New Dual Quark Models*

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On the basis of new representations of the projective group, we construct some new dual quark models whose spin and internal symmetry are not multiplicative. One model is a factorized theory of exotic states with broken exchange degeneracy, ninth mesons being optional. The exotic states are suppressed three units below the Pomeranchon. In another model, with spin-orbit coupling and curved trajectories, both spin ghosts and orbital ghosts are involved in the Ward identities.

I. INTRODUCTION

IN this paper, we report our work on a new dual[1] quark model which improves on the original (multiplicative) quark model[2,3] in a number of respects. The basic approach is to study representations of the projective group on a space spanned by the usual "orbital" operators, plus new quarklike operators carrying spin and internal symmetry labels. (a) Attaching, for example, just $SU(3)$ labels to the quarks, we construct a factorized theory of exotic states, suppressed three units below the Pomeranchon, in which ninth mesons are optional. In this model, the old (orbital) operators correspond to the Pomeranchon. If, to obtain all the Virasoro[4] identities, we set the Pomeranchon intercept equal to unity, then the "meson" octet occurs at zero mass. This formalism works as well for any Lie-algebraic symmetry. (b) Attaching just spin labels to the quarks, we construct a model with spin-orbit coupling (coupling between the quarks and the usual orbital operators). Our motivation for this lies in the fact that, now, both spin ghosts and orbital ghosts are involved in Ward identities, just as orbital ghosts are in the usual model. This involves us in a gauge problem of considerable complexity, but generally of the same sort found by Virasoro.[4] Proofs are difficult to make firm, but we find a strong suggestion that a realistic model [with spin-orbit coupling and $SU(3)$] may be free of ghosts and perhaps even tachyons. This latter is connected with the fact that trajectories are necessarily curved in the presence of spin-orbit coupling. We have not yet calculated the coefficients of the realistic model.

Before outlining the material, a few general comments are in order. (a) We are impressed that the requirement of an $O(2,1)$ group forms, in general, a set of "bootstrap" equations determining all parameters except the intercept of one trajectory and a universal slope.[5] Then, requiring the maximal number of Ward identities fixes the intercept. Thus broken symmetries in general turn out quantized in our approach. For example, at fixed Schwinger term (see Sec. IV A below), the curvature of the trajectories in the model of Sec. IV is fixed. Alternately, we can think of this model as a discrete breaking of a spin symmetry, which will map onto a discretely broken $SU(6)$ in the realistic model referred to above. (b) As far as we can tell, the only way to eliminate spin ghosts in our approach is through Ward identities with spin-orbit coupling. (c) In the spin-orbit models, as discussed in Sec. IV, the natural group structure turns out really to be $O(2,1)\otimes O(2,1)$, so that there are in general two infinite sets of Ward identities, just enough, it is hoped, to eliminate both spin and orbital ghosts. As mentioned in Sec. IV, we believe that this particular group structure is unavoidable, and that it is the natural group for models with spin. This group will, for example, appear in the realistic model as well. (d) In all our models, a candidate for the Pomeranchon (trajectory with vacuum quantum numbers) arises inescapably. This trajectory presumably has little to do with the diffractive part of the Pomeranchon which, we presume, arises from unitarity.

In Sec. II, we review the usual dual model in the local-field-theoretic formalism of Fubini and Veneziano.[6] We emphasize the general principle that a representation of the projective group containing a scalar field with c-number commutation relations automatically yields a set of factorized n-point functions. In Sec. III, we construct our first simple "additive" models, so called because there is no coupling between orbital and internal or spin operators. Concentrating on $SU(3)$, we obtain the exotic-state model mentioned above. We also give a simple broken-symmetry model. The formalism for the analogous additive model with spin is set up at the beginning of Sec. IV but, because this model has all the usual spin ghosts, the bulk of Sec. IV is devoted to the construction of the spin-orbit coupling model.

Having no internal symmetry, the model includes a number of curved trajectories: Pomeranchon and ninth vector mesons. The mass operator is explicitly diagonalized in the first two "sectors" (the natural excitation number for the model). The connections among the $O(2,1)\otimes O(2,1)$ Ward identities and con-

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[2] S. Mandelstam, Phys. Rev. **183**, 1374 (1969); Phys. Rev. D **1**, 1734 (1970); see also H. Harari, Phys. Rev. Letters **22**, 562 (1969); J. L. Rosner, *ibid.* **22**, 689 (1969).
[3] K. Bardakci and M. B. Halpern, Phys. Rev. **183**, 1456 (1969).
[4] M. A. Virasoro, Phys. Rev. D **1**, 2933 (1970).
[5] A single exception is the linear symmetry-breaking model of Sec. III, which contains one other arbitrary parameter.
[6] S. Fubini and G. Veneziano, Nuovo Cimento **67A**, 29 (1970); MIT report (unpublished).

comitant (gauge) degeneracy problems are studied. We should repeat here that we have not shown our prototype models to be free of tachyons and ghosts, but the situation is very suggestive. To be precise, we have studied two separate models: a four-dimensional (4-d) model in Sec. IV, and a five-dimensional analog in Sec. V. Their properties are as follows: 4-d model—no tachyons in the first two sectors, but only "leading" ghosts eliminated; 5-d model—a vacuum tachyon and at least one other which is eliminated by signature, but all Ward identities working. The presumed properties of the "realistic" model are sketched in Sec. VI.

Appendix A gives some sample commutators among quartics used in Sec. IV. Appendix B details our arguments that our scalar field in Sec. IV satisfies c-number commutation relations. Appendix C is a discussion of cyclic symmetry, twists, and single-valuedness problems relevant to Secs. IV and V. Appendix D argues a technical consistency condition between the scalar field and the vacuum.

II. USUAL OPERATOR FORMALISM

To begin, we summarize the usual operator formalism for dual-resonance models. In our notation, we introduce[7]

$$\begin{aligned}&[\pi_n{}^\mu,\pi_m{}^\nu]=g^{\mu\nu}m\delta_{n,-m}\,,\\&(\pi_n{}^\mu)^\dagger=\pi_{-n}{}^\mu,\quad -\infty<n<\infty\,,\ \mu=0,1,2,3.\end{aligned}\tag{2.1}$$

This is only a unified notation for the usual harmonic-oscillator operators:

$$\begin{aligned}&\pi_n{}^\mu\equiv-(\sqrt{n})a_n{}^\mu,\quad \pi_{-n}{}^\mu\equiv(\sqrt{n})a_n{}^{\mu\dagger},\quad n>0\\&[a_n{}^\mu,a_m{}^{\nu\dagger}]=g^{\mu\nu}\delta_{nm},\quad \pi_0{}^\mu=\sqrt{2}p^\mu,\end{aligned}\tag{2.2}$$

where p^μ is the external momentum.[8] Also, we list the local fields,

$$\begin{aligned}&Q^\mu(z)=-\frac{q^\mu}{\sqrt{2}}-\pi_0{}^\mu\ln z-\sum_{n=1}^{\infty}\frac{\pi_n{}^\mu}{n}z^n+\sum_{n=1}^{\infty}\frac{\pi_{-n}{}^\mu}{n}z^{-n}\,,\\&\pi^\mu(z)=-z\frac{d}{dz}Q^\mu(z)=\sum_{n=-\infty}^{\infty}\pi_n{}^\mu z^n\,,\\&[p^\mu,q^\nu]=g^{\mu\nu}.\end{aligned}\tag{2.3}$$

For purposes of Hermitian conjugation and for elegance of notation, we think of z on the unit circle $z=e^{i\theta}$, with $z^*=z^{-1}$. The local algebra of π and Q is

$$\begin{aligned}&[Q^\mu(z),Q^\nu(z')]=i\pi\epsilon(\theta-\theta')g^{\mu\nu}\,,\\&[Q^\mu(z),\pi^\nu(z')]=g^{\mu\nu}2\pi iz\delta(z-z')=2\pi\delta(\theta-\theta')g^{\mu\nu}\,,\\&[\pi^\mu(z),\pi^\nu(z')]=-2\pi izz'g^{\mu\nu}\frac{d}{dz}\delta(z-z')\\&\qquad=2\pi i\frac{d}{d\theta}\delta(\theta-\theta')g^{\mu\nu}\,,\end{aligned}\tag{2.4}$$

where $\epsilon(\theta)=\theta/|\theta|$. From this algebra, it is easy to show that the generators of the projective group $O(2,1)$ are

$$\begin{aligned}&L_0=-\frac{1}{4\pi}\int_0^{2\pi}d\theta\,{:}\pi^2(\theta){:}\,,\quad \pi^2(\theta)=\pi^\mu(\theta)\pi_\mu(\theta)\,,\\&L_\pm=-\frac{1}{4\pi}\int_0^{2\pi}d\theta\,e^{\mp i\theta}\pi^2(\theta)\,,\\&[L_0,L_\pm]=\mp L_\pm\,,\quad [L_+,L_-]=2L_0\,,\end{aligned}\tag{2.5}$$

where colons indicate normal ordering. Thus, $\pi^2(\theta)$ acts as a local density for the algebra. In fact, the generators of the full conformal group can be constructed in the same manner as

$$\begin{aligned}&L_m=-\frac{1}{4\pi}\int_0^{2\pi}d\theta\,e^{-im\theta}{:}\pi^2(\theta){:}\,,\\&L_+\equiv L_1\,,\quad L_-\equiv L_{-1}.\end{aligned}\tag{2.6}$$

For reference, we list explicitly

$$L_0=-\sum_{n=1}^{\infty}\pi_{-n}{}^\mu\pi_{\mu,n}-p^2=R-p^2\,,$$

$$L_+=-\sum_{n=0}^{\infty}\pi_{-n}{}^\mu\pi_{\mu,n+1}\,,\quad L_-=-\sum_{n=0}^{\infty}\pi_{-n-1}{}^\mu\pi_{\mu,n}\,.$$

Q and π transform as a projective scalar and vector, respectively:

$$\begin{aligned}&[L_0,Q^\mu(\theta)]=i\partial_\theta Q^\mu(\theta),\quad [L_\pm,Q^\mu(\theta)]=ie^{\mp i\theta}\partial_\theta Q^\mu(\theta)\,,\\&[L_0,\pi^\mu(\theta)]=i\partial_\theta\pi^\mu(\theta),\quad [L_\pm,\pi^\mu(\theta)]\\&\qquad=e^{\mp i\theta}[i\partial_\theta\pi^\mu(\theta)\pm\pi^\mu(\theta)]\,,\end{aligned}\tag{2.7}$$

and, in particular, $Q^\mu(z)=z^{-L_0}Q^\mu(1)z^{L_0}$.

