Chiral Loops*

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The correct Feynman rules for chiral-invariant theories differ from the naive rules. The difference is such that the soft-pion theorems, including the vanishing of the pion mass, are preserved to each order of perturbation theory. In particular, there are special definitions of the pion field for which all higher-order corrections to the off-shell pion-pion scattering amplitude cancel at low momenta.

I. INTRODUCTION

CHIRAL-INVARIANT effective Lagrangians were introduced[1] as a compact device for deriving the low-energy restrictions placed on S-matrix elements by current algebra and partial conservation of axial-vector current (PCAC). The techniques for implementing this are well known and amount to using only the tree graphs in the perturbation expansion for any physical process. Lagrangians obtained from each other by making general point transformations of the pion field are all equivalent and yield the same S-matrix elements.

The usefulness of such Lagrangians leads to the speculation that they might be the basis of a dynamical theory. For example, restricting attention to pion Lagrangians, we may define conserved vector and axial-vector currents, which generate the chiral $SU(2)\times SU(2)$ algebra[2]; the pion field itself is massless in such a theory, so that all of the current algebra and PCAC conditions will presumably be satisfied by the exact solution and hence by the perturbation solution in *all* orders, not just in the tree approximation. This would possibly provide a method for making unitarity corrections to current-algebra low-energy theorems.

The question we address in this paper is the following: Are the masslessness of the pion, the Adler zeros, and the general-current algebra theorems maintained, order by order, in perturbation theory? There is reason to believe they might not be, since the theory appears to be highly divergent, and examples of anomalous behavior for commutators and divergences of currents are known to exist in perturbation theory for some models.[3]

A calculation which seems to indicate such anomalous behavior has been carried out by Charap.[4] He finds, using naive[5] perturbation theory, that the worst divergences in perturbation theory[6] do indeed violate the Adler condition for π-π scattering. Moreover, the pion seems to acquire a mass. However, there is one definition of the pion field for which Charap shows that these divergences vanish and for which, therefore, current algebra is maintained in perturbation theory (at least to sixth order, which is as far as he carries the calculation). His procedure does not determine the pion transformation function $f(\pi^2)$ completely.[7] Only the first four terms in a power-series expansion of $f(\pi^2)$ are given[8]:

$$f(\pi^2)=1-\tfrac{2}{5}(\lambda^2\pi^2)-\frac{9}{7\times5^2}(\lambda^2\pi^2)^2-\frac{184}{7\times5^3\times3^2\times2}(\lambda^2\pi^2)^3+\cdots. \quad (1.1)$$

A possible interpretation of Charap's work is that indeed anomalies do exist which render the chiral invariance of the theory invalid except for the particular field of Eq. (1.1). This might be anticipated because

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[1] We shall use the notation and conceptual framework of S. Weinberg, Phys. Rev. **166**, 1568 (1968). References to other papers will also be found in this article.

[2] S. L. Adler and R. F. Dashen, *Current Algebra* (Benjamin, New York, 1968).

[3] See, e.g., J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969); S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[4] J. Charap, Phys. Rev. D **2**, 1554 (1970).

[5] By naive perturbation theory we mean the theory for which $\mathcal{H}_I=-\mathcal{L}_I$ and the propagator

$$\Delta_{\mu\nu}=\int d^4x\, e^{-ikx}\langle 0|T\partial_\mu\varphi(x)\partial_\nu\varphi(0)|0\rangle$$

is taken to be $ik_\mu k_\nu/k^2$.

[6] In this calculation the interaction Lagrangian contains two derivatives. These derivatives, when acting on external lines, yield just the powers of momentum needed to obtain the Adler zeros. In higher-order calculations, however, the derivative can act on internal lines which are integrated over. These terms, which violate the Adler theorem, are typically quartically divergent or are products of quartic divergences.

[7] See Ref. 1 or our Eq. (2.2) for the definition of $f(\pi^2)$.

[8] In Ref. 4 the numerator of the fourth term is given as -163. This is an error which comes from Eq. (2.4) of that paper. The quantity $(1+2\alpha_1)^3$ must be subtracted from the right-hand side. When this is done, the values of α_i and β_i computed there lead to our Eq. (1.1).

the theory as it stands is meaningless unless a cutoff is introduced to deal with ultraviolet divergences, and this cutoff might be the agency which breaks chiral invariance. Our investigation shows (perhaps unfortunately, depending on one's point of view) that this suggestion is false. In fact, Charap's results, for all but the field defined by Eq. (1.1), are due to an inapplicable use of naive perturbation theory.[9] We show in Sec. II that a careful use of the canonical quantization procedure yields rules for expanding the S matrix for which the leading divergences which plague the naive calculation never appear, this result being true for *all* pion fields. Since it is precisely the leading divergences which appear to give the pion its mass, violate the Adler theorems, etc., in Charap's calculation, we are now able to affirm that a correct set of perturbation theory rules *does* maintain current algebra order by order, all pion fields being on the same footing.

