma\delta\sigma'\delta')^k},$$
$$\Delta=\frac{(\sigma^2-t)(\delta^2-t)(\sigma'^2-t)(\delta'^2-t)}{\sigma^2\delta^2\sigma'^2\delta'^2}, \quad (7)$$

[6] L. Durand, P. M. Fishbane, S. A. Klein, and L. M. Simmons, Phys. Rev. Letters **23**, 201 (1969).

and $\gamma(k,t)$ is the conventional reduced residue multiplied by t^k.[7] The full amplitude must be analytic at $t=0$ for all x, so the analyticity conditions implied by Eq. (6) are

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial^q}{\partial t^q}\left\{\frac{[\alpha(k,t)-\alpha_0+k]^s R(k,t)\Delta^n}{\Gamma(-\alpha(k,t)+n+\frac{1}{2})\Gamma(r-k-2n+1)\Gamma(\alpha(k,t)-r+k+1)n!2^{2n}}\right\}_{t=0}=0\quad(0\leqslant q<r;\,0\leqslant s).\tag{8}$$

Equation (8) is analogous to Eq. (I11).

We can easily cast Eq. (8) into a form similar to Eq. (I14):

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{w=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{q!g(n,l)R^{(w)}(k,\alpha_0-k)f(k,\alpha_0,q-l,p+s)}{(p-w)!w!n!l!(q-l)!\Gamma(r-k-2n+1)2^{2n}}\frac{\partial^{p-w}}{\partial\alpha_0^{p-w}}\times[\Gamma(\alpha_0-r+1)\Gamma(-\alpha_0+k+n+\tfrac{1}{2})]^{-1}=0\quad(0\leqslant q<r;\,0\leqslant s).\tag{9}$$

Here

$$g(n,l)\equiv\frac{\partial^l}{\partial t^l}\Delta^n|_{t=0},\quad R^{(w)}(k,\alpha_0-k)\equiv\frac{\partial^w}{\partial\alpha(k,t)^w}R(k,\alpha(k,t))|_{\alpha_0-k},$$
$$f(k,\alpha_0,q-l,p+s)\equiv\frac{\partial^{q-l}}{\partial t^{q-l}}[\alpha(k,t)-\alpha_0+k]^{p+s}|_{t=0}.\tag{10}$$

The only modification involved in deriving Eq. (9) is an extra application of Leibniz's theorem because of the explicit t dependence of Δ. This gives rise to the sum over l. Continuing the manipulations of Paper I, we obtain an analog of Eq. (I18):

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{w=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^u g(n,l)}{p!w!n!l!u!(q-l)!\Gamma(r-k-2n+1)2^{2n}}\frac{\partial^p}{\partial\alpha_0^p}\left\{[\Gamma(\alpha_0-r+1)\Gamma(-\alpha_0+k+n+\tfrac{1}{2})]^{-1}\times\frac{\partial^u}{\partial\alpha_0^u}[R^{(w)}(k,\alpha_0-k)f(k,\alpha_0,q-l,w+u+p+s)]\right\}=0\quad(0\leqslant q<r;\,0\leqslant s).\tag{11}$$

We find, by the argument presented in Paper I, that the sum over p is redundant in Eq. (11). We therefore set $p=0$. At this point we evaluate the sum over n, and find the analyticity conditions in the form

$$\sum_{k=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^u(\partial^l/\partial t^l)\{F(\frac{1}{2}k-\frac{1}{2}r,\frac{1}{2}k-\frac{1}{2}r+\frac{1}{2},-\alpha_0+k+\frac{1}{2},\Delta)\}_{t=0}}{w!l!u!(q-l)!(r-k)!\Gamma(-\alpha_0+k+\frac{1}{2})}\frac{\partial^u}{\partial\alpha_0^u}\times[R^{(w)}(k,\alpha_0-k)f(k,\alpha_0,q-l,w+u+s)]=B_r^{(q,s)},\tag{12}$$

$$B_r^{(q,s)}=\begin{cases}0 & (0\leqslant q<r,\ \text{or}\ q<s)\\ \text{arbitrary} & (\text{otherwise}).\end{cases}$$

$B_r^{(q,s)}$ vanishes for $q<s$ by Eq. (I16).

B. Step II

We multiply Eq. (12) by t^q and sum over q. On the left-hand side the sums over q, l, and w can be done. The analyticity conditions take the form

$$\sum_{k=0}^{r}N_{rk}(\alpha_0,\Delta)\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}\frac{\partial^u}{\partial\alpha_0^u}\{R(k,\alpha(k,t,\alpha_0))[\alpha(k,t,\alpha_0)-\alpha_0+k]^{u+s}\}=\sum_{q=0}^{\infty}t^qB_r^{(q,s)},\tag{13}$$

where

$$N_{rk}(\alpha_0,\Delta)\equiv\frac{F(\frac{1}{2}k-\frac{1}{2}r,\frac{1}{2}k-\frac{1}{2}r+\frac{1}{2},-\alpha_0+k+\frac{1}{2},\Delta)}{(r-k)!\Gamma(-\alpha_0+k+\frac{1}{2})}.\tag{14}$$

The inverse of N is

$$N^{-1}{}_{ir}(\alpha_0,\Delta)=\frac{(-1)^{i-r}\Gamma(-\alpha_0+i+\frac{1}{2})F(\frac{1}{2}r-\frac{1}{2}i,\frac{1}{2}r-\frac{1}{2}i+\frac{1}{2},\alpha_0-i+\frac{3}{2},\Delta)}{(i-r)!}.\tag{15}$$

[7] In Paper I, $\gamma(k,t)$ is taken to be the coefficient of $u^{\alpha(k,t)}$, which is a slightly different definition.