Finally, as shown in Ref. 6, the n-point functions are of the form

$$\begin{aligned}&B_n(k_1,\dots,k_n)\\&\quad=\lim_{z_1\to0;\,z_2\to1;\,z_n\to\infty}\int dz_3\cdots dz_{n-1}\rho(z_1,\dots,z_n)\\&\qquad\times{}_{p=0}\langle0|\Gamma(k_1,z_1)\cdots\Gamma(k_n,z_n)|0\rangle_{p=0}\,,\end{aligned}\tag{2.8}$$

where $|0\rangle_{p=0}$ is the projective vacuum,

$$\pi_l{}^\mu|0\rangle_{p=0}=p^\mu|0\rangle_{p=0}=\mathbf{L}|0\rangle_{p=0}=0\quad(l>0)\,,$$

and Γ is the ground-state vertex

$$\Gamma(k,z)=z^{-L_0}{:}e^{\sqrt{2}k\cdot Q(1)}{:}z^{L_0}z^{-k^2}.\tag{2.9}$$

Here ρ is a suitable projective invariant volume factor, and the k's are the external momenta. Factorization is obvious in this formalism. The proof of cyclic symmetry rests on two facts. (a) That we have a projective scalar E (here Q) under some algebra $\mathbf{J}$ (here $\mathbf{L}$), which is exponentiated in forming Γ. Thus the integrand is

[7] Our metric is $-g^{ii}=+g^{00}=+1$, $p^2=p_0{}^2-|\mathbf{p}|^2$.

[8] More precisely, p^μ is the external momentum if such are all taken outgoing. If momenta are ingoing, then p^μ is the negative of the external momentum.

projective invariant. (b) That E satisfies c-number commutation relations with itself. It is this prescription, namely, that projective invariance plus c-number commutation relations imply cyclic duality, that we shall apply to find other models.

The n-point functions can be written in an alternative form with all the integrations carried out. Using the result

$$\lim_{z\to\infty}\Gamma(k,z)|0\rangle_{p=0}=|0\rangle_{k^2}\equiv e^{-k\cdot q}|0\rangle_{p=0}, \qquad (2.10)$$

we have

$$B_n(k_1,\ldots,k_n)=\langle 0|V(k_2)\Delta(s_{12})V(k_3)\cdots V(k_{n-1})|0\rangle,$$

where

$$s_{1i}=(k_1+\cdots+k_i)^2, \quad V(k)=\Gamma^{(-)}(k)\Gamma^{(+)}(k),$$

$$\Gamma^{(+)}(k)=\exp\left(-\sqrt{2}k\cdot\sum_{n=1}^{\infty}\frac{\pi_n}{n}\right),$$

$$\Gamma^{(-)}(k)=\exp\left(\sqrt{2}k\cdot\sum_{n=1}^{\infty}\frac{\pi_{-n}}{n}\right),$$

$$\Delta(s)=\int_0^1 dz\, z^{J_0-1-a_0}(1-z)^{a_0-1} =\frac{\Gamma(J_0-a_0)\Gamma(a_0)}{\Gamma(J_0)}, \qquad (2.11)$$

with a_0 the intercept of the trajectory.

Finally, a useful trick is worth noting. If $E(1)$ satisfies the "stability condition" at $z=1$,

$$[(J_\pm-J_0),E(1)]=0,$$

then

$$E(z)=z^{-J_0}E(1)z^{J_0} \qquad (2.12)$$

is a scalar under $\mathbf{J}$.

Our program in this paper will then be to find new representations of the projective group which allow (a) and (b) above.

III. ADDITIVE MODELS

In this section, we shall develop a formalism for incorporating an arbitrary symmetry group into the projective formalism. To be concrete, we shall work with $SU(3)$, limiting ourselves to occasional remarks on the general problem.

We begin by defining a set of new[9] quarklike operators (to be used in conjunction with $\pi_l{}^\mu$) satisfying the following anticommutation relations:

$$\begin{aligned}&[b_r(n),b_s{}^\dagger(m)]_+=\delta_{rs}\delta_{nm},\\ &[d_r(n),d_s{}^\dagger(m)]_+=\delta_{rs}\delta_{nm}, \quad n\geq 0\end{aligned} \qquad (3.1)$$

while the b's and d's anticommute among themselves. The label r runs from 1 to 3, corresponding to the quark representation of $SU(3)$ for $b_r(n)$, and antiquark representation for $d_r(n)$. The vacuum satisfies

$$b_r(n)|0\rangle=d_r(n)|0\rangle=0.$$

Now define a field

$$\psi_r(z)\equiv\sum_{n=0}^{\infty}[b_r(n)z^{n+1/2}+d_r{}^\dagger(n)z^{-n-1/2}], \qquad (3.2)$$

which satisfies the local anticommutation relation[4]

$$[\psi_r(z),\psi_s{}^\dagger(z')]_+=\delta_{rs}2\pi iz\delta(z-z')=2\pi\delta(\theta-\theta')\delta_{rs}. \qquad (3.3)$$

The purpose of the half-integral powers in ψ is, for the moment, to avoid introducing a zero-fermion mode.[10] Later, we shall see that such is necessary to have ψ transform as a projective spinor and $\psi^\dagger\psi$ as a vector.

We can now immediately construct an $O(2,1)$ algebra:

$$\begin{aligned}S_0&=-\frac{i}{4\pi}\int_0^{2\pi}d\theta:\psi_r{}^\dagger\overleftrightarrow{\partial_\theta}\psi_r:,\\ S_\pm&=-\frac{i}{4\pi}\int_0^{2\pi}d\theta\, e^{\mp i\theta}\psi_r{}^\dagger\overleftrightarrow{\partial_\theta}\psi_r,\\ (S_+,S_-)&=2S_0, \quad [S_0,S_\pm]=\mp S_\pm.\end{aligned} \qquad (3.4)$$

Explicitly, we have

$$\begin{aligned}S_0&=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})[b^\dagger(n)b(n)+d^\dagger(n)d(n)],\\ S_+&=\sum_{n=0}^{\infty}(n+1)[b^\dagger(n)b(n+1)+d^\dagger(n)d(n+1)],\\ S_-&=\sum_{n=0}^{\infty}(n+1)[b^\dagger(+n+1)b(n)+d^\dagger(+n+1)d(n)].\end{aligned} \qquad (3.4')$$

Under the group, ψ transforms as a projective spinor,

$$\begin{aligned}&[S_0,\psi(\theta)]=i\partial_\theta\psi(\theta),\\ &[S_\pm,\psi(\theta)]=e^{\mp i\theta}[i\partial_\theta\psi(\theta)\pm\tfrac{1}{2}\psi(\theta)],\end{aligned} \qquad (3.5)$$

so that $\psi^\dagger\psi$ and $\psi^\dagger\lambda^\alpha\psi$ [λ^α are the usual[11] 3×3 matrices of $SU(3)$] transform as vectors.

We define the full model as being generated by the sum of the two projective angular momentum operators given by Eqs. (2.5) and (3.4),

$$\mathbf{J}=\mathbf{L}+\mathbf{S}; \qquad (3.6)$$

hence the name additive.

The J's also generate $O(2,1)$, and Q, π, and ψ transform under this new algebra as originally stated. In particular, $Q(z)$ is still a scalar under $\mathbf{J}$. Thus, our n-

[9] It would not do to put spin and internal symmetry labels directly on the orbital operators (e.g., $a_{\alpha,n}{}^\mu$), because this leads directly to exotic states, etc., on the leading trajectory. Hence we need new operators.

[10] We do not know what external additive quantity could couple to a zero-fermion mode, especially when the mode carries spin and internal symmetry.

[11] We normalize the λ's as usual to $\mathrm{Tr}(\lambda^\alpha\lambda^\beta)=2\delta^{\alpha\beta}$.

TABLE I. Low-lying states of the additive model.

Mass	State	Particle
$p^2=-a_0$	$\|0\rangle$	Scalar Pomeranchon
$p^2=-a_0+\frac{1}{2}$	$b^{r\dagger}(0)\|0\rangle$	Quark
$p^2=-a_0+\frac{1}{2}$	$d^{r\dagger}(0)\|0\rangle$	Antiquark
$p^2=-a_0+1$	$\pi_{-1}{}^{\mu}\|0\rangle$	Vector Pomeranchon
$p^2=-a_0+1$	$b^{r\dagger}(0)b^{r'\dagger}(0)\|0\rangle$	Di-quark
$p^2=-a_0+1$	$d^{r\dagger}(0)d^{r'\dagger}(0)\|0\rangle$	Di-antiquark
$p^2=-a_0+1$	$\sum_r b^{r\dagger}(0)d^{r\dagger}(0)\|0\rangle$	Singlet meson
$p^2=-a_0+1$	$\sum_{rr'}(\lambda_\alpha)^{rr'}b^{r\dagger}(0)d^{r'\dagger}(0)\|0\rangle$	Octet meson

point amplitudes can be constructed as before. Before we undertake this construction, we digress to the particle content of the theory.

A. Spectrum

The model has a variety of parallel straight trajectories. The states $|0\rangle$, $\pi_{-1}{}^{\mu}|0\rangle$, and $\pi_{-1}{}^{\mu}\pi_{-1}{}^{\nu}|0\rangle$, obtained by the successive application of π's on the vacuum, correspond to a vacuum trajectory (Pomeranchon). By Pomeranchon, we mean a trajectory like any other one, only with quantum numbers of vacuum. Of course, we do not expect it to correspond to the "diffractive" part of the Pomeranchon arising from unitarity corrections. There is also a leading quark-antiquark (octet and singlet) trajectory one unit below the Pomeranchon,

$$b_r{}^{\dagger}(0)d_s{}^{\dagger}(0)|0\rangle,\ b_r{}^{\dagger}(0)d_s{}^{\dagger}(0)\pi_{-1}{}^{\mu}|0\rangle,\ldots. \tag{3.7}$$

Thus, if we take the Pomeranchon with unit intercept, guaranteeing all the Virasoro identities, then the lowest meson octet occurs at zero mass. "Nucleons" also occur in the model as states of three quarks. A brief tabulation of the lowest masses and their properties is given in Table I (with Pomeranchon intercept a_0). In the meson sector, there are exotic states constructed out of four or more quarks. For example,

$$b_{r_1}{}^{\dagger}(n_1)b_{r_2}{}^{\dagger}(n_2)d_{r_3}{}^{\dagger}(n_3)d_{r_4}{}^{\dagger}(n_4)|0\rangle \tag{3.8}$$

contains in general exotic-state components. The lowest possible exotic state (2 units below Pomeranchon) would be in (3.8) with $n_1=n_2=n_3=n_4=0$. However, this state is antisymmetric in r_1 and r_2 and again in r_3 and r_4 (by Fermi statistics), and hence gives rise only to nonexotic states. Thus the lowest-lying exotic states in the model are 3 units below the Pomeranchon.