As a byproduct of our investigation we ask, in Sec. III, whether there is a particular definition of the pion field for which the correct rules as derived in Sec. II are just the naive rules. The answer is yes, for the field defined by

$$f(\pi^2)=(\lambda^2/\pi^2)^{1/2}\cot\tfrac{1}{2}y(\pi^2)\,, \tag{1.2}$$

where $y(\pi^2)$ satisfies

$$y(\pi^2)-\sin y(\pi^2)=\tfrac{4}{3}(\lambda^2\pi^2)^{3/2}. \tag{1.3}$$

Since this field gives rise to a Lagrangian for which the naive perturbation theory is correct, it must be the field that Charap found, and indeed the expansion of Eqs. (1.2) and (1.3) in powers of $\lambda^2\pi^2$ is exactly Eq. (1.1).

In Sec. IV we consider whether current algebra may be used to give meaningful theorems for off-shell amplitudes, as well as for physical processes, in a chirally invariant theory. We find that there are such theorems for particular definitions of the pion field (for which the double commutator of π with Q_5, the generator of chiral rotations, is linear in π). These theorems show that to order q^2 the loop contributions in perturbation theory can be interpreted as renormalization effects.

II. PERTURBATION THEORY FOR CHIRAL DYNAMICS

We begin by summarizing the relevant parts of the theory of chiral dynamics following Ref. 1. The generators of $SU(2)\times SU(2)$ transformations are Q^a, $Q_5{}^a$ with the usual commutation relations. The pion field satisfies

$$[Q^a,\pi^b]=i\epsilon^{abc}\pi^c\,, \tag{2.1}$$

$$[Q_5{}^a,\pi^b]=-i\lambda^{-1}[f(\pi^2)\delta^{ab}+g(\pi^2)\pi^a\pi^b]\,, \tag{2.2}$$

where $\lambda^{-1}=f_0{}^\pi$ is the unrenormalized pion decay amplitude. The function $g(\pi^2)$ is

$$g(\pi^2)=[\lambda^2+2f'(\pi^2)f(\pi^2)][f(\pi^2)-2\pi^2f'(\pi^2)]^{-1} \tag{2.3}$$

and $f(\pi^2)$, normalized by $f(0)=1$, is arbitrary and defines the pion field. It is convenient for our purposes to define

$$[Q_5{}^{a_1},[Q_5{}^{a_2},\ldots,[Q_5{}^{a_n},\pi^b]\cdots]]\equiv\sigma^{a_1a_2\cdots a_nb}.$$

According to Eq. (2.2),

$$\sigma^{ab}=-i\lambda^{-1}[f(\pi^2)\delta^{ab}+g(\pi^2)\pi^a\pi^b]\,, \tag{2.4}$$

and it is easy to show that

$$\sigma^{abc}=\pi^a\delta^{bc}-\lambda^{-2}\{(\delta^{ab}\pi^c+\delta^{ac}\pi^b+\delta^{bc}\pi^a)f(\pi^2)g(\pi^2) +2\pi^a\pi^b\pi^c(g^2(\pi^2)+g'(\pi^2)[f(\pi^2)+\pi^2g(\pi^2)])\}\,. \tag{2.5}$$

The invariant Lagrangian for the interaction of these massless pions has the form

$$\mathcal{L}=\tfrac{1}{2}(D^\mu\pi)\cdot(D_\mu\pi)\,, \tag{2.6}$$

where the covariant derivative is given by

$$(D_\mu\pi)_a=d_{ab}(\pi)\partial_\mu\pi_b\,, \tag{2.7}$$

$$d_{ab}(\pi)\equiv d_1(\pi^2)\delta_{ab}+d_2(\pi^2)\pi_a\pi_b\,, \tag{2.8}$$

$$d_1(\pi^2)=[f^2(\pi^2)+\lambda^2\pi^2]^{-1/2}\,, \tag{2.9}$$

$$d_2(\pi^2)=-[f^2(\pi^2)+\lambda^2\pi^2]^{-1} \times[2f'(\pi^2)+\lambda^2\{f(\pi^2)+[f^2(\pi^2)+\lambda^2\pi^2]^{1/2}\}^{-1}]. \tag{2.10}$$

Thus if we write Eq. (2.6) as

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\pi^aG_{ab}(\pi)\partial^\mu\pi^b\,, \tag{2.11}$$

we have

$$G_{ab}(\pi)=d^2{}_{ab}(\pi)\,, \tag{2.12}$$

which, in terms of the function $f(\pi^2)$, becomes

$$G_{ab}(\pi)=[f^2(\pi^2)+\lambda^2\pi^2]^{-1}\delta_{ab}+[f^2(\pi^2)+\lambda^2\pi^2]^{-2} \times[4\pi^2f'(\pi^2)f'(\pi^2)-4f(\pi^2)f'(\pi^2)-\lambda^2]\pi_a\pi_b. \tag{2.13}$$

We now develop the perturbation-theory rules appropriate to this Lagrangian. Calling Π_a the momentum conjugate to the field π_a, we have

$$\Pi_a=\delta\mathcal{L}/\delta\partial_0\pi_a=G_{ab}(\pi)\partial_0\pi_b\,, \tag{2.14}$$

$$\partial_0\pi_a=G^{-1}{}_{ab}(\pi)\Pi_b. \tag{2.15}$$

The Hamiltonian is therefore

$$\mathcal{H}=\Pi_aG^{-1}{}_{ab}(\pi)\Pi_b-\mathcal{L}\,, \tag{2.16}$$

$$\mathcal{H}=\tfrac{1}{2}\Pi_aG^{-1}{}_{ab}(\pi)\Pi_b-\tfrac{1}{2}\partial^i\pi_aG_{ab}(\pi)\partial_i\pi_b. \tag{2.17}$$