The fact that these matrices are inverses can be checked by expanding the hypergeometric functions, computing $N^{-1}N$, and collecting coefficients of each power of Δ. We can now "invert" Eq. (13) to obtain the analog of Eq. (I28):

$$\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}\frac{\partial^u}{\partial\alpha_0{}^u}\{R(k,\alpha(k,t,\alpha_0))[\alpha(k,t,\alpha_0)-\alpha_0+k]^{u+s}\}=\sum_{q=s}^{\infty}t^q\sum_{i=0}^{q}B_i{}^{(q,s)}N^{-1}{}_{ki}(\alpha_0,\Delta). \quad (16)$$

We can proceed from this point as in Paper I, and formally sum over u. We note in Appendix B that the summation is a form of a result known in analysis as Lagrange's theorem. Applying Lagrange's theorem, Eq. (16) becomes

$$\frac{R(k,\alpha_0-k)[-z_0(k,t,\alpha_0)]^s}{\partial\alpha(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_0(k,t,\alpha_0)}}=\sum_{q=s}^{\infty}t^q\sum_{i=0}^{q}B_i{}^{(q,s)}N^{-1}{}_{ki}(\alpha_0,\Delta), \quad (17)$$

where $z_0(k,t,\alpha_0)$ is determined by

$$\alpha(k,t,\alpha_0+z_0(k,t,\alpha_0))=\alpha_0-k,\qquad \lim_{t\to 0}z_0(k,t,\alpha_0)=0. \quad (18)$$

An expansion of $-z_0(k,t,\alpha_0)$ in powers of t can be obtained from Eqs. (17) for $s=0$ and $s=1$. It is shown in Appendix A that

$$\sum_{q=s}^{\infty}t^q\sum_{i=0}^{q}B_i{}^{(q,s)}N^{-1}{}_{ki}(\alpha_0,\Delta)=(-1)^k2^{-k}\Delta^{k/2}\sum_{q=s}^{\infty}t^q\sum_{i=0}^{q}\tilde{B}_i{}^{(q,s)}M^{-1}{}_{ki}(\alpha_0). \quad (19)$$

Here the coefficients $\tilde{B}$ are independent of k and t, like the B's. The matrix M^{-1} appears extensively in Paper I, and is given by Eq. (I25) (we have here corrected a typographical error in Paper I):

$$M^{-1}{}_{ki}(\alpha_0)=\frac{(-1)^{k-i}(2\alpha_0-2k+1)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}. \quad (20)$$

Using Eq. (19), the discussion given in Paper I goes through, and the analyticity conditions achieve the form of Eq. (I36).

$$\frac{R(k,\alpha_0-k)}{\partial\alpha(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_0(k,t,\alpha_0)}}=\sum_{q=0}^{\infty}t^q\sum_{i=0}^{q}B_i{}^{(q,0)}N^{-1}{}_{ki}(\alpha_0,\Delta),$$
$$-z_0(k,t,\alpha_0)=\mathcal{A}(k,t,\alpha_0). \quad (21)$$

$\mathcal{A}(k,t,\alpha_0)$ is defined in Eq. (1).

C. Step III

We continue to follow Paper I, forming the series indicated there, and summing them by Lagrange's theorem. The final constraints imposed by analyticity on the trajectories and residues are

$$R(k,t)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}\{\mathcal{B}(k,t,\alpha_0)[\mathcal{A}(k,t,\alpha_0)]^n\}, \quad (22a)$$

$$\alpha(k,t)=\alpha_0-k+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{\partial^n}{\partial\alpha_0{}^n}[\mathcal{A}(k,t,\alpha_0)]^{n+1}, \quad (22b)$$

where

$$\mathcal{B}(k,t,\alpha_0)=\sum_{q=0}^{\infty}t^q\sum_{i=0}^{q}\frac{B_i{}^q}{(\sigma\delta\sigma'\delta')^i}N^{-1}{}_{ki}(\alpha_0,\Delta). \quad (23)$$

Equation (22b) is a direct restatement of Eq. (I46b). Equation (22a) is an improvement over Eq. (I46a) in that the power series in t does not diverge at the pseudothresholds. It will converge in a circle bounded by the thresholds, or by an intersection of trajectories. In any event, it has a finite radius of convergence in the equal-mass limit, with all the complications of this limit explicitly displayed in Eq. (23) in the explicit mass factor, or the matrix N^{-1}. The mass factor in Eq. (23) has been extracted from the expansion coefficients so that the $B_i{}^q$, as defined in Eq. (23), are analytic functions of σ, δ, σ', and δ' when these mass parameters vanish. The analyticity of the $B_i{}^q$ may be established by means of Eqs. (7) and (13), and the observation that the reduced residues have no kinematic singularities where the mass parameters vanish. We prove in Appendix C that $A_i{}^q$ and $B_i{}^q$ may be taken to be independent of α_0 without loss of generality. This result was asserted in Paper I without proof.

It is of interest to examine the content of Eq. (22a) in the equal-mass limit $\delta=\delta'=0$. Of course, there is no need for daughters to restore analyticity when the initial and final particles are pairwise equal. However, since daughters are required when either $\delta\neq 0$ or $\delta'\neq 0$, it is conceivable that constraints on equal-mass scattering could result by demanding continuity in δ and δ'. Equation (22a) is just the formula needed to test this hypothesis because the parameters $B_i{}^q$ are known to be analytic in δ and δ', with all the complications explicitly displayed. However, one readily finds that the conventional reduced residue is arbitrary at $t=\delta=\delta'=0$:

$$\gamma(k,t)/t^k|_{t=\delta=\delta'=0}=\sum_{i=0}^{\infty}\theta(k-2i+\tfrac{1}{2})C(i,k)B_{k-2i}{}^{k-2i}. \quad (24)$$

The constants $C(i,k)$ are known. The additional requirement of factorization is therefore necessary as well as sufficient to produce a Toller pole in equal-mass scattering.