The scheme evidently accommodates states of nonzero triality (e.g., quark and di-quark states). However, if the external states have zero triality, so do the internal ones, and we can restrict ourselves to a model of only zero-triality particles. Finally, we note that we have a choice concerning whether we want ninth mesons. If we choose to have them, they are degenerate with the octets, but couple only to themselves. That we have this choice is clearly related to the existence of nonleading exotic states in the model.

An elegant way of summarizing this discussion is to note that the model can be rewritten in terms of "currents." Define the projective vectors

$$J^\alpha(\theta)\equiv\psi^\dagger(\theta)\lambda^\alpha\psi(\theta),\quad J(\theta)\equiv\psi^\dagger(\theta)\psi(\theta). \tag{3.9}$$

These objects satisfy current-algebraic commutation relations,

$$\begin{aligned}[][J^\alpha(\theta),J^\beta(\theta')]&=4\pi i f^{\alpha\beta\gamma}J^\gamma(\theta)\delta(\theta-\theta')\\&\quad-4\pi i\delta^{\alpha\beta}\partial_\theta\delta(\theta-\theta'),\\ [J^\alpha(\theta),J(\theta')]&=0,\quad [J(\theta),J(\theta')]=-6\pi i\partial_\theta\delta(\theta-\theta').\end{aligned} \tag{3.10}$$

In "momentum" space these become

$$\begin{aligned}[][J_l{}^\alpha,J_m{}^\beta]&=2if^{\alpha\beta\gamma}J_{l+m}{}^\gamma+2l\delta^{\alpha\beta}\delta_{l,-m},\\ J^\alpha(\theta)&=\sum_{m=-\infty}^{+\infty}e^{im\theta}J_m{}^\alpha,\quad J_m{}^\alpha|0\rangle=0\text{ for }m\geqq 0.\end{aligned} \tag{3.11}$$

After a little algebra, it can be shown[12] that the following quartic structures reduce precisely to the previously given bilinear forms for $\mathbf{S}$:

$$\begin{aligned}S_0&=\frac{1}{32\pi}\int_0^{2\pi}d\theta[:J^\alpha(\theta)J^\alpha(\theta):+:J(\theta)J(\theta):]\\&=\tfrac{1}{32}[J_0{}^\alpha J_0{}^\alpha+J_0J_0\\&\qquad+2\sum_{n=1}^{\infty}(J_{-n}{}^\alpha J_n{}^\alpha+J_{-n}J_n)],\\ S_\pm&=\frac{1}{32\pi}\int_0^{2\pi}d\theta\, e^{\mp i\theta}[J^\alpha(\theta)J^\alpha(\theta)+J(\theta)J(\theta)]\end{aligned} \tag{3.12}$$

(α summed from 1 to 8). Thus, if we wish, we can stay entirely in the subspace of these operators. Finally, the ninth meson can be consistently dropped from the above equations if so desired. These results will be useful in what follows.[13]

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[12] This observation is due to S. Mandelstam (private communication).

[13] One might ask whether the orbital operators themselves have a quark representation, so that one could rewrite even the original dual model in terms of anticommuting operators. The answer, interestingly enough, is yes, but it is not very illuminating. In terms of scalar anticommuting quark fields with no indices, we construct

$$S(\theta)=\psi^\dagger(\theta)\psi(\theta)=\sum_{l=-\infty}^{+\infty}e^{il\theta}S_l;$$

then $[S_l,S_m]=i\delta_{l,-m}$, so that we could represent $\pi_l\leftrightarrow iS_l$ if π had no Lorentz index. This is amusing because representation of commuting operators in terms of anticommuting operators can only be possible for a system with an infinite number of degrees of freedom. The Lorentz index is a problem. The best we can do is to introduce four quarks ψ^μ, where $\mu=0$, 1, 2, 3, and write $\pi_l{}^\mu=(\psi^{+\mu}\psi^\mu)_l$; of course, these quarks have no simple Lorentz-transformation properties.

B. Vertices

First consider the case of n Pomeranchons. The vertex for each is (for unit intercept)

$$\begin{aligned}\Gamma_P(k,z) &= z^{+1}z^{-J_0}{:}e^{\sqrt{2}Q(1)\cdot k}{:}z^{J_0}\\ &= z^{+1}z^{-L_0}{:}e^{\sqrt{2}Q(1)\cdot k}{:}z^{L_0}, \quad k^2=-1 \end{aligned}\tag{3.13}$$

which is the usual vertex of the ordinary dual models. Thus the n-Pomeranchon amplitudes are the usual n-point beta functions, and the n Pomeranchons couple only to themselves.

Now we turn to excited-state vertices. We list (still for unit Pomeranchon intercept)

$$\begin{aligned}\Gamma_{\text{quark}}(k,z) &= z^{+1}z^{-J_0}{:}e^{\sqrt{2}Q(1)\cdot k}{:}\psi^\dagger(1)z^{J_0}\\ &= z^{+1}z^{-L_0}{:}e^{\sqrt{2}Q(1)\cdot k}{:}z^{L_0}\psi^\dagger(z) \quad (k^2=-\tfrac{1}{2}),\\ \Gamma^\alpha{}_{\text{meson}}(k,z) &= z^{+1}z^{-J_0}{:}e^{\sqrt{2}Q(1)\cdot k}{:}J^\alpha(1)z^{J_0}\\ &= z^{+1}z^{-L_0}{:}e^{\sqrt{2}Q(1)\cdot k}{:}z^{L_0}J^\alpha(z) \quad (k^2=0).\end{aligned}\tag{3.14}$$

There are a number of ways of constructing these vertices: (a) through consideration of their projective transformation properties, or (b) by writing, e.g., $\langle 0|J_1{}^\alpha\Gamma_P\cdots\Gamma_P J_{-1}{}^\beta|0\rangle$ and making a cyclic change of variables, thus determining Γ_{meson}. In any case, the vertices can easily be guessed through the consistency condition on the vacuum, e.g.,

$$\begin{aligned}&\lim_{z\to\infty}\Gamma_{\text{quark}}(k,z)|0\rangle_{p=0}=b^\dagger(0)|0\rangle_{k^2=-1/2},\\ &\lim_{z\to\infty}\Gamma^\alpha{}_{\text{meson}}(k,z)|0\rangle_{p=0}=J_{-1}{}^\alpha|0\rangle_{k^2=0}\equiv|\alpha\rangle.\end{aligned}\tag{3.15}$$

The generalization to arbitrary intercept is left as an exercise for the reader.

With the local commutation (and anticommutation) relations for J and ψ, we can prove commutation relations for the vertices. These differ from the algebra of orbital or Pomeranchon vertices by terms involving extra factors of $\delta(z-z')$ and $(d/dz)\delta(z-z')$ from the J's and ψ's. Such terms are of no consequence, because the rest of the integrand (the "orbital skeleton") vanishes when any two z's are equal. Hence the proof of cyclic symmetry for excited-state n-point functions is essentially the same as in the usual model.[14]

C. Some Explicit Amplitudes

Taking volume elements appropriate to the external masses, we find, in the case of two-meson–two-Pomeranchon scattering (Fig. 1),

$$\begin{aligned}B_{\alpha\beta}(k_1,k_2,k_3,k_4) &= \langle\alpha|\Gamma^{(+)}(k_2)\Delta(s)\Gamma^{(-)}(k_3)|\beta\rangle\\ &= \delta_{\alpha\beta}\int_0^1 du\, u^{-s-a_0}(1-u)^{-t-1-a_0},\end{aligned}\tag{3.16}$$

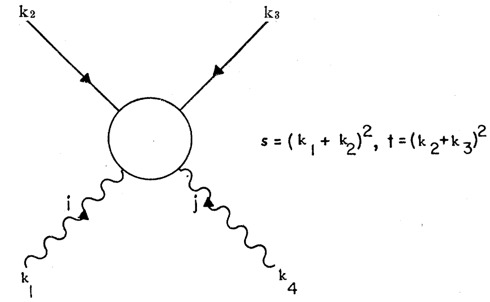

FIG. 1. Meson-Pomeranchon scattering.

where $s=(k_1+k_2)^2$, $t=(k_2+k_3)^2$, and $|\alpha\rangle$, $|\beta\rangle$ are defined in (3.15). Similarly, in the case of the four-meson amplitude (Fig. 2), we find

$$\begin{aligned}B_{\alpha\beta\gamma\delta}(s,t) &= \int_0^1 du\, u^{-s-a_0-1}(1-u)^{-t-a_0-1}\\ &\times[\delta_{\alpha\beta}\delta_{\gamma\delta}(1-u)^2+\delta_{\beta\delta}\delta_{\alpha\gamma}u^2(1-u)^2+\delta_{\alpha\delta}\delta_{\beta\gamma}u^2\\ &-4f^{\alpha\beta\sigma}f^{\gamma\delta\sigma}u(1-u)^2-4f^{\beta\gamma\sigma}f^{\delta\alpha\sigma}u^2(1-u)].\end{aligned}\tag{3.17}$$

The cyclic symmetry can easily be verified by letting $u\to 1-u$. Examining the s channel, we see that the vacuum has the lowest mass, and the octet, with an extra factor of u, has a mass higher by one unit. Exotic states come with extra factors of u and higher masses.

D. Symmetry Breaking

We comment that the algebra of $\mathbf{S}$ is unchanged if we add to it the following terms:

$$\begin{aligned}\Delta S_0 &= \lambda J_0{}^8+\lambda^2,\\ \Delta S_\pm &= \lambda J_{\pm 1}{}^8,\end{aligned}\tag{3.18}$$

where λ is an arbitrary constant. This introduces symmetry breaking along the eighth direction, and the modifications of the previous formalism are quite easy to find. Such a term, of course, does not split different isospin multiplets with the same hypercharge and so it is not by itself very interesting. We have also found

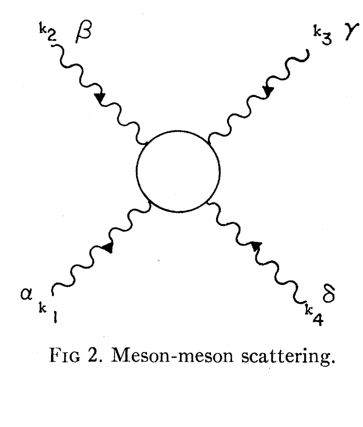

FIG 2. Meson-meson scattering.

[14] The proof of cyclic symmetry for excited-state n-point functions in the ordinary model has been studied by K. Bardakci. The formalism is essentially identical to this section, with excited-state vertices being of the form $:\pi^\mu(z)e^{iQ(z)\cdot k}:$, etc.

quartic symmetry-breaking models of the form

$$S_0=\sum_{\alpha=1}^{8} C^\alpha J^\alpha J^\alpha .$$

Solutions for C^α are determined by requiring projective-algebra commutation relations, and they yield quantized $SU(3)$-breaking models. Unfortunately, these models all suffer from infinite-degeneracy problems.[15] Rather than discuss these models, we will go on to a more sophisticated model, which may offer another route to symmetry breaking, free of such defects.