It is convenient, for the purpose of dividing the Lagrangian and Hamiltonian into free and interaction parts, to write

$$G_{ab}(\pi)=\delta_{ab}+\bar{G}_{ab}(\pi)\,, \tag{2.18}$$

so that Eq. (2.17) becomes

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_I\,,$$

$$\mathcal{H}_0=\tfrac{1}{2}\Pi_a\Pi_a-\tfrac{1}{2}\partial^i\pi_a\partial_i\pi_a\,, \tag{2.19}$$

$$\mathcal{H}_I=-\mathcal{L}_I-\tfrac{1}{2}\partial_0\pi_a\bar{G}^2{}_{ab}(\pi)\partial_0\pi_b\,, \tag{2.20}$$

[9] By inapplicable we mean that it violates chiral symmetry. J. Reif and M. Veltman [Nucl. Phys. **B13**, 545 (1969)] give a regulator procedure which justifies the use of naive perturbation theory, but the regulator mass is not chirally invariant.

where the interaction Lagrangian is simply

$$\mathcal{L}_I=\tfrac{1}{2}\partial_\mu\pi_a\bar{G}_{ab}(\pi)\partial^\mu\pi_b. \tag{2.21}$$

We see that, as in all cases with derivative coupling, the interaction Hamiltonian is not simply the negative of the interaction Lagrangian but there is an additional, noncovariant piece. Of course the Feynman-Dyson expansion of the S matrix is always covariant; noncovariant contributions to the propagator of pion field derivatives ensure this.

The interaction Hamiltonian, Eq. (2.20), is written in terms of the Heisenberg-picture field operators and so is not the operator which enters into the perturbation expansion. To write $\mathcal{H}_I$ as a function of the interaction picture (IP) field operators, we first write it as a function of π_a, $\partial_i\pi_a$, and Π_a, and recognize that these operators transform simply into the IP,

$$\pi_a\longrightarrow\varphi_a\,,$$
$$\partial_i\pi_a\longrightarrow\partial_i\varphi_a\,,$$
$$\Pi_a\longrightarrow\partial_0\varphi_a\,,$$

where φ_a is the IP pion field. Making these substitutions, we have the interaction Hamiltonian in the IP

$$\mathcal{H}_I{}^{(\mathrm{IP})}=-\tfrac{1}{2}\partial_\mu\varphi_a\bar{G}_{ab}(\varphi)\partial^\mu\varphi_b +\tfrac{1}{2}\partial_0\varphi_a\{\bar{G}^2(\varphi)[1+\bar{G}(\varphi)]^{-1}\}_{ab}\partial_0\varphi_b. \tag{2.22}$$

The usual perturbation rules are to expand in powers of $\mathcal{H}_I{}^{(\mathrm{I.P.})}$ the object

$$\langle 0|O(x)|0\rangle=\langle 0|TO(x)^{(\mathrm{IP})}\times\exp\left(-i\int d^4y\,\mathcal{H}_I{}^{(\mathrm{IP})}(y)\right)|0\rangle_C\,, \tag{2.23}$$

where $O(x)$ is any operator. Here the S matrix is

$$S=T\exp\left(-i\int d^4y\,\mathcal{H}_I{}^{(\mathrm{IP})}(y)\right), \tag{2.24}$$

and the subscript C indicates that only connected diagrams are relevant.

Upon expanding Eq. (2.23) in the usual way, we shall encounter, for interactions of the type in Eq. (2.13), the following IP propagators:

$$\delta_{ab}\Delta(k^2)=\int d^4x\,e^{ikx}\langle 0|T\varphi_a(x)\varphi_b(0)|0\rangle\,, \tag{2.25a}$$

$$\delta_{ab}\Delta_\mu(k)=\int d^4x\,e^{ikx}\langle 0|T\partial_\mu\varphi_a(x)\varphi_b(0)|0\rangle\,, \tag{2.25b}$$

$$\delta_{ab}\Delta_{\mu\nu}(k)=\int d^4x\,e^{ikx}\langle 0|T\partial_\mu\varphi_a(x)\partial_\nu\varphi_b(0)|0\rangle\,. \tag{2.25c}$$

The explicit form of these is

$$\Delta(k^2)=i/(k^2+i\epsilon)\,, \tag{2.26a}$$

$$\Delta_\mu(k)=k_\mu/(k^2+i\epsilon)\,, \tag{2.26b}$$

$$\Delta_{\mu\nu}(k)=ik_\mu k_\nu/(k^2+i\epsilon)-ig_{\mu 0}g_{\nu 0}\,. \tag{2.26c}$$

Equations. (2.22), (2.23), and (2.26) completely specify the Feynman-Dyson expansion. Although both Eqs. (2.22) and (2.26c) have noncovariant pieces, the perturbation theory will be covariant in every order.