When either $\delta=0$ or $\delta'=0$ (equal-unequal scattering), the even and odd daughter residues are independent of each other for all t. For $t=0$, the even (or odd) residues are not arbitrary, but have ratios in agreement with previous results.[4]

III. FACTORIZATION AND EQUAL-MASS LIMIT

A. Factorization of Residues

Equation (22a) can be summed by Lagrange's theorem, and put in the form

$$R(k,t,\sigma,\delta,\sigma',\delta')=\frac{\mathcal{B}(k,\,t,\,\alpha_0+z_1(k,t,\alpha_0),\,\sigma,\,\delta,\,\sigma',\,\delta')}{[1-\partial\mathcal{A}(k,t,\alpha_0)/\partial\alpha_0]_{\alpha_0+z_1(k,t,\alpha_0)}},\qquad(25)$$

where $z_1(k,t,\alpha_0)$ is the solution of

$$\mathcal{A}(k,\,t,\,\alpha_0+z_1(k,t,\alpha_0))=z_1(k,t,\alpha_0)\,,\qquad \lim_{t\to 0} z_1(k,t,\alpha_0)=0\,.\qquad(26)$$

Factorization can be imposed on Eq. (25), and it yields

$$R(k,t,\sigma,\delta,\sigma',\delta')=F(k,t,\sigma,\delta)F(k,t,\sigma',\delta'),\qquad(27)$$

where by Lagrange's theorem

$$F(k,t,\sigma,\delta)=\left[\frac{\mathcal{B}(k,\,t,\,\alpha_0+z_1(k,t,\alpha_0),\,\sigma,\,\delta,\,\sigma,\,\delta)}{1-\partial\mathcal{A}(k,t,\alpha_0)/\partial\alpha_0\,|_{\,\alpha_0+z_1(k,t,\alpha_0)}}\right]^{1/2}$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}\left\{\left[\left(1-\frac{\partial\mathcal{A}(k,t,\alpha_0)}{\partial\alpha_0}\right)\times\mathcal{B}(k,t,\alpha_0,\sigma,\delta,\sigma,\delta)\right]^{1/2}[\mathcal{A}(k,t,\alpha_0)]^n\right\}.\qquad(28)$$

In Appendix D we show that $\mathcal{B}^{1/2}$ has the representation

$$[\mathcal{B}(k,t,\alpha_0,\sigma,\delta,\sigma,\delta)]^{1/2}\equiv\mathfrak{F}(k,t,\alpha_0,\sigma,\delta)$$
$$=[(-1)^k M^{-1}{}_{k0}(\alpha_0)2^{-k}]^{-1/2}\sum_{q=0}^{\infty}t^q\sum_{i=0}^{q}\frac{b_i{}^q}{(\sigma\delta)^i}\times N^{-1}{}_{ki}(\alpha_0,\sqrt{\Delta})\,,\qquad(29)$$

where $\sqrt{\Delta}=(\sigma^2-t)(\delta^2-t)/\sigma^2\delta^2$. $b_i{}^q$ is seen to be analytic at $\sigma=0$ and $\delta=0$ by the argument applied in Sec. II C. Finally, we point out in Appendix C that the $b_i{}^q$ may be taken to be independent of α_0 without loss of generality. Equations (27)–(29) are recapitulated in Eq. (2).

B. Equal-Unequal and Equal-Equal Mass Limits

We can now display the behavior of the factorized residues in the limits of equal-unequal mass channels ($\delta=0$, $\delta'\neq0$), and equal-equal mass channels. Since our goal here is the modest one of verifying the presence of a Toller pole in the equal-equal case, we confine ourselves to the point $t=0$. In the case of an equal-mass channel, we first let $\delta\to0$ in Eq. (29), and find a leading behavior δ^{-k}. Because of the compensating factor in Eq. (7), this corresponds to a finite value of the reduced residue. Next we let $t\to0$, and find

$$F(k,t,\sigma,\delta)\to[(-1)^k2^{-k}M^{-1}{}_{k0}(\alpha_0)]^{-1/2}(\alpha_0-k+\tfrac{1}{2})\pi\delta^{-k}(\cos\pi\alpha_0)^{-1}\times\begin{cases}[t^{k/2}b_0{}^0(-1)^{k/2}2^{-k}]/[(\tfrac{1}{2}k)!\,\Gamma(\alpha_0-\tfrac{1}{2}k+\tfrac{3}{2})] & (k\text{ even})\\ [t^{(k+1)/2}(b_1{}^1/\sigma)(-1)^{(k-1)/2}2^{1-k}]/[(\tfrac{1}{2}k-\tfrac{1}{2})!\,\Gamma(\alpha_0-\tfrac{1}{2}k+1)] & (k\text{ odd})\,.\end{cases}\qquad(30)$$

Note that the $t=0$ couplings of the even and odd daughters to an equal-mass channel are independent, and that the odd-daughter couplings vanish more rapidly at $t=0$ than do the even-daughter couplings. This relative decoupling has the consequence that the odd daughters do not contribute to equal-equal mass scattering processes at $t=0$.

In the case of an unequal-mass channel, we simply let $t\to0$ into Eq. (29), with σ and δ finite. In this limit we find

$$F(k,t,\sigma,\delta)\to[(-1)^k2^{-k}M^{-1}{}_{k0}(\alpha_0)]^{-1/2}\frac{(-1)^k\pi^{1/2}\Gamma(\alpha_0+1)b_0{}^02^{2\alpha_0-k+1}(\alpha_0-k+\tfrac{1}{2})}{(\cos\pi\alpha_0)k!\,\Gamma(2\alpha_0-k+2)}\,.\qquad(31)$$

We can use Eqs. (30) and (31) to compute the ratios of the reduced residues at $t=0$. For completeness, we include the case of unequal-unequal mass channels.