IV. MODEL WITH SPIN-ORBIT COUPLING

A. Spin Formalism

We begin by using the formalism of Sec. III, now for the case of spin (but no internal symmetry). The relevant quark operators are again denoted by $b_r(n)$ and $d_r(n)$ where n ranges from 0 to ∞, but now r is a Dirac 4-spinor index, ranging from 1 to 4. Under Lorentz transformations, b_r and d_r transform as 4-spinors. As usual, $b|0\rangle=d|0\rangle=0$.

It is convenient to introduce a unified notation for b and d through the definition (all n)

$$b_r(n)\equiv d_r^\dagger(-1-n),\quad b_r^\dagger(-n-1)=d_r(n).$$

Then we can write the anticommutation relations as

$$[b_r(n),b_s(m)]_+=[b_r^\dagger(n),b_s^\dagger(m)]_+=0\,,\qquad [b_r(n),b_s^\dagger(m)]_+=\delta_{n,m}(\gamma_0)_{rs}\,, \tag{4.1}$$

where γ_0 is the ordinary fourth γ matrix. Following ordinary Dirac theory, we define further

$$\bar b_r(n)\equiv\sum_s b_s^\dagger(n)(\gamma_0)_{sr}\,,\quad \bar b\equiv b^\dagger\gamma_0$$

so that

$$[b_r(n),\bar b_s(m)]_+=\delta_{rs}\delta_{nm}. \tag{4.2}$$

It is clear that the factor γ_0 in (4.1) is necessary for consistency with Lorentz transformations, and this, of course, introduces an indefinite metric.

In the same manner as before, we define

$$\psi_r(z)=\sum_{-\infty}^{\infty} z^{n+1/2}b_r(n)\,,$$

$$V^\mu(z)=\sum_{-\infty}^{\infty} V_n{}^\mu z^n=:\bar\psi(z)\gamma^\mu\psi(z):\,, \tag{4.3}$$

$$V_n{}^\mu=\sum_{k=-\infty}^{\infty} :\bar b(k)\gamma^\mu b(k+n):.$$

The algebra of these vector currents is

$$[V_n{}^\mu,V_m{}^\nu]=-2iT_{n+m}{}^{\mu\nu}+cg^{\mu\nu}n\delta_{n,-m}\,, \tag{4.4}$$

where

$$T_n{}^{\mu\nu}=\sum_{k=-\infty}^{\infty} :\bar b(k)\sigma^{\mu\nu}b(k+n):$$

and

$$T^{\mu\nu}(\theta)=\sum_{-\infty}^{\infty} e^{in\theta}T_n{}^{\mu\nu}.$$

The rest of the commutators are given by

$$\begin{aligned}
[V_n{}^\mu,T_m{}^{\lambda\eta}]&=2i(g^{\mu\lambda}V_{m+n}{}^\eta-g^{\mu\eta}V_{m+n}{}^\lambda)\,,\\
[T_n{}^{\mu\nu},T_m{}^{\lambda\eta}]&=cn(g^{\mu\lambda}g^{\nu\eta}-g^{\mu\eta}g^{\nu\lambda})\delta_{n,-m}\\
&\quad+2i(g^{\nu\lambda}T_{n+m}{}^{\mu\eta}-g^{\nu\eta}T_{n+m}{}^{\mu\lambda}+g^{\mu\eta}T_{n+m}{}^{\nu\lambda}\\
&\quad-g^{\mu\lambda}T_{n+m}{}^{\nu\eta})\,.
\end{aligned} \tag{4.5}$$

Of course, the π's commute with all the above-listed operators. Here c is equal to the number of dimensions of the space in which γ matrices operate. For the usual 4×4 matrices, $c=4$, and for the higher-dimensional representations of γ's, c can be any positive-integer multiple of 4. It is also possible to change the sign of c; to do this, we have to change the anticommutation relations of Eq. (4.1) into commutation relations. Under such a change, the algebra of currents remains the same, except that c then can be any negative-integer multiple of 4. Changing the quark statistics from Fermi to Bose alters the norms throughout the theory, e.g.,

$$\|V_{-1}{}^\mu|0\rangle\|^2=cg^{\mu\nu}\,, \tag{4.6}$$

so with Fermi quarks, the space components are ghosts and time components are particles. With Bose quarks, the situation is reversed. To avoid committing ourselves as to the nature of the quark space, we shall leave c as a free parameter.[16]

One can now write an additive theory of the kind studied in Sec. III. It is clear from the preceding discussion that such a model is infested with ghosts. Thus we attempt to find other representations of the projective group in this space.

B. Projective Group with Spin-Orbit Coupling

As explained in the Introduction, our main purpose in this paper is to look for representations of the projective group for which states with spin are involved in the Ward identities, thus giving us a chance of eliminating spin ghosts in the same manner as orbital ghosts. This implies some sort of spin-orbit coupling. Thus we

[15] The degeneracy is such that, at a given mass and spin, states of arbitrarily large isospin and/or hypercharge occur. At present, we do not see how to interpret these models.

[16] As seen below, to avoid complex solutions, we have to take c in a range that does not correspond to any simple bilinear quark representation for the currents. See also Ref. 20.

look for representations of the projective group in the form[17]

$$J_0=\frac{1}{2\pi}\int_0^{2\pi}d\theta:\mathcal{J}(\theta):,$$

$$J_\pm=\frac{1}{2\pi}\int_0^{2\pi}d\theta\, e^{\mp i\theta}\mathcal{J}(\theta), \tag{4.7}$$

$$\mathcal{J}(\theta)=a_1\pi^\mu(\theta)\pi_\mu(\theta)+a_2\pi^\mu(\theta)V_\mu(\theta) +a_3V^\mu(\theta)V_\mu(\theta)+a_4T^{\mu\nu}(\theta)T_{\mu\nu}(\theta).$$

If we can find solutions with $a_2\neq 0$, then states like $p\cdot V_{-1}|0\rangle$ will be involved in the Ward identities.

Requiring $\mathbf{J}$ to generate $O(2,1)$ (see Appendix A for some sample commutators), we arrive at the following equations for $a_1,\ldots,a_4$:

$$\begin{aligned} 2(a_1^2-\tfrac{1}{4}ca_2^2)&=-a_1,\\ 2a_1a_2-a_2a_3(2c+12)-24a_2a_4&=-a_2,\\ \tfrac{1}{2}a_2^2-(2c+24)a_3^2&=-a_3,\\ 4a_3^2-16a_3a_4-(4c+32)a_4^2&=-a_4. \end{aligned} \tag{4.8}$$

There are several discrete sets of solutions to these equations. All the solutions are fixed; there are no arbitrary parameters in them except for c. Some of the solutions require either $a_2=0$, or both $a_2=0$ and $a_1=0$. We discard the first possibility since it does not give rise to useful Ward identities, as explained later. The second possibility, in addition, corresponds to an absence of orbital angular momentum and is rejected. Then we are left with only two solutions:

$$\begin{aligned} a_1&=-\tfrac{1}{4}\mp\frac{c-1}{4(1-4c)^{1/2}},\\ a_2&=\pm\frac{1}{2}\left(\frac{2+c}{1-4c}\right)^{1/2},\\ a_3&=\frac{1}{4(12+c)}\left[1\mp\frac{5+c}{(1-4c)^{1/2}}\right],\\ a_4&=\frac{1}{8(12+c)}\left[1\pm\frac{7}{(1-4c)^{1/2}}\right], \end{aligned} \tag{4.9}$$

where the top signs form one solution, and the bottom another. Reality of $a_1a_3a_4$ dictates that $c<\frac{1}{4}$. Reality of a_2 dictates, in addition, that $-2<c$. This second restriction is not strictly necessary, since it affects only norms of states, which in any case are not positive definite. We shall decide the precise range of c below on the basis of a timelike spectrum.

[17] We have also looked at representations that involve spin-orbit coupling via the density $p\cdot V(\theta)$ instead of $\pi(\theta)\cdot V(\theta)$, but these appear to have kinematical singularities in the n-point functions.

C. $O(2,1)\otimes O(2,1)$

We denote the top solution by $\mathbf{J}$ and the bottom solution by $\mathbf{K}$. Both sets generate $O(2,1)$ among themselves:

$$\begin{aligned} [J_0,J_\pm]&=\mp J_\pm, & [K_0,K_\pm]&=\mp K_\pm,\\ [J_+,J_-]&=2J_0, & [K_+,K_-]&=2K_0, \end{aligned} \tag{4.10}$$

by construction. Moreover, one easily calculates that

$$[\mathbf{J},\mathbf{K}]=0, \tag{4.10$'$}$$

so that, taken together, we have the algebra of $O(2,1)\otimes O(2,1)$. In fact, all the higher moments commute as well:

$$[J_l,K_m]=0,$$

so we can think of two commuting "conformal groups" on the same space. We will, of course, have to choose which one of these two groups will be used to boost fields. We will do this below also on the basis of the spectrum.

Of course, the sum of $\mathbf{J}$ and $\mathbf{K}$ also generates an $O(2,1)$:

$$\mathbf{N}=\mathbf{J}+\mathbf{K}. \tag{4.11}$$

These generators are much simpler than $\mathbf{J}$ or $\mathbf{K}$ separately, and will be useful. Thus, for reference, we give explicitly

$$N_0=-\tfrac{1}{2}\sum_{n=-\infty}^{+\infty}:\pi_{-n}{}^\mu\pi_{\mu n}:+\frac{1}{2(12+c)}\sum_{n=-\infty}^{+\infty}:V_{-n}{}^\mu V_{\mu n}: +\frac{1}{4(12+c)}\sum_{n=-\infty}^{+\infty}:T_{-n}{}^{\mu\nu}T_{\mu\nu,n}:, \tag{4.12}$$

with similar expressions for $N_\pm$. Notice that $\mathbf{N}$ contains no spin-orbit terms, and is in fact the spin-analog of the additive model of Sec. III. We can emphasize this fact by recording the transformation properties of the various objects in the theory under $\mathbf{N}$:

$$\begin{aligned} [N_0,Q^\mu(\theta)]&=i\partial_\theta Q^\mu(\theta),\\ [N_\pm,Q^\mu(\theta)]&=ie^{\mp i\theta}\partial_\theta Q^\mu(\theta),\\ Q^\mu(z)&=z^{-N_0}Q^\mu(1)z^{N_0},\\ [N_0,V^\mu(\theta)]&=i\partial_\theta V^\mu(\theta),\\ [N_\pm,V^\mu(\theta)]&=e^{\mp i\theta}[i\partial_\theta V^\mu(\theta)\pm V^\mu(\theta)],\\ [N_\pm,T^{\mu\nu}(\theta)]&=e^{\mp i\theta}[i\partial_\theta T^{\mu\nu}(\theta)\pm T^{\mu\nu}(\theta)], \end{aligned} \tag{4.13}$$

and so on—exactly the relations of the additive model. Of course, N_0 has integrally spaced eigenvalues. All this is in marked contrast, as we shall see, to the properties of $\mathbf{J}$ and/or $\mathbf{K}$.