III. COVARIANT PERTURBATION THEORY

It is somewhat inconvenient to use the formalism developed in Sec. II to do higher-order calculations. It is frequently supposed that one can ignore the complications which arise from the noncovariant pieces. It can be shown, for certain interactions with derivative coupling, that not only do the noncovariant pieces of the Hamiltonian and propagator combine to give a covariant result but these extra pieces cancel. That is, it is correct to do perturbation theory using the naive rules.[10]

For nonlinear Lagrangians, of the type we are considering, this result is not true. This may easily be seen from the explicit calculation of Appendix B. In fact, it is precisely the terms which violate chiral invariance, using the naive rules, which are canceled by the additional terms in the Hamiltonian and propagator.

In order to see more clearly what these extra terms contribute, it is convenient to reformulate the theory in terms of a covariant interaction Hamiltonian. We show in Appendix A that for a system defined by Eqs. (2.22) and (2.25) we can drop the noncovariant piece of the propagator Eq. (2.25c) by using as the interaction Hamiltonian

$$\mathcal{H}_I{}^{(\mathrm{IP})}=-\tfrac{1}{2}\partial^\mu\varphi_a\bar{G}_{ab}(\varphi)\partial_\mu\varphi_b +\tfrac{1}{2}i\delta^4(0)\,\mathrm{Tr}\ln[1+\bar{G}(\varphi)]\,, \tag{3.1}$$

which, of course, is covariant. Explicit calculation verifies that the two theories give the same results. A similar structure was first discovered by Lee and Yang in a related context.[11]

We are now in a position to understand Charap's result that, using naive perturbation theory, there is a unique pion field, for which the leading divergences, which are responsible for the violation of chiral invariance, vanish. Since these terms are not present for any theory using the correct Hamiltonian, Eq. (3.1), and since this Hamiltonian is to be used with the naive rules, it is clear that Charap's field is the one for which Eq. (3.1) reduces to the naive Hamiltonian, i.e.,

$$\mathrm{Tr}\ln(1+\bar{G})=\mathrm{Tr}\ln G=0\,. \tag{3.2}$$

[10] Y. Nambu, Progr. Theoret. Phys. (Kyoto) **7**, 131 (1952).
[11] T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962).

To find the pion field corresponding to Eq. (3.2), we use the identity

$$\mathrm{Tr}\ \ln G = \ln \det G\,,$$

so that Eq. (3.2) becomes

$$\det G = 1\,. \tag{3.3}$$

The use of Eq. (2.12) for G, gives

$$\det d = 1\,, \tag{3.4}$$

where the positive sign has been chosen so that (ultimately) the boundary condition $f(0)=1$ can be imposed. Finally, from Eq. (2.8) we see that the following is equivalent to (3.4):

$$d_1{}^3(\pi^2)+\pi^2 d_1{}^2(\pi^2)d_2(\pi^2)=1\,. \tag{3.5}$$

This yields a differential equation for the function $f(\pi^2)$ which determines the transformation law of the pion field. From Eqs. (2.9) and (2.10), we obtain this differential equation,

$$[f^2(\pi^2)+\lambda^2\pi^2]^2 = f(\pi^2)-2\pi^2 f'(\pi^2)\,. \tag{3.6}$$

In order to solve (3.6) we let

$$f(\pi^2)=(\lambda^2\pi^2)^{1/2}\cot\tfrac{1}{2}y(\pi^2) \tag{3.7}$$

and find that $y(\pi^2)$ satisfies the transcendental equation

$$y(\pi^2)-\sin y(\pi^2)=\tfrac{4}{3}(\lambda^2\pi^2)^{3/2}\,. \tag{3.8}$$

As pointed out in Sec. I, this yields an $f(\pi^2)$ whose power-series expansion agrees with the terms computed by Charap.

It is worthwhile emphasizing that our approach shows that this field is not special in the sense of being the unique one which satisfies chiral invariance in perturbation theory. *All* definitions of the pion field are on an equal footing when the correct perturbation rules are used and they all yield a chirally invariant S matrix. It might be convenient to attempt to use Charap's field in calculations since the perturbation theory using it is the naive one, but the complexity of Eqs. (3.7) and (3.8) appears to negate this convenience.

IV. OFF-SHELL CURRENT-ALGEBRA THEOREMS

We have seen in Secs. I–III that a chiral-invariant perturbation theory exists for all definitions of the pion field. The hope has often been expressed that such a theory could be used to carry out dynamical calculations that go beyond the tree approximation. However, in such calculations one generally has to deal with soft pions that are off the mass shell.[12] We consider, in this section, the current-algebra theorems which can be used to validate the tree approximation even off the mass shell.