$$\left.\frac{\gamma_{EE}(2n)/t^{2n}}{\gamma_{EE}(0)}\right|_{t=0}=\frac{(\sigma\sigma')^{2n}(2n)!}{2^{6n}(n!)^2}\frac{\Gamma(\alpha_0-n+1)\Gamma(\alpha_0+\tfrac{3}{2})}{\Gamma(\alpha_0+1)\Gamma(\alpha_0-n+\tfrac{3}{2})},$$
$$\left.\frac{\gamma_{EU}(2n)/t^{n}}{\gamma_{EU}(0)}\right|_{t=0}=\frac{(\sigma\sigma'\delta')^{2n}(-1)^n}{2^{4n}n!}\frac{\Gamma(\alpha_0+\tfrac{3}{2})}{\Gamma(\alpha_0-n+\tfrac{3}{2})},\qquad(32)$$
$$\left.\frac{\gamma_{UU}(k)}{\gamma_{UU}(0)}\right|_{t=0}=\frac{(\sigma\delta\sigma'\delta')^k}{2^kk!}\frac{\Gamma(\alpha_0+1)\Gamma(\alpha_0+\tfrac{3}{2})(-1)^k}{\Gamma(\alpha_0-\tfrac{1}{2}k+1)\Gamma(\alpha_0-\tfrac{1}{2}k+\tfrac{3}{2})}.$$

Factorization is not necessary in order to get the equal-unequal and unequal-unequal results, whereas it is an essential ingredient for obtaining the equal-equal result. Note that the conventional reduced residues are nonsingular at $t=0$ in equal-equal scattering, and only half as singular in equal-unequal scattering as they are in the general case. The ratio of residues for equal-equal scattering corresponds to a single Toller pole.[4,8] For other mass configurations, we have computed the contribution of a Regge family at $t=0$ by summing the daughter contributions. The result is finite (since daughters were adjusted to remove the singularity at $t=0$), but much more complicated than a single Toller pole. The complication stems from the fact that the result depends upon all the constants $b_q{}^q$, $b_{q'}{}^{q'}$, and $A_{\bar{q}}{}^{\bar{q}}$, instead of only one. Probably the only noteworthy point about the formula is that only those constants appear whose upper and lower indices are the same. Of course, in the equal-equal limit the Toller pole emerges.

APPENDIX A

In this Appendix we sketch the proof of Eq. (19). The proof for $s>0$ follows from the proof for $s=0$, so we examine the case $s=0$. The relation to be established is equivalent to

$$\sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} \tilde{B}_i{}^q \sum_{k=0}^{\infty} N_{pk}(\alpha_0,\Delta) M^{-1}{}_{ki}(\alpha_0)(-1)^k 2^{-k}\Delta^{k/2} = \sum_{q=p}^{\infty} t^q B_p{}^q. \quad \text{(A1)}$$

Equation (A1) holds for arbitrary $\tilde{B}_i{}^q$ if and only if we can establish the behavior

$$\sum_{k=0}^{\infty} N_{pk}(\alpha_0,\Delta) M^{-1}{}_{ki}(\alpha_0)(-1)^k 2^{-k}\Delta^{k/2} = O(t^{p-i}) \quad (0\leqslant i\leqslant p). \quad \text{(A2)}$$

To prove Eq. (A2), we transform the hypergeometric function in N_{pk} so that $-\eta$ is its argument, where $\eta\equiv(1-\Delta)/\Delta$. Thus, we obtain the formula

$$\Delta^{k/2}N_{pk}(\alpha_0,\Delta) = \left[\frac{2^{-2\alpha_0+k+p-1}\Gamma(-\alpha_0+p)}{\pi^{1/2}(p-k)!\Gamma(-2\alpha_0+k+p)}\right]\frac{1}{(1+\eta)^{p/2}} \times F(\tfrac{1}{2}k-\tfrac{1}{2}p, \alpha_0-\tfrac{1}{2}k-\tfrac{1}{2}p+\tfrac{1}{2}, \alpha_0-p+1, -\eta). \quad \text{(A3)}$$

Since η is proportional to t for small t, Eq. (A2) is true if and only if the equation

$$\sum_{k=0}^{p} \frac{(-1)^k M^{-1}{}_{ki}(\alpha_0)}{(p-k)!\Gamma(-2\alpha_0+k+p)} \times F(\tfrac{1}{2}k-\tfrac{1}{2}p, \alpha_0-\tfrac{1}{2}k-\tfrac{1}{2}p+\tfrac{1}{2}, \alpha_0-p+1, -\eta) = O(\eta^{p-i}) \quad \text{(A4)}$$

is true. When we expand the hypergeometric function in a series, the coefficients of all powers of η with index less than $p-i$ must vanish. Thus, we must prove the equations

$$\sum_{k=0}^{p} M_{pk}(\alpha_0)M^{-1}{}_{ki}(\alpha_0) \times \frac{\Gamma(\tfrac{1}{2}k-\tfrac{1}{2}p+n)\Gamma(\alpha_0-\tfrac{1}{2}p-\tfrac{1}{2}k+\tfrac{1}{2}+n)}{\Gamma(\tfrac{1}{2}k-\tfrac{1}{2}p)\Gamma(\alpha_0-\tfrac{1}{2}p-\tfrac{1}{2}k+\tfrac{1}{2})} = 0 \quad (0\leqslant n<p-i), \quad \text{(A5)}$$

where

$$M_{pk}(\alpha_0) = \Gamma(2\alpha_0-p-k+1)/(p-k)! \quad \text{(A6)}$$

is the inverse matrix to M^{-1}. Equation (A5) is valid because of the relation

$$\frac{\Gamma(\tfrac{1}{2}k-\tfrac{1}{2}p+n)\Gamma(\alpha_0-\tfrac{1}{2}p-\tfrac{1}{2}k+\tfrac{1}{2}+n)}{\Gamma(\tfrac{1}{2}k-\tfrac{1}{2}p)\Gamma(\alpha_0-\tfrac{1}{2}p-\tfrac{1}{2}k+\tfrac{1}{2})} = \sum_{q=0}^{n} e_{pqn}\frac{M_{p-q,k}(\alpha_0)}{M_{pk}(\alpha_0)}, \quad \text{(A7)}$$

where the e's are independent of k. Equation (A7) is proved by the argument presented in the Appendix of Paper I.