We want to define a particularly simple and useful operator, the "sector operator"

$$N=N_0+p^2 \tag{4.14}$$

which senses the mode-excitation number of a state:

$$N\{\pi_{-1}{}^{i_1}\cdots\pi_{-n}{}^{i_n}V_{-1}{}^{j_1}\cdots V_{-m}{}^{j_m}\}|0\rangle$$
$$=(i_1+2i_2+\cdots+ni_n+j_1+\cdots+mj_m)$$
$$\times\{\pi_{-1}{}^{i_1}\cdots\pi_{-n}{}^{i_n}V_{-1}{}^{j_1}\cdots V_{-m}{}^{j_m}\}|0\rangle. \quad (4.15)$$

Because

$$(N,J_0)=(N,K_0)=0\,,$$

we can diagonalize each Hamiltonian (mass operator) sector by sector.

D. *K*-Degeneracy

Note that each eigenstate of the Hamiltonian is degenerate with an infinite family of the same spin and mass. For example, if

$$J_0(p_\alpha)|\alpha\rangle=a_0|\alpha\rangle\,,$$

then

$$J_0K_{-1}{}^{l_1}\cdots K_{-n}{}^{l_n}|\alpha\rangle=a_0K_{-1}{}^{l_1}\cdots K_{-n}{}^{l_n}|\alpha\rangle. \quad (4.16)$$

We call this degeneracy the K-degeneracy, and say that the spectrum breaks up into K-degenerate families.[18] At first, this looks like a disaster. However, we now have two sets of Ward identities, those generated by **J** and those generated by **K**. As will be discussed in Sec. IV G, the K-identities are intimately related to this degeneracy, and they tend to eliminate it.

Moreover, we might have expected something like this $O(2,1)\otimes O(2,1)$ structure on physical grounds. After all, we hope to eliminate not only $p\cdot\pi_{-l}|0\rangle$ but also $p\cdot V_{-l}|0\rangle$, and so on with the Ward identities. Thus we would need many more than the usual model. Indeed, we feel that $O(2,1)\otimes O(2,1)$ is the natural, perhaps inescapable, group for "good" models with spin: Suppose we have one solution [one $O(2,1)$ group], say **J**, which we know transforms as an angular momentum under the sector generators **N**. Then it is trivial to construct the commuting $O(2,1)$ group **K** as **N**$-$**J**.

E. Local Field $E^\mu(z)$

In this section, we will, for simplicity, assume we are boosting with J_0, but, in fact, all statements can be read with $J_0\to K_0$.

We note that $Q^\mu(1)$ satisfies the "stability conditions"

$$[J_\pm-J_0,\,Q^\mu(1)]=0. \quad (4.17)$$

This is essentially a consequence of the fact that our generators are constructed out of a local density function of local operators. Thus

$$E^\mu(z)=z^{-J_0}Q^\mu(1)z^{J_0} \quad (4.18)$$

is a scalar under **J**. In Appendix B, it is argued that E^μ is c-number local,

$$[E^\mu(z),E^\nu(z')]=i\pi\epsilon(z-z')g^{\mu\nu}\,, \quad (4.19)$$

just as in the original dual model. Thus, we can take the ground-state vertex for the model as

$$\Gamma(k,z)=z^{-k^2}z^{-J_0}{:}e^{\sqrt{2}Q(1)\cdot k}{:}z^{J_0}. \quad (4.20)$$

As will carefully be discussed in Sec. IV G, we must choose $k^2=0$ in order to be able to use the **K**-identities (and hope to eliminate the K-degeneracy). This consideration then fixes our intercept ($a_0=0$).

The n-point functions are constructible immediately in the form (2.8) with $k^2=0$. Invariance of the integrand under infinitesimal projective transformations follows immediately because E^μ is a scalar. Thus the integrand is a function of cross ratios, and hence invariant under finite transformations.[19] This, together with c-number locality of E^μ, proves cyclic symmetry in the usual manner. Finally, as argued in Appendix D,

$$\lim_{z\to\infty}\Gamma(k,z)|0\rangle_{k^2=0}\,, \quad (4.21)$$

so that we may pass to the integrated form in the usual manner, resulting in n-point functions of the form (2.11) with $a_0=0$.

F. Spectrum

Having determined the intercept, we begin studying the lower states in the spectra of J_0 and K_0. For this purpose, we adopt a common name $L_0(a_1a_2a_3a_4)$ for either J_0 or K_0, depending on the a's. Because of the form of the propagator at $a_0=0$, mass-shell states of mass $p_\alpha{}^2$ are the solutions to

$$L_0(p_\alpha)|\alpha\rangle=-n|\alpha\rangle,\quad n=0,\,1,\,\ldots\,. \quad (4.22)$$

For simplicity, we will concentrate on $n=0$. Now we begin to work our way up sector by sector.

N=0

There is only one state $|0\rangle_{k^2=0}$ in the zeroth sector. This is, of course, the external ground state.

N=1

We write the general state in this sector as

$$|\psi\rangle=\{\alpha^\mu\pi_{-1}{}^\mu+\beta^\mu V_{-1}{}^\mu+\beta^{\mu\nu}T_{-1}{}^{\mu\nu}\}|0\rangle\,, \quad (4.23)$$

where α^μ, β^μ, $\beta^{\mu\nu}$ are c numbers. Then requiring $L_0(p)|\psi\rangle=0$, we obtain

$$2(p^2-1)a_1\alpha^\mu+[2a_1(2a_1+1)/a_2]\beta^\mu=0\,,$$
$$[2a_1(p^2+1)+1]\beta^\mu-a_2\alpha^\mu-4i\sqrt{2}a_2p^\nu\beta^{\mu\nu}=0\,, \quad (4.24)$$
$$(2p^2a_1+1-8a_3+4a_3{}^2/a_4)\beta^{\mu\nu}-i\sqrt{2}a_2(p^\mu\beta^\nu-p^\nu\beta^\mu)=0\,.$$

The solutions to these equations are tabulated in Table II. Solutions (5) and (6) correspond to the two different roots of the quadratic equation for p^2, and ϵ^μ is a polarization vector satisfying $\epsilon\cdot p=0$.

We now require that no state has complex or spacelike mass. This forces us to choose the **J** system as mass

[18] For more discussion of K-degenerate families, see Appendix C.

[19] For further discussion of cyclic symmetry, see Appendix C.

TABLE II. Lowest-lying states of the spin-orbit model.

	Mass	State	Spin and parity
(1)	$p^2=-1/2a_1$	$p_\mu\{2a_1\pi_{-1}{}^\mu-a_2V_{-1}{}^\mu\}\|0\rangle$	Scalar
(2)	$p^2=0$	$p_\mu\{(2a_1+1)\pi_{-1}{}^\mu-a_2V_{-1}{}^\mu\}\|0\rangle$	Scalar
(3)	$p^2=\dfrac{8a_3a_4-a_4-4a_3{}^2}{2a_1a_4}$	$\epsilon_{\mu\nu\lambda\eta}p^\nu T_{-1}{}^{\lambda\eta}\|0\rangle$	Abnormal pseudovector
(4)	$p^2=0$	$\epsilon_\mu\left\{\dfrac{2a_1+1}{a_2}\pi_{-1}{}^\mu-V_{-1}{}^\mu-\dfrac{2\sqrt{2}ia_2}{1-8a_3+4a_3{}^2/a_4}p_\nu T_{-1}{}^{\mu\nu}\right\}\|0\rangle$	Vector
(5)	$(2a_1p^2+1)\left(2a_1p^2+1-8a_3+\dfrac{4a_3{}^2}{a_4}\right)-8a_2{}^2(p^2-1)=0$	$\epsilon_\mu\left\{\dfrac{2a_1+1}{a_2(1-p^2)}\pi_{-1}{}^\mu-V_{-1}{}^\mu-\dfrac{2\sqrt{2}ia_2}{2p^2a_1+1-8a_3+4a_3{}^2/a_4}p_\nu T_{-1}{}^{\mu\nu}\right\}\|0\rangle$	Vector
(6)	$(2a_1p^2+1)\left(2a_1p^2+1-8a_3+\dfrac{4a_3{}^2}{a_4}\right)-8a_2{}^2(p^2-1)=0$	$\epsilon_\mu\left\{\dfrac{2a_1+1}{a_2(1-p^2)}\pi_{-1}{}^\mu-V_{-1}{}^\mu-\dfrac{2\sqrt{2}ia_2}{2p^2a_1+1-8a_3+4a_3{}^2/a_4}p_\nu T_{-1}{}^{\mu\nu}\right\}\|0\rangle$	Vector

operator, etc. Further, we must restrict the Schwinger term to the range $-2<c<0$. Note that we are then outside of any simple quark range. That is, no simple bilinear quark representation exists for these currents.[20] It is by no means clear whether this situation will persist with the inclusion of internal symmetry.

Finally, we remark that the spectrum remains timelike for $n=1, 2, 3, \ldots$ [see Eq. (4.22)], but we have not studied higher sectors ($N=2, 3, \ldots$). It is also worth remarking that there is no limit in which $\mathbf{J}$ can reduce to the ordinary dual model: The only way to eliminate the spin-orbit coupling smoothly is to take $c=-2$, in which case $\mathbf{K}$ reduces to the usual generators, but $\mathbf{J}$ does not.

G. Ward Identities

We begin by listing the relevant identities:

$$[K_n(p)-K_0(p)]\Gamma^{(-)}(k)\Gamma^{(+)}(k)=\Gamma^{(-)}(k)\Gamma^{(+)}(k)\times[K_n(p+k)-K_0(p+k)+2a_1k^2n], \quad (4.25)$$
$$[K_n(p)-K_0(p), \Delta(p)]=0,$$

because Δ is a function of J_0. Thus, if we are to be able to use the K-identities to eliminate states, i.e.,

$$0=\langle\psi|(K_n-K_0)\Gamma^{(-)}(k_2)\Gamma^{(+)}(k_2)\Delta(s_{12})\cdots\Gamma^{(+)}(k_{n-1})|0\rangle,$$

we must take $k^2=0$ for the external ground state. In this case, we find

$$[J_1(p)-J_0(p)]\Gamma^{(-)}(k)\Gamma^{(+)}(k)=\Gamma^{(-)}(k)\Gamma^{(+)}(k)[J_1(p+k)-J_0(p+k)],$$
$$[J_1(p)-J_0(p)]\Delta(p)=[J_1(p)-J_0(p)]\int_0^1 dz\, z^{J_0-1}(1-z)^{-1}=\int_0^1 dz\, z^{J_0}(1-z)^{-1}[J_1(p)-J_0(p)]. \quad (4.26)$$

Thus we find the full infinite set of K-identities working, but only the first J-identity in play. Below, we shall construct another model in which both infinite sets of identities come into play.