We shall derive a theorem for the π-π scattering amplitude with two pions off the mass shell. It has previously been established that this is sufficient to construct the entire amplitude to second order in the four-momentum.[13] The comments we make are easily generalized to the n-point amplitude with two off-shell pions, although we shall not give the details for these processes here. As usual, we define

$$T_{\mu\nu}{}^{abcd}(q_1,q_2,q_3)=\int d^4x d^4y d^4z\, e^{-iq_1x-iq_2y-iq_3z} \times\langle 0|TA_\mu{}^a(x)A_\nu{}^b(y)\pi^c(z)\pi^d(0)|0\rangle\,, \tag{4.1a}$$

$$T_\mu{}^{abc}(q_1,q_2)=\int d^4x d^4y\, e^{-iq_1x-iq_2y} \times\langle 0|TV_\mu{}^a(x)\pi^b(y)\pi^c(0)|0\rangle\,, \tag{4.1b}$$

$$T^{[ab][cd]}(q)=\int d^4x\, e^{-iqx}\langle 0|T\sigma^{ab}(x)\sigma^{cd}(0)|0\rangle\,, \tag{4.1c}$$

$$T^{a[bcd]}(q)=\int d^4x\, e^{-iqx}\langle 0|T\pi^a(x)\sigma^{bcd}(0)|0\rangle\,. \tag{4.1d}$$

Here the vector current $V_\mu{}^a$ and the axial-vector current $A_\mu{}^a$ are related to the generators of chiral transformations of Eqs. (2.1) and (2.2) by

$$Q^a=\int d^3x\, V_0{}^a(x)\,, \tag{4.2a}$$

$$Q_5{}^a=\int d^3x\, A_0{}^a(x)\,. \tag{4.2b}$$

It is easy to derive the Ward identity (recall that in a chiral-invariant theory, of the type we are considering, the axial-vector current is conserved):

$$\begin{aligned} iq_1{}^\mu iq_2{}^\nu T_{\mu\nu}{}^{abcd}(q_1,q_2,q_3) &= i\epsilon^{abc'}\tfrac{1}{2}i(q_2-q_1)^\mu T_\mu{}^{c'cd}(q_1+q_2,q_3)+\tfrac{1}{2}T^{d[abc]}(q_4)\\ &\quad+\tfrac{1}{2}T^{d[bac]}(q_4)+\tfrac{1}{2}T^{c[abd]}(q_3)+\tfrac{1}{2}T^{c[bad]}(q_3)\\ &\quad+T^{[bc][ad]}(q_2+q_3)+T^{[ac][bd]}(q_1+q_3)\,, \end{aligned} \tag{4.3}$$

where

$$q_4=-q_1-q_2-q_3\,.$$

We now let $q_1{}^\mu$, $q_2{}^\nu\to 0$, remembering to pick up the pion poles in the axial-vector current lines on the left-hand side of (4.3),

$$\begin{aligned} \lim_{q_1\to 0;\, q_2\to 0} iq_1{}^\mu iq_2{}^\nu T_{\mu\nu}{}^{abcd}(q_1,\,q_2,\,q_3=q) &= (q^2)^{-2}\Lambda^{-2}M^{abcd}(q_1=0,\,q_2=0,\,q_3=q)\\ &= \tfrac{1}{2}T^{d[abc]}(q)+\tfrac{1}{2}T^{d[bac]}(q)+\tfrac{1}{2}T^{c[abd]}(q)+\tfrac{1}{2}T^{c[bad]}(q)\\ &\quad+T^{[bc][ad]}(q)+T^{[ac][bd]}(q)\,. \end{aligned} \tag{4.4}$$

In this equation the constant Λ is the inverse of the physical pion decay rate, $\Lambda=f_\pi{}^{-1}$, and $M^{abcd}(q_1,q_2,q_3)$

[12] S. Weinberg, Phys. Rev. D **2**, 674 (1970).

[13] S. Weinberg, Phys. Rev. Letters **17**, 617 (1966).

is the usual Lehmann-Symanzik-Zimmermann (LSZ) off-shell pion scattering amplitude. The σ propagator terms have no pole at $q^2=0$, and hence start off like a constant in a power-series expansion in q^2, while the amplitudes $T^{a[bcd]}$ *do* have such a pole and so start off like q^{-2}. Therefore, we see that we have the theorem

$$M^{abcd}(0,0,q)=\Lambda^2q^2 \lim_{q^2\to 0} \tfrac{1}{2}q^2[T^{d[abc]}(q)+T^{d[bac]}(q)+T^{c[abd]}(q)+T^{c[bad]}(q)]+O(q^4). \quad (4.5)$$

In general, a theorem of this type is not very useful, beyond telling us that the pion scattering amplitude vanishes as $q^2\to 0$, which is simply the Adler condition. This is because σ^{abc}, as given by Eq. (2.5), is a complicated function of π, and we have no way of evaluating the right-hand side of Eq. (4.5). To make further progress, we must ask for what definitions of the pion field is σ^{abc} simple enough for us to find the right-hand side of Eq. (4.5) exactly? The answer is clear; σ^{abc} must be linear in the pion field. We see from Eq. (2.5) that to implement this we require

$$f(\pi^2)g(\pi^2)=\text{const}=c\,, \quad (4.6\text{a})$$

$$g^2(\pi^2)+g'(\pi^2)[f(\pi^2)+\pi^2g(\pi^2)]=0\,. \quad (4.6\text{b})$$

Equations (4.6) have two solutions [$g(\pi^2)$ is defined by Eq. (2.3)]. In the first place, we have

$$g(\pi^2)=0\,, \quad (4.7)$$

which satisfies Eqs. (4.6), and so

$$\sigma^{abc}=\pi^a\delta^{bc}. \quad (4.8)$$

From Eq. (2.3), then, we find

$$f(\pi^2)=(1-\lambda^2\pi^2)^{1/2}. \quad (4.9)$$

This solutions corresponds to the well-known nonlinear version of the σ model.