APPENDIX B

In this series of papers we repeatedly have to sum series like Eq. (16) to obtain Eq. (17). Such sums are applications of Lagrange's theorem[9]:

"Let $f(z)$ and $\phi(z)$ be functions of z analytic on and inside a contour C surrounding a point a, and let t be such that the inequality

$$|t\phi(z)| < |z-a|$$

is satisfied on the perimeter of C; then the equation

$$\zeta = a + t\phi(\zeta),$$

regarded as an equation in ζ, has one root in the interior of C; and further any function of ζ analytic on and inside C can be expanded in a power series in t by the formula

$$f(\zeta) = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!}\frac{d^{n-1}}{da^{n-1}}\{f'(a)[\phi(a)]^n\}."$$

As an example of the use of this theorem, we derive Eq. (17). We first note that the derivatives can be rearranged to put the last equation in the form

$$f(\zeta) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{da^n}\left[f(a)\left(1-\frac{d}{da}[t\phi(a)]\right)[t\phi(a)]^n\right]. \quad \text{(B1)}$$

[8] The modification in Ref. 7 must be taken into account in reading Ref. 4.

[9] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge U. P., Cambridge, England, 1962), p. 133.

Next, we make the identifications

$$\zeta \to \alpha_0 + z_0(k,t,\alpha_0), \qquad a \to \alpha_0,$$

$$t\phi(a) \to -\alpha(k,t,\alpha_0)+\alpha_0-k,$$

$$f(a) \to \frac{R(k,\alpha(k,t,\alpha_0))[\alpha(k,t,\alpha_0)-\alpha_0+k]^s}{\partial\alpha(k,t,\alpha_0)/\partial\alpha_0}.$$

The equation for ζ now becomes an equation determining z_0:

$$\alpha(k,t,\alpha_0+z_0(k,t,\alpha_0))=\alpha_0-k. \tag{B2}$$

The equation for f becomes the desired sum

$$\sum_{u=0}^{\infty} \frac{(-1)^u}{u!} \frac{\partial^u}{\partial\alpha_0{}^u}\{R(k,\alpha(k,t,\alpha_0))[\alpha(k,t,\alpha_0)-\alpha_0+k]^{u+s}\} = \frac{R(k,\alpha_0-k)[-z_0(k,t,\alpha_0)]^s}{[\partial\alpha(k,t,q)/\partial q]_{q=\alpha_0+z_0(k,t,\alpha_0)}}. \tag{B3}$$

APPENDIX C

Here we prove that the parameters $A_i{}^q$ and $B_i{}^q$ in Eq. (22) may be taken to be independent of α_0 without loss of generality. We begin with the relation

$$\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} D_i{}^q(\alpha_0,\bar{\alpha}_0) \frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{\partial^n}{\partial\bar{\alpha}_0{}^n}\left[\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} \bar{D}_i{}^q(\bar{\alpha}_0) \times \frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}\right]^{n+1}. \tag{C1}$$

The validity of Eq. (C1) is established by means of the formulas

$$\frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)} = (-1)^i \frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)},$$
$$\frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)} = \sum_{q=0}^{i} c_{q,p}(\alpha_0) \frac{M^{-1}{}_{k,p+q}(\alpha_0)}{M^{-1}{}_{k,p}(\alpha_0)}. \tag{C2}$$

The second of these relations is proved in the Appendix of Paper I. Equation (C1) generates a relation between the D's and $\bar{D}$'s of the form

$$D_i{}^q(\alpha_0,\bar{\alpha}_0) = \bar{D}_i{}^q(\bar{\alpha}_0) + f_i{}^q\left(\alpha_0, \frac{\partial^p}{\partial\bar{\alpha}_0{}^p}\bar{D}_{i'}{}^{q'}(\bar{\alpha}_0)\right). \tag{C3}$$

Examination of Eqs. (C1) and (C2) shows that the coefficients which occur among the arguments of the $f_i{}^q$ will have $p<q$, $q'<q$, and $i'<i$. We use Eq. (C3) to define a new set of $A_i{}^q$'s from the set occurring in Eq. (22b):

$$A_i{}^q(\alpha_0) = \bar{A}_i{}^q(\alpha_0) + f_i{}^q\left(\alpha_0, \frac{\partial^p}{\partial\alpha_0{}^p}\bar{A}_{i'}{}^{q'}(\alpha_0)\right). \tag{C4}$$

Because of the restrictions on the arguments mentioned above, these are *not* differential equations for the $\bar{A}_i{}^q$'s. Rather, when evaluated for successively larger values of q and i, Eqs. (C4) define the $\bar{A}_i{}^q$ as functions of the $A_i{}^q$'s and their derivatives.

We define the functions

$$\mathfrak{A}(k,t,\alpha_0,\bar{\alpha}_0) = \sum_{q=1}^{\infty} t^q \sum_{i=0}^{q}\left[\bar{A}_i{}^q(\bar{\alpha}_0) + f_i{}^q\left(\alpha_0, \frac{\partial^p}{\partial\alpha_0{}^p}\bar{A}_{i'}{}^{q'}(\bar{\alpha}_0)\right)\right] \times \frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}, \tag{C5}$$

$$\mathfrak{A}^*(k,t,\alpha_0,\bar{\alpha}_0) = \sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} \bar{A}_i{}^q(\bar{\alpha}_0) \frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}.$$

We also introduce $z(k,t,\alpha_0,\bar{\alpha}_0)$, which satisfies the equation

$$z(k,t,\alpha_0,\bar{\alpha}_0) = \mathfrak{A}(k,t,\alpha_0+z(k,t,\alpha_0,\bar{\alpha}_0),\bar{\alpha}_0+z(k,t,\alpha_0,\bar{\alpha}_0)). \tag{C6}$$