As a result of these identities, states (1) and (2) of Table II are spurious. This is because state (2) is proportional to

$$[K_{-1}(p)-K_0(p)]|0\rangle_{p^2=0}=K_{-1}(p)|0\rangle_{p^2=0},$$

while the spurious state

$$[J_{-1}(p)-J_0(p)]|0\rangle_{p^2=-1/2a_1} \quad (4.27)$$

cancels state (1) against the first recurrence of the vacuum [$n=1$ in Eq. (4.22)]. (This is precisely the way the first Ward identity works for zero intercept in the usual dual model.)

Note that we thus eliminate the two ghosts associated with the time components of $\pi_{-1}{}^\mu$ and $V_{-1}{}^\mu$.

Now we come to the question of removal of the infinite K-degeneracy. Clearly, the K-identities are tending to eliminate the very degenerate states they generate. For example, we noted above that state (2), itself K-degenerate with the ground state, is removed by the first K-Ward identity. In general, the picture is more complicated, and best discussed through operator equations for real states.

[20] When c is arbitrary, the currents cannot in general be written in terms of the quark operators b and d. In this case, however, one can introduce the quark operators as separate entities anyway. For example, postulate that

$$[V_n{}^\mu, b^r(m)]=-(\gamma^\mu)^{rr'}b^{r'}(n+m), \text{ etc.},$$

which would be true at $c=\pm4$, and thus cannot violate any Jacobi identities, etc. In this fashion, the Hilbert space may be enlarged to include quarks. Similarly one may introduce three-index symbols for baryons.

If $\langle\alpha(p)|$ is any state whatsoever, then

$$\langle\alpha(p)|[K_n(p)-K_0(p)],\quad n>0 \tag{4.28}$$

is spurious. Real states $|R(p)\rangle$, those not eliminated by Ward identities, are orthogonal to spurious states,

$$[K_n(p)-K_0(p)]|R(p)\rangle=0,\quad n>0.$$

Because of the algebra of the K_n among themselves, only the first two equations need be enforced[21]:

$$\begin{aligned}[][K_1(p)-K_0(p)]|R(p)\rangle&=0,\\ [K_2(p)-K_0(p)]|R(p)\rangle&=0.\end{aligned} \tag{4.29}$$

Spurious states associated with J are best removed sector by sector in the usual manner.

There is one simple solution to (4.29), namely, the ground state $|0\rangle_{k^2=0}$, so we may be sure it is real. That is, the K-identities will eliminate all states degenerate with it (the K-degenerate family of the vacuum; see Appendix C). This is consistent with the argument of Appendix D. In general, however, these equations are very complicated, requiring contributions from an entire K-degenerate family (every sector). For example, state (5) of Table II does not solve (4.29); only some linear combination of state (5) plus its entire K-degenerate family can solve it. We have studied an iterative procedure for the solution, but a closed form appears beyond the scope of our present work. We note, however, that for the model to be satisfactory, the K-identities must eliminate all but a finite degree of degeneracy. At present, we do not know whether this is in fact what happens.

Finally, we make a remark about the norms of real states. For example, $T_{-1}{}^{ij}|0\rangle$ (ij spatial) is the only remaining ghost in the first sector. On the other hand, this is not a real state, because it does not solve (4.29). The norm of the first K-degenerate state $[K_{-1}T_{-1}{}^{ij}|0\rangle]$ is $2(1-p^2)$ times the norm of the original state. This feature we find to be generally true among the K-degenerate states. For timelike p^2, there is much oscillation of the norm. Thus, without detailed analysis, we cannot state the norms of real states.

V. MODEL WITH DOUBLY INFINITE SET OF WARD IDENTITIES

We were not able to use more than the first J-identity in the previous model. By introducing a fifth operator in an essentially trivial manner, however, we can bring both sets of identities into full play.

We introduce fifth[22] operators $Q^5(z)$, $\pi^5(z)$ with positive metric,

$$(\pi_n{}^5,\pi_m{}^5)=n\delta_{n,m},\ \pi_0{}^5=\sqrt{2}p^5,\ (q^5,p^5)=+1 \tag{5.1}$$

and fifth operators commute with all other operators previously introduced. We incorporate this extra dimension into the algebra via

$$\begin{aligned}\hat{J}_0&=J_0+\tfrac{1}{2}\sum_{n=-\infty}^{+\infty}:\pi_{-n}{}^5\pi_n{}^5:,\quad \hat{K}_0=K_0\\ \hat{J}_+&=J_++\tfrac{1}{2}\sum_{n=-\infty}^{+\infty}\pi_{-n}{}^5\pi_{n+1}{}^5,\quad \hat{K}_+=K_+\end{aligned} \tag{5.2}$$

etc., where $\mathbf{J}$ and $\mathbf{K}$ are the generators of Sec. IV. Clearly we still have $O(2,1)\otimes O(2,1)$, etc., and we will now boost with $\hat{J}_0$. The vertices become correspondingly 5-d, e.g.,

$$\hat{\Gamma}^{(+)}(k)=\exp\left(-\sqrt{2}k^\mu\sum_{n=1}^{\infty}\frac{\pi_{\mu n}}{n}+\sqrt{2}k^5\sum_{n=1}^{\infty}\frac{\pi_n{}^5}{n}\right), \tag{5.3}$$

etc. As is well known, p_5 becomes a conserved additive quantum number.

To obtain all the Virasoro identities for J, we must take the propagator

$$\Delta=1/(\hat{J}_0-1), \tag{5.4}$$

and we must have

$$[\hat{J}_n(p)-\hat{J}_0(p)]\hat{\Gamma}(k)=\hat{\Gamma}(k)[\hat{J}_n(p+k)-\hat{J}_0(p+k)+n].$$

On the other hand, we calculate directly

$$\begin{aligned}[][\hat{J}_n(p)-\hat{J}_0(p)]\hat{\Gamma}(k)=\hat{\Gamma}(k)[\hat{J}_n(p+k)&-\hat{J}_0(p+k)\\ &+(2a_1k^2+k_5{}^2)n],\end{aligned} \tag{5.5}$$

so we need to require

$$2a_1k^2+k_5{}^2=1. \tag{5.6}$$

But we also need the K-identities. Because K_l contains no fifth operators, we still have $(K_l,\Delta)=0$, so we must still insist that

$$[K_n(p)-K_0(p)]\hat{\Gamma}(k)=\hat{\Gamma}(k)[K_n(p+k)-K_0(p+k)],$$

which can be done only for $k^2=0$. Thus, (5.6) becomes

$$k_5=\pm1 \tag{5.7}$$

for the external scalars. If we arrange our n-point functions (with an even number of these particles) alternating in sign of k_5, then we find by factorization only two types of trajectories in the system, whose (spin-0, vacuum) masses are

$$\begin{aligned}&\text{odd:}\quad k_5=\pm1,\ k^2=0,\\ &\text{even:}\quad k_5=0,\ k^2=+1/2a_1,\end{aligned} \tag{5.8}$$

so that k_5 acts like a multiplicative quantum number. In fact, by factorization one can go to just $k_5=0$ external particles, which couple only to $k_5=0$ internal trajectories; in this latter model, the fifth operator will no longer appear at all. We have analyzed the first sector of this $k_5=0$ model, and found that at least one other state (besides the vacuum) is spacelike. On the other hand, all the first sector states are eliminated by Bose symmetrization.

---

[21] Zero-norm states must in general be handled separately. We want to thank Charles Thorn for emphasizing this point to us.

[22] Y. Nambu, in Proceedings of the International Conference on Symmetries and Quark Models, Wayne University (unpublished).

VI. DIRECTIONS

In addition to our relatively straightforward "additive" models, we have studied here two nontrivial models which indicate a definite direction for future work. To approach a realistic model, we must look for representations of $O(2,1)$ in terms of the set of operator densities

$$\begin{aligned}&\pi^\mu(\theta)\pi_\mu(\theta),\ V^\mu(\theta)V_\mu(\theta),\ A^\mu(\theta)A_\mu(\theta),\\&T^{\mu\nu}(\theta)T_{\mu\nu}(\theta),\ P(\theta)P(\theta),\ \pi^\mu(\theta)V_\mu(\theta),\\&V_\mu{}^\alpha(\theta)V_\alpha{}^\mu(\theta),\ T_\alpha{}^{\mu\nu}(\theta)T_{\mu\nu}{}^\alpha(\theta),\ P^\alpha(\theta)P^\alpha(\theta),\\&\qquad A_\mu{}^\alpha(\theta)A_\alpha{}^\mu(\theta),\end{aligned}\tag{6.1}$$

where P, A, etc., are quartics which transform like the pion and axial-vector current, respectively, and $\alpha=1,\ldots,8$ is an $SU(3)$ index. This is not entirely trivial algebraically, but we expect, as in Secs. IV and V, discrete solutions and the $O(2,1)\otimes O(2,1)$ structure. On the other hand, as we can see from Secs. IV and V, many details may change considerably on this introduction of $SU(3)$. In particular, one is interested in knowing how many tachyons are required to have all the Ward identities in force, and what the mass spectrum looks like in general.

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APPENDIX A: QUARTIC COMMUNICATION RELATIONS

We discuss here briefly the calculation of the commutators necessary to determine the coefficients $a_1\cdots a_4$ in the model of Sec. IV. Because we are dealing with singular operator products in z space, a naive calculation in z space will miss certain operator Schwinger terms. Such terms can be properly treated with an ϵ-limiting procedure commonly used in analogous field-theory calculations, or more simply, the calculation can be done directly in "momentum" space.