The other solution obtains when $g(\pi^2)\neq 0$, so that [recalling that $f(\pi^2)$ cannot be zero since $f(0)=1$] $c\neq 0$. It is easy to combine Eqs. (4.6) and (2.3) and obtain $c=-\lambda^2$, so that for this model

$$\sigma^{abc}=2\pi^a\delta^{bc}+\pi^b\delta^{ac}+\pi^c\delta^{ab}. \quad (4.10)$$

In addition we have from Eq. (4.6b)

$$g^2(\pi^2)+g'(\pi^2)[-\lambda^2g^{-1}(\pi^2)+\pi^2g(\pi^2)]=0\,,$$

which has the solution

$$g(\pi^2)=-2\lambda^2[1+(1-4\lambda^2\pi^2)^{1/2}]^{-1}\,, \quad (4.11)$$

$$f(\pi^2)=\tfrac{1}{2}[1+(1-4\lambda^2\pi^2)^{1/2}]. \quad (4.12)$$

This solution has the property that the pion field transforms as a member of the (1,1) representation of $SU(2)\times SU(2)$.

Returning now to Eq. (4.5), we see that we have an exact current-algebra theorem for these two cases, (4.9) and (4.12). Respectively, this theorem is

$$M^{abcd}(0,0,q)=\Lambda^2q^2\begin{cases}(\delta^{ad}\delta^{bc}+\delta^{ac}\delta^{bd})\\(2\delta^{ab}\delta^{cd}+3\delta^{ad}\delta^{bc}+3\delta^{ac}\delta^{bd})\end{cases}+O(q^4). \quad (4.13)$$

We see, of course, that the off-mass-shell theorems differ for the two different definitions of the pion field. Equation (4.13) represents a kind of renormalization theorem for these two models. In the tree approximation, using the Lagrangian appropriate to either Eq. (4.9) or Eq. (4.12), it is easy to find

$$M^{abcd}(0,0,q)=\lambda^2q^2\begin{cases}(\delta^{ad}\delta^{bc}+\delta^{ac}\delta^{bd})\\(2\delta^{ab}\delta^{cd}+3\delta^{ad}\delta^{bc}+3\delta^{ac}\delta^{bd}).\end{cases} \quad (4.14)$$

The total effect of all the higher-order (in λ^2) graphs is simply (to order q^2) to renormalize multiplicatively $\lambda^2=(f_\pi{}^0)^{-2}$. This can be verified by explicit calculation using either version of the chiral-invariant perturbation theory which we have developed.

Alternatively, we may say that for these two theories the tree approximation is as good off the mass shell as on the mass shell. The effect of loops, to order q^2, is taken into account by simply replacing the unrenormalized pion decay constant by its physical value. Of course current-algebra theorems of this type are inherently low-energy theorems, so we are not implying anything about the validity of the tree approximation for hard pions.

Note added in manuscript. While this work was being prepared for publication, we received papers [J. Honerkamp and K. Meetz, Phys. Rev. D **3**, 1996 (1971), and J. Charap, Phys. Rev. D **3**, 1998 (1971)] which present essentially the same material as our Sec. III. Also we have learned of the following related investigations: D. G. Boulware, Ann. Phys. (N. Y.) **56**, 140 (1970); A. Salam and J. Strathdee, Phys. Rev. D **2**, 2869 (1970).

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APPENDIX A

We derive the effective Hamiltonian [Eq. (3.1)], which permits the use of naive Feynman rules in the Dyson expansion of the S matrix. The result was obtained by Lee and Yang,[11] in another context, by explicitly summing the relevant graphs in perturbation theory. Here we use an elegant functional technique.[14]

The S matrix is given by

$$S=T\,\exp\left(\frac{i}{2}\int d^4x\,\varphi^\mu{}_a(x)H_{ab,\mu\nu}(x)\varphi^\nu{}_b(x)\right), \quad (\text{A1a})$$

$$\varphi^\mu{}_a=\partial^\mu\varphi_a\,,$$

[14] The formalism employed here is due to J. Schwinger. We learned how to exploit it from K. Johnson, whose instruction is gratefully acknowledged.

$$H_{ab,\mu\nu}=\bar{G}_{ab}g_{\mu\nu}-[\bar{G}^2(1+\bar{G})^{-1}]_{ab}n_\mu n_\nu. \quad \text{(A1b)}$$

A unit timelike vector n_μ has been introduced. For future use, we determine $H^{-1}{}_{ab,\mu\nu}$, the inverse of $H_{ab,\mu\nu}$:

$$H_{ab,\mu\nu}H^{-1}{}_{bc,\nu'\omega}g^{\nu\nu'}=\delta_{ac}g_{\mu\omega}, \quad \text{(A2a)}$$

$$H^{-1}{}_{ab,\mu\nu}=\bar{G}^{-1}{}_{ab}g_{\mu\nu}+\delta_{ab}n_\mu n_\nu. \quad \text{(A2b)}$$