In view of Eqs. (22b), (C3), and (C5), and of Lagrange's theorem,

$$\alpha(k,t) = \alpha_0 - k + z(k,t,\alpha_0,\alpha_0). \tag{C7}$$

Also, Eqs. (C1), (C3), (C5), and Lagrange's theorem imply

$$\mathfrak{A}(k,t,\alpha_0,\bar{\alpha}_0) = \mathfrak{A}^*(k,t,\alpha_0,\bar{\alpha}_0 - \mathfrak{A}(k,t,\alpha_0,\bar{\alpha}_0)). \tag{C8}$$

Equation (C8) may be put in the form

$$\mathfrak{A}^*(k,t,\alpha_0,\bar{\alpha}_0) = \mathfrak{A}(k,t,\alpha_0,\bar{\alpha}_0 + \mathfrak{A}^*(k,t,\alpha_0,\bar{\alpha}_0)). \tag{C9}$$

At this point we introduce $\bar{z}(k,t,\alpha_0,\bar{\alpha}_0)$, which satisfies

$$\bar{z}(k,t,\alpha_0,\bar{\alpha}_0) = \mathfrak{A}^*(k,t,\alpha_0+\bar{z}(k,t,\alpha_0,\bar{\alpha}_0),\bar{\alpha}_0). \tag{C10}$$

Because of Eq. (C9), $\bar{z}$ also satisfies Eq. (C6), so that $\bar{z}=z$. Thus, Eqs. (C7) and (C10), taken with Lagrange's theorem, establish the representation

$$\alpha(k,t) = \alpha_0 - k + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{\partial^n}{\partial\alpha_0{}^n}\left[\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} \bar{A}_i{}^q(\bar{\alpha}_0) \frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}\right]^{n+1}_{\bar{\alpha}_0=\alpha_0}. \tag{C11}$$

This is the statement that the $A_i{}^q$'s may be taken to be independent of α_0 in Eq. (22b) without loss of generality.

We next prove that the $A_i{}^q$'s may be taken independent of α_0 in Eq. (22a). Lagrange's theorem can be used to evaluate the sum

$$R(k,t)=\frac{\mathcal{B}(k,\,t,\,\alpha_0+z(k,t,\alpha_0,\alpha_0))}{[1-\partial\mathcal{A}(k,t,q,p)/\partial q-\partial\mathcal{A}(k,t,q,p)/\partial p]_{q=p=\alpha_0+z(k,t,\alpha_0,\alpha_0)}}$$
$$=\frac{\mathcal{B}(k,\,t,\,\alpha_0+\bar{z}(k,t,\alpha_0,\bar{\alpha}_0))[1+\partial\mathcal{A}^*(k,t,q,\bar{\alpha}_0)/\partial\bar{\alpha}_0]}{1-\partial\mathcal{A}^*(k,t,q,\bar{\alpha}_0)/\partial q}\Bigg|_{q=\alpha_0+\bar{z}(k,t,\alpha_0,\bar{\alpha}_0);\ \bar{\alpha}_0=\alpha_0} \tag{C12}$$

The second form has been derived by differentiating Eq. (C8) with respect to α_0 and $\bar{\alpha}_0$. Equation (C12) shows that in the numerator $[1+\partial\mathcal{A}^*/\partial\bar{\alpha}_0]$ may be absorbed into $\mathcal{B}$. Lagrange's theorem now produces a representation which shows that the $A_i{}^q$ may be taken to be independent of α_0 in Eq. (22a):

$$R(k,t)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}\{\mathcal{B}(k,t,\alpha_0)[\mathcal{A}^*(k,t,\alpha_0,\bar{\alpha}_0)]^n\}_{\bar{\alpha}_0=\alpha_0}. \tag{C13}$$

Henceforth we use $\mathcal{A}(k,t,\alpha_0)$ of Eq. (1), with the $A_i{}^q$ independent of α_0.

The final point is to show that the $B_j{}^r$'s may be taken to be independent of α_0 in Eq. (22a). We begin with a relation analogous to Eq. (C1):

$$\sum_{r=0}^{\infty}t^r\sum_{j=0}^{r}E_j{}^r(\alpha_0,\hat{\alpha}_0)M^{-1}{}_{kj}(\alpha_0)=\sum_{n=0}^{\infty}\frac{[-\mathcal{A}(k,t,\alpha_0)]^n}{n!}\frac{\partial^n}{\partial\hat{\alpha}_0{}^n}\{\sum_{r=0}^{\infty}t^r\sum_{j=0}^{r}\hat{E}_j{}^r(\hat{\alpha}_0)M^{-1}{}_{kj}(\alpha_0)\}. \tag{C14}$$

Equation (C2) can be used to establish the validity of this relation. The relation between the E's and $\hat{E}$'s is

$$E_j{}^r(\alpha_0,\hat{\alpha}_0)=\hat{E}_j{}^r(\hat{\alpha}_0)+g_j{}^r\left(\alpha_0,\frac{\partial^p}{\partial\hat{\alpha}_0{}^p}\hat{E}_{j'}{}^{r'}(\hat{\alpha}_0)\right). \tag{C15}$$

The E's which occur among the arguments of g will have $p<r$, $r'<r$, and $j'<j$. We use Eq. (C15) to define a new set of $B_j{}^r$'s from the set occurring in Eq. (22a):

$$B_j{}^r(\alpha_0)=\hat{B}_j{}^r(\alpha_0)+g_j{}^r\left(\alpha_0,\frac{\partial^p}{\partial\alpha_0{}^p}\hat{B}_{j'}{}^{r'}(\alpha_0)\right). \tag{C16}$$

We define the sums

$$\mathcal{B}(k,t,\alpha_0,\hat{\alpha}_0)=\sum_{r=0}^{\infty}t^r\sum_{j=0}^{r}\left[\hat{B}_j{}^r(\hat{\alpha}_0)+g_j{}^r\left(\alpha_0,\frac{\partial^p}{\partial\hat{\alpha}_0{}^p}\hat{B}_{j'}{}^{r'}(\hat{\alpha}_0)\right)\right]M^{-1}{}_{kj}(\alpha_0),$$
$$\mathcal{B}^*(k,t,\alpha_0,\hat{\alpha}_0)=\sum_{r=0}^{\infty}t^r\sum_{j=0}^{r}\hat{B}_j{}^r(\hat{\alpha}_0)M^{-1}{}_{kj}(\alpha_0). \tag{C17}$$