We will discuss the latter. As a sample calculation, consider $[A_0,A_+]$, where

$$\begin{aligned}A_0&=\sum_{n=-\infty}^{+\infty}:V_{-n}{}^\mu V_{\mu n}:=2\sum_{n=1}^{\infty}V_{-n}{}^\mu V_{\mu n}+V_0{}^\mu V_{\mu 0},\\A_+&=\sum_{n=-\infty}^{+\infty}V_{-n}{}^\mu V_{\mu,n+1}=2\sum_{n=0}^{\infty}V_{-n}{}^\mu V_{\mu,n+1}.\end{aligned}\tag{A1}$$

Using (4.4) and (4.5), we get after a somewhat tedious calculation

$$[A_0,A_+]=-2(c+12)\sum_{n=-\infty}^{+\infty}V_{-n}{}^\mu V_{\mu,n+1}+4\sum_{n=-\infty}^{+\infty}T_{-n}{}^{\mu\nu}T_{\mu\nu,n+1}.$$

Similarly, we list

$$\begin{aligned}&\Big[\sum_{n=-\infty}^{+\infty}:\pi_{-n}{}^\mu V_{\mu,n}:,\ \sum_{m=-\infty}^{+\infty}\pi_{-m}{}^\nu V_{\nu,m+1}\Big]\\&\qquad=-\tfrac{1}{2}c\sum_{n=-\infty}^{+\infty}\pi_{-n}{}^\mu\pi_{\mu,n+1}+\tfrac{1}{2}\sum_{n=-\infty}^{+\infty}V_{-n}{}^\mu V_{\mu,n+1},\\&\Big[\sum_{n=-\infty}^{+\infty}:T_{-n}{}^{\mu\nu}T_{\mu\nu,n}:,\ \sum_{m=-\infty}^{+\infty}T_{-m}{}^{\mu\nu}T_{\mu\nu,m+1}\Big]\\&\qquad=-(4c+32)\sum_{n=-\infty}^{+\infty}T_{-n}{}^{\mu\nu}T_{\mu\nu,n+1},\\&\Big[\sum_{n=-\infty}^{+\infty}:V_{-n}{}^\mu V_{\mu,n}:,\ \sum_{m=-\infty}^{+\infty}T_{-m}{}^{\lambda\rho}T_{\lambda\rho,m+1}\Big]\\&\qquad+\Big[\sum_{n=-\infty}^{+\infty}:T_{-n}{}^{\lambda\rho}T_{\lambda\rho,n}:\ \sum_{m=-\infty}^{+\infty}V_{-n}{}^\mu V_{\mu,m+1}\Big]\\&\qquad=-16\sum_{n=-\infty}^{+\infty}T_{-n}{}^{\mu\nu}T_{\mu\nu,n+1}.\end{aligned}\tag{A2}$$

The commutator of $\pi_n{}^\mu$ with various operators is also needed, for example, in deriving the final equations. We have

$$\begin{aligned}(\pi_n{}^\mu,J_0)&=-2a_1\pi_n{}^\mu-a_2nV_n{}^\mu,\\(\pi_n{}^\mu,J_+)&=-2a_1\pi_{n+1}{}^\mu-a_2nV_{n+1}{}^\mu,\end{aligned}\tag{A3}$$

which immediately leads to (4.8).

APPENDIX B: c-NUMBER LOCALITY OF $E^\mu(Z)$

Here we argue that $E^\mu(z)$ satisfies a c-number commutation relation with itself,

$$[E^\mu(z),E^\nu(z)]=i\pi g^{\mu\nu}\epsilon(z-z'),\tag{B1}$$

which is precisely the commutator of $[Q^\mu(z),Q^\nu(z')]$ of the usual dual model. We have been able to show (B1) only through various series expansions, whose convergence we cannot actually prove. With this understanding, we will study

$$\rho^{\mu\nu}(z)=[E^\mu(z),Q^\nu(1)],\tag{B2}$$

which loses no generality; the general case is reached by boosting $\rho^{\mu\nu}$ in the usual fashion.

Using (4.18) and (4.11), we write $E^\mu(z)$ as

$$\begin{aligned}E^\mu(z)&=z^{K_0}Q^\mu(z)z^{-K_0}\\&=Q^\mu(z)+\ln z\,[K_0,Q^\mu(z)]\\&\qquad+\frac{\ln^2 z}{2!}[K_0,[K_0,Q^\mu(z)]]+\cdots,\end{aligned}\tag{B3}$$

and we will need

$$\begin{aligned}[V^\mu(z),V^\nu(z')]&=2\pi iz(-2i)T^{\mu\nu}(z)\delta(z-z')\\&\qquad+2\pi iczz'\delta'(z'-z).\end{aligned}\tag{B4}$$

We will not be concerned with precise numerical coefficients here but rather only the types of terms. We list

$$
\begin{aligned}
&[K_0,Q^\mu(z)]\propto \pi^\mu(z)+V^\mu(z)\,,\\
&[K_0,[K_0,Q^\mu(z)]]\propto z\frac{d}{dz}\pi^\mu(z)+z\frac{d}{dz}V^\mu(z)\\
&\qquad+\pi_\nu(z)T^{\nu\mu}(z)+V_\nu(z)T^{\nu\mu}(z)\,,\\
&[K_0,[K_0,[K_0,Q^\mu(z)]]]\propto\left(z\frac{d}{dz}\right)^2\pi^\mu(z)\\
&\qquad+\left(z\frac{d}{dz}\right)^2V^\mu(z)+\pi_\nu(z)z\frac{d}{dz}T^{\nu\mu}(z)\\
&\qquad+T^{\mu\nu}(z)\frac{d}{dz}\pi_\nu(z)+V_\nu(z)z\frac{d}{dz}T^{\nu\mu}(z)\\
&\qquad+T^{\mu\nu}(z)z\frac{d}{dz}V_\nu(z)+\pi^\lambda(z)\pi_\lambda(z)V^\mu(z)+\cdots\,,
\end{aligned}
\tag{B5}
$$

and so on. Thus we can calculate the commutators contributing to $\rho^{\mu\nu}$:

$$
\begin{aligned}
&[[K_0,Q^\mu(z)],Q^\nu(1)]\propto\delta(z-1)g^{\mu\nu}\,,\\
&[[K_0[K_0,Q^\mu(z)]],Q^\nu(1)]\propto\delta'(z-1)g^{\mu\nu}\\
&\qquad+\delta(z-1)T^{\mu\nu}(z)\,,\\
&[[K_0,[K_0,[K_0,Q^\mu(z)]]],Q^\nu(1)]\propto\delta''(z-1)g^{\mu\nu}\\
&\qquad+z\frac{d}{dz}T^{\mu\nu}(z)\delta(z-1)+T^{\mu\nu}(z)\delta'(z-1)\\
&\qquad+\pi^\mu(z)V^\nu(z)\delta(z-1)+\cdots\,.
\end{aligned}
\tag{B6}
$$

The crucial observation is that the nth term $(n>0)$ $[[K_0[K_0\cdots(nK_0\text{'s})]],\ Q^\mu(z)]$ involves at most the $(n-1)$th derivative of $\pi(z)$. Consequently, this term when commuted with $Q(1)$ involves at most the $(n-1)$th derivative of $\delta(z-1)$. When multiplied by $(\ln z)^n$, which vanishes as $(z-1)^n$, each of these contributions vanishes, thus leaving only the first term

$$\rho^{\mu\nu}(z)=[Q^\mu(z),Q^\nu(1)]=i\pi\epsilon(z)g^{\mu\nu}\,,\tag{B7}$$

which proves our initial relation (B1). We also comment that we reach the same conclusion if we had constructed E by boosting with K_0 instead of J_0.

Finally, we mention that we have checked this formal method by expanding E in a number of different ways. For example, we break up

$$J_0=-2a_1L_0+\Delta\,,\tag{B8}$$

where L_0 is just the ordinary dual-model mass operator of Sec. II, and expand in powers of Δ. Because $[L_0,\Delta]\neq0$, we obtain an "interaction-picture" expansion, which at any finite order in Δ sums infinite subsets of the $(\ln z)^n$ expansion presented above. Clearly there are many possible expansions, and they all verify (B1).

APPENDIX C: CYCLIC TRANSFORMATIONS, TWISTS, AND SINGLE VALUEDNESS

We have to be very careful about finite projective transformations in the theories with spin-orbit coupling. This is because $E^\mu(z)$ contains in general infinite numbers of nonintegral powers of z, which in turn is related to the fact that the trajectories are no longer linear. Thus we have to worry, in general, about slipping off onto another sheet. Let us first see why E contains nonintegral powers. We expand

$$E^\mu(z)=z^{-J_0}Q^\mu(1)z^{J_0}\tag{C1}$$

in complete sets of eigenstates of J_0:

$$
\begin{aligned}
&E^\mu(z)=\sum_{\alpha,\beta}z^{(\beta-\alpha)}|\alpha\rangle\langle\alpha|Q^\mu(1)|\beta\rangle\langle\beta|\,,\\
&J_0|\alpha\rangle=\alpha|\alpha\rangle.
\end{aligned}
\tag{C2}
$$

It is easily verified that J_0 has (some) nonintegrally spaced eigenvalues. (It also has, of course, families of integrally spaced eigenvalues going up from each eigenvalue.) Thus, for example, we have no reason to expect that $e^{-2\pi iJ_0}E^\mu(z)e^{2\pi iJ_0}$ returns to $E(z)$.

In proving cyclic symmetry of the n-point functions for external ground states, we can argue that the sheet structure causes no trouble. Our path is first to note infinitesimal projective invariance—which implies that the integrand

$$I={}_{p=0}\langle0|e^{\sqrt{2}E(z_1)\cdot k_1}\cdots e^{\sqrt{2}E(z_n)\cdot k_n}|0\rangle_{p=0}$$

is a function only of cross ratios. Then the integrand is invariant under finite projective transformations. In particular, the projective transformation which takes $\theta_i\to\theta_i+2\pi$ leaves the integrand invariant. That is to say, although each $E(z_i)$ has terrible cut structure, it all cancels out of the integrand.