The propagators which are used in evaluating (A1a) are

$$\langle 0|T\varphi_a(x)\varphi_b(y)|0\rangle=i\Delta_{ab}(x-y), \quad \text{(A3a)}$$

$$\langle 0|T\varphi^\mu{}_a(x)\varphi_b(y)|0\rangle=i\partial^\mu\Delta_{ab}(x-y), \quad \text{(A3b)}$$

$$\begin{aligned}\langle 0|T\varphi^\mu{}_a(x)\varphi^\nu{}_b(y)|0\rangle&=-i\partial^\mu\partial^\nu\Delta_{ab}(x-y)\\&\quad -in^\mu n^\nu\delta_{ab}\delta^4(x-y)\\&\equiv i\Delta^{\mu\nu}{}_{ab}(x-y).\end{aligned} \quad \text{(A3c)}$$

In order to study the effect of replacing the noncovariant propagator $\Delta^{\mu\nu}{}_{ab}(x-y)$ by the covariant object

$$\begin{aligned}\bar{\Delta}^{\mu\nu}{}_{ab}(x-y)&=\Delta^{\mu\nu}{}_{ab}(x-y)+n^\mu n^\nu\delta_{ab}\delta^4(x-y)\\&=-\partial^\mu\partial^\nu\Delta_{ab}(x-y),\end{aligned}$$

we exhibit the dependence of S on $\Delta^{\mu\nu}{}_{ab}(x-y)$; i.e., we perform all the contractions in (A1a) between $\varphi^\mu{}_a$ and $\varphi^\nu{}_b$. The computation may be performed by a trick: $\varphi^\mu{}_a(x)$ is replaced by

$$\varphi^\mu{}_a(x)+i\int d^4x'\Delta^{\mu\mu'}{}_{aa'}(x-x')\frac{\delta}{\delta\varphi^{\mu'}{}_{a'}(x')},$$

$$\begin{aligned}S=:\exp\Bigg\{\frac{i}{2}\int d^4x\Bigg[&\varphi^\mu{}_a(x)\\&+i\int d^4x'\Delta^{\mu\mu'}{}_{aa'}(x-x')\frac{\delta}{\delta\varphi_{a'}{}^{\mu'}(x')}\Bigg]\\&\times H_{ab,\mu\nu}(x)\Bigg[\varphi^\nu{}_a(x)\\&+i\int d^4x''\Delta^{\nu\nu'}{}_{bb'}(x-x'')\frac{\delta}{\delta\varphi^{\nu'}{}_{b'}(x'')}\Bigg]\Bigg\}:.\end{aligned} \quad \text{(A4)}$$

Note that $H_{ab,\mu\nu}$ does not depend on $\varphi^\mu{}_a$. Since we are uninterested in contractions with φ_a, the dependence of $H_{ab,\mu\nu}$ on that variable may be ignored. The colons in (A4) indicate that normal ordering, *with respect to* $\varphi^\mu{}_a$, has been performed; however, contractions with φ_a have not been evaluated.

What follows will be vastly simplified by using matrix notation. The field $\varphi^\mu{}_a(x)$ is considered to be the vector $\mathbf{Q}$, labeled by the isospin index a, space-time index μ, and coordinate "index" x. Also the matrix $\mathbf{\Delta}$ is introduced: $\mathbf{\Delta}=\Delta^{\mu\nu}{}_{ab}(x-y)$. Finally, we define the matrix $\mathbf{H}$ and its inverse $\mathbf{H}^{-1}$

$$\mathbf{H}=H_{ab,\mu\nu}(x,y)=H_{ab,\mu\nu}(x)\delta^4(x-y), \quad \text{(A5a)}$$

$$\mathbf{H}^{-1}=H^{-1}{}_{ab,\mu\nu}(x,y)=H^{-1}{}_{ab,\mu\nu}(x)\delta^4(x-y). \quad \text{(A5b)}$$

Thus the formula for S is

$$S=:\exp\left\{\frac{i}{2}\left[\mathbf{Q}+i\left(\mathbf{\Delta}\frac{\delta}{\delta\mathbf{Q}}\right)\right]\mathbf{H}\left[\mathbf{Q}+i\left(\mathbf{\Delta}\frac{\delta}{\delta\mathbf{Q}}\right)\right]\right\}:. \quad \text{(A6)}$$

S satisfies a differential equation in $\mathbf{Q}$ which is derived by differentiating (A6) by $\mathbf{Q}$:

$$\frac{\delta S}{\delta\mathbf{Q}}=\left(i\mathbf{HQ}-\mathbf{H\Delta}\frac{\delta}{\delta\mathbf{Q}}\right)S. \quad \text{(A7a)}$$

The solution is

$$S=C:\exp[\tfrac{1}{2}i\mathbf{Q}(\mathbf{H}^{-1}+\mathbf{\Delta})^{-1}\mathbf{Q}]:. \quad \text{(A7b)}$$

Here C is a constant independent of $\mathbf{Q}$. To determine C, we differentiate (A7b) with respect to $\mathbf{H}$ and set $\mathbf{Q}$ equal to zero:

$$\left.\frac{\delta S}{\delta\mathbf{H}}\right|_{\mathbf{Q}=0}=\frac{\delta C}{\delta\mathbf{H}}. \quad \text{(A8a)}$$