By Eqs. (C14)–(C16),

$$\mathcal{B}(k,t,\alpha_0,\hat{\alpha}_0)=\mathcal{B}^*(k,\,t,\,\alpha_0,\,\hat{\alpha}_0-\mathcal{A}(k,t,\alpha_0)). \tag{C18}$$

Lagrange's theorem and Eqs. (C13), (C16), and (C17) show that

$$R(k,t)=\frac{\mathcal{B}(k,\,t,\,\alpha_0+z(k,t,\alpha_0),\,\alpha_0+z(k,t,\alpha_0))}{1-\partial\mathcal{A}(k,t,q)/\partial q\,|_{q=\alpha_0+z(k,t,\alpha_0)}},$$
$$z(k,t,\alpha_0)=\mathcal{A}(k,\,t,\,\alpha_0+z(k,t,\alpha_0)). \tag{C19}$$

Using Eq. (C18) and Lagrange's theorem,

$$R(k,t)=\frac{\mathcal{B}^*(k,\,t,\,z(k,t,\alpha_0)+\alpha_0,\,\alpha_0)}{1-\partial\mathcal{A}(k,t,q)/\partial q\,|_{q=\alpha_0+z(k,t,\alpha_0)}}$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}\{[\sum_{r=0}^{\infty}t^r\sum_{j=0}^{r}\hat{B}_j{}^r(\hat{\alpha}_0)M^{-1}{}_{kj}(\alpha_0)][\mathcal{A}(k,t,\alpha_0)]^n\}_{\hat{\alpha}_0=\alpha_0}. \tag{C20}$$

Thus, the B_j^{r}'s may be taken to be independent of α_0 in Eq. (22a). Stated differently, the B_j^{r}'s may be given *any* α_0 dependence without affecting the content of our equations. We can use this freedom to make the b_i^{q}'s of Eq. (29) independent of α_0. Altogether, then, the α_0 dependence of the parameters appearing in the final analyticity constraints may be ignored.

APPENDIX D

In this Appendix, we verify by explicit multiplication that the square root of $\mathcal{B}(k,t,\alpha_0,\sigma,\delta,\sigma,\delta)$ has the form given in Eq. (29). We begin with the identity

$$\sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} \frac{b_i^{q}}{(\sigma\delta)^i} N_{ki}^{-1}(\alpha_0,\sqrt{\Delta}) = (-1)^k 2^{-k} (\sqrt{\Delta})^{k/2} \sum_{q=0}^{q} t^q \sum_{i=0}^{q} \tilde{b}_i^{q} M^{-1}{}_{ki}(\alpha_0). \quad \text{(D1)}$$

The proof of Eq. (D1) is very similar to that given in Appendix A.

Equation (I33) reads

$$M^{-1}{}_{ki}(\alpha_0) M^{-1}{}_{ki'}(\alpha_0) = M^{-1}{}_{k0}(\alpha_0) \sum_{q=0}^{i} c_{q,i'}(\alpha_0) M^{-1}{}_{k,i'+q}(\alpha_0). \quad \text{(D2)}$$

Using Eq. (D2), it is easy to show that

$$\left[\sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} \tilde{b}_i^{q} M^{-1}{}_{ki}(\alpha_0)\right]^2 = M^{-1}{}_{k0}(\alpha_0) \sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} \tilde{B}_i^{q} M^{-1}{}_{ki}(\alpha_0). \quad \text{(D3)}$$

Therefore,

$$\left[(-1)^k 2^{-k} M^{-1}{}_{k0}(\alpha_0)\right]^{-1} \left[\sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} \frac{b_i^{q}}{(\sigma\delta)^i} N^{-1}{}_{ki}(\alpha_0,\sqrt{\Delta})\right]^2 = (-1)^k 2^{-k} \Delta^{k/2} \sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} \tilde{B}_i^{q} M^{-1}{}_{ki}(\alpha_0) = \sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} \frac{B_i^{q}}{(\sigma^2\delta^2)^i} N^{-1}{}_{ki}(\alpha_0,\Delta). \quad \text{(D4)}$$

We have used Eq. (19) to make the final step.

APPENDIX E

Here we show that our results for the trajectory and residue functions are equivalent to those given by Durand *et al.*[6]

Trajectory Functions

Equation (I34) is valid for all α_0. We use this freedom to replace α_0 by $\alpha(k,t)+k$, noting that the A_i^{q} can be taken to be independent of α_0 (Appendix C).

$$\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} A_i^{q} \frac{k!\Gamma(2\alpha(k,t)+k+2)}{(k-i)!\Gamma(2\alpha(k,t)+k-i+2)} = -z_0(k,t,\alpha(k,t)+k). \quad \text{(E1)}$$

We use Eqs. (I39) and (I42) to obtain the implicit formula for the trajectory functions given in Ref. 6.

$$\alpha(k,t)+k = \sum_{i=0}^{k} \frac{k!\Gamma(2\alpha(k,t)+k+2)}{(k-i)!\Gamma(2\alpha(k,t)+k-i+2)} t^i a_i(t). \quad \text{(E2)}$$

Here we have made the identification $a_0(0)=\alpha_0$.