This makes us wonder if we cannot implement the $z_1\to0$, $z_2\to1$, $z_n\to\infty$ finite projective transformations (and later twists) by single-valued generators. The only candidate for this in the theory is the set of generators $\mathbf{N}$ whose eigenvalues are integrally spaced. Let us see what kind of transformations are generated by $\mathbf{N}$. We study the single-valued transformation

$$U=e^{-i\boldsymbol{\xi}\cdot\mathbf{N}},\quad \mathbf{N}|0\rangle_{p=0}=0,\quad \mathbf{N}=\mathbf{J}+\mathbf{K}\tag{C3}$$

for arbitrary $\boldsymbol{\xi}$. This gives us the identity

$$I={}_{p=0}\langle0|e^{\sqrt{2}E[\bar z_1(\xi),\xi]\cdot k_1}\cdots e^{\sqrt{2}E[\bar z_n(\xi),\xi]\cdot k_n}|0\rangle_{p=0}\,,\tag{C4}$$

where

$$E^\mu(z,\boldsymbol{\xi})=e^{-i\boldsymbol{\xi}\cdot\mathbf{K}}E^\mu(z)e^{+i\boldsymbol{\xi}\cdot\mathbf{K}}.\tag{C5}$$

Choosing $\boldsymbol{\xi}$ to map $\bar z_i\to0$, $\bar z_2\to1$, $\bar z_n\to\infty$ results in

$$I={}_{p=0}\langle0|e^{\sqrt{2}E(0,\xi)\cdot k_1}\cdots e^{\sqrt{2}E(\infty,\xi)\cdot k_n}|0\rangle_{p=0}.\tag{C6}$$

Now we can formally undo the $\mathbf{K}\cdot\boldsymbol{\xi}$ transformation, which annihilates on the vacuum, giving finally the $\boldsymbol{\xi}$-independent result. This last step is only formal, since we should really worry about the sheet structure for the finite K transformation. On the other hand, we have at least given a plausible argument that cyclic transformations should be implemented by $\mathbf{N}$. We will risk the conjecture that such is in fact so, and furthermore that "twists" should also be implemented by $\mathbf{N}$. An interesting consistency check on this conjecture is the following. The twist operator defines a "signature" for the various states; it would be a disaster if the signature varied among the K-degenerate states necessary to form "real" states. In fact, we shall show that twisting by $\mathbf{N}$ defines a unique signature for all the contributions to a real state. Moreover, of course, the signature will be integral (± 1), and the "square" of the $\mathbf{N}$ twist is unity.

The twist transformation in the usual dual model is

$$\Omega = e^{i\pi(L_0+p^2)}e^{L_+}. \tag{C7}$$

So, we find, assuming transformation by $\mathbf{N}$, in our case

$$\Omega = e^{i\pi N}e^{N_+}, \tag{C8}$$

where $N=N_0+p^2$ is the "sector operator" introduced in Sec. IV. The derivation of this formula is identical to the usual derivation, since it relies only on projective invariance and field commutation relations. Now we need a language to discuss all the states K-degenerate with a given state.

We define the lowest state $|S\rangle_0$ (lowest sector number) in a K-degenerate family by the equation

$$K_{+l}|S\rangle_0 = 0, \quad l>0. \tag{C9}$$

If this were not zero, then there would be a K-degenerate state n sectors lower. We also take $|S\rangle_0$ as a mass-shell state[23] (zero-intercept theory), $J_0|S\rangle_0=0$. Note that $|S\rangle_0$ is in general not real because

$$K_0|S\rangle_0 = N_0|S\rangle_0 \neq 0.$$

Finally, we also require that $|S\rangle_0$ is free of J spurious states:

$$J_{+1}|S\rangle_0 = 0. \tag{C10}$$

Then, we generate this state's K-degenerate family via

$$\{|K\rangle\} = (K_{-1})^{j_1}\cdots(K_{-l})^{j_l}|S\rangle_0.$$

What is the signature of $|S\rangle_0$? Noting that

$$N_{+1}|S\rangle_0 = (J_{+1}+K_{+1})|S\rangle_0 = 0,$$

we have

$$\Omega|S\rangle_0 = (-1)^n|S\rangle_0, \tag{C11}$$

where n is the sector number of $|S\rangle_0$. Now we claim that

$$\langle\text{scalars}|\Omega|K\rangle = (-1)^n\langle\text{scalars}|K\rangle \tag{C12}$$

for all the states in the family. As a simple example, we calculate

$$\begin{aligned}\langle\text{scalars}|\Omega K_{-l}|S\rangle_0 &= \langle\text{scalars}|e^{i\pi N}e^{K_{+1}}K_{-l}|S\rangle_0\\ &= \langle\text{scalars}|e^{-K_{+1}}e^{i\pi N}K_{-l}|S\rangle_0\\ &= (-1)^{n+l}\langle\text{scalars}|e^{-K_{+1}}K_{-l}|S\rangle_0\\ &= (-1)^{n+l}\langle\text{scalars}|\sum_{\kappa=0}^{l}\binom{l+1}{\kappa}(-1)^\kappa K_{-l+\kappa}|S\rangle_0\\ &= (-1)^{n+l}\left[\sum_{\kappa=0}^{l}\binom{l+1}{\kappa}(-1)^\kappa\right]\langle\text{scalars}|K_0|S\rangle_0. \end{aligned} \tag{C13}$$

In this last step, we used the fact that

$$\langle\text{scalars}|K_{-m}-K_0|\alpha\rangle = 0$$

for $m\geq 0$ and arbitrary $|\alpha\rangle$ (Ward identities). The sum over binomial coefficients is precisely $(-1)^l$. We have checked this result also for $(K_{-1})^l|S\rangle_0$, etc., but we do not have a general proof to present. Thus we claim that any state $|S\rangle_0$ with odd sector number, together with its entire K-degenerate family, decouples after Bose symmetrization.

APPENDIX D: CONSISTENCY CONDITION ON $E^\mu(z)$ AND THE VACUUM

Here we want to argue for the identity

$$\lim_{z\to\infty}\Gamma(k,z)|0\rangle_{p=0} = |0\rangle_{k^2=0},$$
$$\Gamma(k,z) = z^{-J_0}e^{\sqrt{2}Q(1)\cdot k}z^{J_0}, \quad k^2=0 \tag{D1}$$

relevant to the integration of the n-point functions for the model of Sec. IV. We study

$$\begin{aligned} z^{-J_0}e^{\sqrt{2}Q(1)\cdot k}z^{J_0}|0\rangle_{p=0} &= z^{-J_0}e^{\sqrt{2}Q(1)\cdot k}|0\rangle_{p=0}\\ &= \sum_\alpha z^{-\alpha(p)}|\alpha(p)\rangle\langle\alpha(p)|e^{\sqrt{2}Q(1)\cdot k}|0\rangle_{p=0}, \end{aligned} \tag{D2}$$

where we have introduced a complete set of eigenstates of J_0,

$$J_0(p)|\alpha(p)\rangle = \alpha(p)|\alpha(p)\rangle. \tag{D3}$$

Clearly, however, we need sum only over states with $p^2=0$. Thus far, in our study of the spectrum of J_0 we have found no reason to suspect that $p^2=0$ states exist with $\alpha<0$; assuming this, then only states with $\alpha=0$, $p^2=0$ survive in the limit $z\to\infty$. Thus,

$$\lim_{z\to\infty}\Gamma(k,z)|0\rangle_{p=0} = \psi e^{\sqrt{2}Q(1)\cdot k}|0\rangle_{p=0}, \tag{D4}$$

where

$$\psi = \sum_k |k\rangle\langle k|$$

is the sum over the $k^2=0$ vacuum and its entire K-

[23] The discussion can be generalized to $J_0|S\rangle_0=-n|S\rangle_0$ but we omit the complications for the sake of brevity. In the analogous discussion of the model in Sec. V, we would of course use only $J_0|S\rangle_0=|S\rangle_0$.

degenerate family. We now claim that

$$\psi e^{\sqrt{2}Q(1)\cdot k}|0\rangle_{p=0}=|0\rangle_{k^2=0}\,, \tag{D5}$$

that is, only the true (sector zero) vacuum contributes. For example, consider

$$\begin{aligned} &{}_{k^2=0}\langle 0|K_l(p)e^{\sqrt{2}Q(1)\cdot k}|0\rangle_{p=0} \quad (l>0) \\ &\qquad = {}_{k^2=0}\langle 0|[K_l(p),e^{\sqrt{2}Q(1)\cdot k}]|0\rangle_{p=0}; \end{aligned} \tag{D6}$$

then using the fact that $Q^\mu(1)$ satisfies the "stability conditions"

$$[K_l-K_0,Q^\mu(1)]=0\,, \tag{D7}$$

this reduces to

$${}_{k^2=0}\langle 0|[K_0(p),e^{\sqrt{2}Q(1)\cdot k}]|0\rangle_{p=0}=0\,. \tag{D8}$$

Similarly, one shows that

$${}_{k^2=0}\langle 0|(K_1)^l e^{\sqrt{2}Q(1)\cdot k}|0\rangle_{p=0}=0\,, \tag{D9}$$

etc. Q.E.D.

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Exact Consequences of Broken $O(4)$ Symmetry. III. Factorization and Mass Dependence*

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We use analyticity arguments to obtain the constraints imposed by factorization and broken $O(4)$ symmetry on Regge daughter sequences coupled to spinless external particles. Our results hold for arbitrary external masses.

I. INTRODUCTION

In Papers I and II of this series, we derived the consequences of broken $O(4)$ symmetry for Regge-daughter sequences corresonding to Toller poles with $M=0$,[1] and integer $M\geqslant 1$.[2] Our results were not based on $O(4)$ symmetry directly, but rather on the requirement that scattering amplitudes be analytic at zero total energy ($t=0$ in our notation). Thus, our work is partially a derivation of new results, and partially a demonstration that the requirement of analyticity is interchangeable with the study of $O(4)$ symmetry.

In Papers I and II we emphasized the trajectory functions $\alpha(k,t)$, $k=0, 1, 2, \ldots$, which make up a daughter sequence. Here we wish to study the reduced residues $\gamma(k,t)$, and the many new complications which arise in a discussion of them. In the present paper we bypass the complications connected with spinning external particles. Among these are conspiracy relations and the requirement of factorization in the helicity indices. We do this by studying reactions involving spinless particles, as in Paper I. Accordingly, we can study only sequences with $M=0$, as coupled to spinless channels. However, even for this restricted case much remains to be shown beyond the results derived in Paper I. First, we want results which are valid for arbitrary external masses, and which show how the odd daughters decouple when either the initial or the final particle pair have equal masses. Second, we must impose the requirement of factorization, and verify that the daughter sequence constitutes a Toller pole when coupled to equal-mass initial and final particle pairs. Third, we want to find out if factorization is *necessary* to get a Toller pole, starting from analyticity requirements. It has been known for some time that analyticity and factorization are *sufficient* to get a Toller pole in equal-mass scattering,[3–5] but a Toller pole might also result from analyticity and continuity in the masses. We verify that factorization is necessary.

A modification of the procedures of Paper I must be made so that our results will be valid for arbitrary masses. We must recognize that a pseudothreshold (a point where the c.m. three-momentum vanishes) moves to $t=0$ when channel masses become equal. We must therefore deal only with functions which are analytic at pseudothresholds, as well as $t=0$, if we want our results to be valid in a neighborhood of $t=0$ for all mass configurations. For example, the function $R(k,t)$ defined in Eq. (I10), which differs from $\gamma(k,t)$ by a kinematic factor, has a square-root singularity at pseudothresholds for odd k because of the kinematic factor. Such kinematic singularities must be avoided. Our procedure for doing this is to use the expansion of $Q_{-\alpha(t)-1}(-z_t)$ in terms of powers of z_t^{-2} instead of the expansion in powers of $(1+z_t)^{-1}$ used in Paper I. The new choice entails one more summation, but is useful for all mass configurations.

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