Residue Functions

The Regge residues $\beta(k,t)$ given in Ref. 6 are related to our reduced residues in the following way:

$$\begin{aligned} \beta(k,t) &= \bar{\beta}(k,t,\sigma,\delta)\bar{\beta}(k,t,\sigma',\delta'), \\ \gamma(k,t) &= \bar{\gamma}(k,t,\sigma,\delta)\bar{\gamma}(k,t,\sigma',\delta'), \\ \bar{\beta}(k,t,\sigma,\delta) &= [2\alpha(k,t)+1]^{1/2} t^{-k/2} q^{\alpha(k,t)} \bar{\gamma}(k,t,\sigma,\delta). \end{aligned} \quad \text{(E3)}$$

The relations among kinematic variables are

$$\cosh\beta_1 = \frac{t+\sigma\delta}{(\sigma+\delta)t^{1/2}}, \quad \sinh\beta_1 = \frac{[(\sigma^2-t)(\delta^2-t)]^{1/2}}{(\sigma+\delta)t^{1/2}}, \quad \sqrt{\Delta} = \frac{(\sigma^2-t)(\delta^2-t)}{\sigma^2\delta^2} = \frac{t(\sigma+\delta)^2}{\sigma^2\delta^2}\sinh^2\beta_1. \quad \text{(E4)}$$

We begin by using Eqs. (2), (7), (28), (29), and (I42) to write our residue in the form

$$\bar{\gamma}(k,t,\sigma,\delta) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial\alpha_0{}^n} \left\{ [\mathcal{A}(k,t,\alpha_0)]^n \left[1 - \frac{\partial\mathcal{A}(k,t,\alpha_0)}{\partial\alpha_0}\right]^{1/2} \times \left[\frac{(\sigma\delta)^k(-1)^{k/2}[k!\Gamma(2\alpha_0-k+2)]^{1/2}}{(2\alpha_0-2k+1)} \times \sum_{i=0}^{\infty} \frac{t^i \bar{b}_i(t,\sigma,\delta)}{(\sigma\delta)^i} N^{-1}{}_{ki}(\alpha_0,\sqrt{\Delta})\right]\right\}. \quad \text{(E5)}$$

Here we have used Lagrange's theorem to make the shift $\alpha(k,t)+k \to \alpha_0$, and have absorbed factors depending upon α_0 into $\bar{b}_i$ (Appendix C). Next we use the identity

$$(-1)^k(\sigma\delta)^k \sum_{i=0}^{\infty} \frac{t^i \bar{b}_i(t,\sigma,\delta)}{(\sigma\delta)^i} N^{-1}{}_{ki}(\alpha_0,\sqrt{\Delta}) = t^{k/2}(\sigma+\delta)^k \Gamma(-\alpha_0+k+\tfrac{1}{2}) \sum_{i=0}^{\infty} \frac{t^{i/2} b_i(t,\sigma,\delta)}{(k-i)!} e^{(k-i)\beta_1} \times F(\alpha_0-k+1,\, i-k,\, 2\alpha_0-2k+2,\, 1-e^{-2\beta_1}). \quad \text{(E6)}$$

This identity is valid when $\bar{b}_i$ and b_i are analytic in δ at $\delta=0$. We substitute Eq. (E6) into (E5), separate factors depending on α_0 from b_i, and use Lagrange's theorem to sum over n. We find

$$\tilde{\beta}(k,t,\sigma,\delta)=\left[1-\frac{\partial\mathfrak{A}(k,t,\alpha(k,t)+k)}{\partial\alpha(k,t)}\right]^{-1/2} q^{\alpha(k,t)}\left[\frac{k!\Gamma(2\alpha(k,t)+k+2)}{2\alpha(k,t)+1}\right]^{1/2}\frac{(-1)^{k/2}\cos\pi\alpha(k,t)\Gamma(-\alpha(k,t)+\frac{1}{2})}{(\sigma+\delta)^{\alpha(k,t)}}$$
$$\times\sum_{i=0}^{\infty}\frac{t^{i/2}b_i(t,\sigma,\delta)e^{\beta_1(k-i)}F(\alpha(k,t)+1,\,i-k,\,2\alpha(k,t)+2,\,1-e^{-2\beta_1})}{(k-i)!\left[\Gamma(\alpha(k,t)+k-i+2)\Gamma(-\alpha(k,t)-k+i)\right]^{1/2}}. \quad \text{(E7)}$$

Equation (E7) is the residue given in Ref. 6 after conversion to our notation.[10]

The verification of Eq. (E6) proceeds differently for $\delta\neq0$ and $\delta=0$. In the unequal-mass case we first use Eq. (D1), and then proceed as in Appendix A. It is ufeful to change the argument of the hypergeometric function to $x=-(e^{2\beta_1}-1)^{-1}$ by means of a suitable transformation. We find that Eq. (E6) is true for $\delta\neq0$ if and only if

$$\sum_{k=0}^{\infty} M_{qk}(\alpha_0)\frac{(-1)^k(2\alpha_0-2k+1)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}F(i-k,\,-2\alpha_0+k+i-1,\,-\alpha_0+i,\,x)=O(x^{q-i})\quad(0\leqslant i\leqslant q). \quad \text{(E8)}$$

Equation (E8) is easily verified by using the hypergeometric series and computing the coefficient of x^n.

In the equal-mass case we apply the operator $\sum_k N_{rk}(\alpha_0,\sqrt{\Delta})$ to Eq. (E6). We take the limit $\delta\to0$ of the result. noting that only the highest term of the series for N_{rk} contributes. We change the argument of the hypergeometric function to $y^{-1}=\tanh^2\beta_1$, and find that Eq. (E6) is true for $\delta=0$ if and only if

$$\sum_{k=0}^{\infty}\frac{(-4y)^{-k}\Gamma(\alpha_0-r+k+\frac{1}{2})F(-\frac{1}{2}r+\frac{1}{2}i+k,\,-\frac{1}{2}r+\frac{1}{2}i+\frac{1}{2}+k,\,\alpha_0-r+2k+\frac{3}{2},\,1/y)}{k!(r-i-2k)!\Gamma(\alpha_0-r+2k+\frac{1}{2})}=O(y^0). \quad \text{(E9)}$$

This equation can be verified by using the hypergeometric series.

[10] See M. H. Rubin, Phys. Rev. **162**, 1551 (1967), for the formula for $d_{j,0,0}{}^{0,\sigma}(\beta)$.