An alternate formula for $\delta S/\delta\mathbf{H}$ may be obtained from (A6):

$$\begin{aligned}\left.\frac{\delta S}{\delta\mathbf{H}}\right|_{\mathbf{Q}=0}&=-\tfrac{1}{2}\mathbf{\Delta}C-\tfrac{1}{2}i\left(\mathbf{\Delta}\frac{\delta}{\delta\mathbf{Q}}\right)\left(\mathbf{\Delta}\frac{\delta}{\delta\mathbf{Q}}\right)S\Big|_{\mathbf{Q}=0}\\&=-\tfrac{1}{2}\mathbf{\Delta}C+\tfrac{1}{2}\mathbf{\Delta}(\mathbf{H}^{-1}+\mathbf{\Delta})^{-1}\mathbf{\Delta}C\\&=-\tfrac{1}{2}(\mathbf{\Delta}^{-1}+\mathbf{H})^{-1}C.\end{aligned} \quad \text{(A8b)}$$

The second equality follows from the first with the help of (A7b); the last term in (A8b) is a rearrangement of the previous expression. The resulting equation for C is

$$\delta C/\delta\mathbf{H}=-\tfrac{1}{2}(\mathbf{\Delta}^{-1}+\mathbf{H})^{-1}C. \quad \text{(A8c)}$$

The solution is given in terms of the Fredholm determinant:

$$C=[\det(\mathbf{1}+\mathbf{\Delta H})]^{1/2}. \quad \text{(A9)}$$

In offering (A9), we have fixed an arbitrary constant by the boundary condition $S|_{\mathbf{H}=0}=1$. Therefore, the explicit form for S is

$$\begin{aligned}S=\exp(-\tfrac{1}{2}\ln\det\mathbf{H})\exp[-\tfrac{1}{2}\ln\det(\mathbf{H}^{-1}+\mathbf{\Delta})]\\\times:\exp[\tfrac{1}{2}i\mathbf{Q}(\mathbf{H}^{-1}+\mathbf{\Delta})^{-1}\mathbf{Q}]:.\end{aligned} \quad \text{(A10)}$$

Let us now rewrite (A10) in terms of $\bar{\mathbf{\Delta}}=\mathbf{\Delta}+\mathbf{n}$, and $\bar{\mathbf{H}}^{-1}=\mathbf{H}^{-1}-\mathbf{n}$. Here $\mathbf{n}$ is the noncovariant matrix $n^\mu n^\nu\delta_{ab}\delta^4(x-y)$.

$$S=\bar{S}\exp[\tfrac{1}{2}\ln\det(1+\bar{\mathbf{H}}\mathbf{n})], \quad \text{(A11a)}$$

$$\begin{aligned}\bar{S}=\exp(-\tfrac{1}{2}\ln\det\bar{H})\exp[-\tfrac{1}{2}\ln\det(\bar{\mathbf{H}}^{-1}+\bar{\mathbf{\Delta}})]\\\times:\exp[\tfrac{1}{2}i\mathbf{Q}(\bar{\mathbf{H}}^{-1}+\bar{\mathbf{\Delta}})^{-1}\mathbf{Q}]:.\end{aligned} \quad \text{(A11b)}$$

From (A2b) we see that $\bar{\mathbf{H}}$ is given by the covariant expression

$$\bar{\mathbf{H}}=\bar{G}_{ab}(x)g_{\mu\nu}\delta^4(x-y), \quad \text{(A12a)}$$

$$\bar{\mathbf{H}}\mathbf{n}=\bar{G}_{ab}(x)n_\mu n_\nu\delta^4(x-y). \quad \text{(A12b)}$$

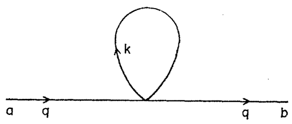

FIG. 1. First-order contribution to pion propagator.

The explicit factor occurring in (A11a) may be evaluated from (A12b).

$$\ln\det(\mathbf{1}+\bar{\mathbf{H}}\mathbf{n})=\mathrm{Tr}\,\ln(\mathbf{1}+\mathbf{H}\mathbf{n})$$
$$=\int d^4x\;\delta^4(0)\;\mathrm{Tr}\,\ln[1+\bar{G}(x)]. \quad \text{(A13)}$$

On the right-hand side of (A13), the trace is performed only over the isospin indices.

The final expression for S is

$$S=\bar{S}\exp\left[-i\int d^4x\,\tfrac{1}{2}i\delta^4(0)\;\mathrm{Tr}\,\ln G(x)\right]. \quad \text{(A14)}$$

According to (A10), (A11b), and (A12a), $\bar{S}$ is the S matrix computed from the interaction Hamiltonian $-\frac{1}{2}\varphi^\mu{}_a\bar{G}_{ab}\varphi_{b\mu}$, with the covariant propagator

$$\bar{\Delta}^{\mu\nu}{}_{ab}(x-y)=-\partial^\mu\partial^\nu\bar{\Delta}_{ab}(x-y).$$

Therefore, S may be evaluated with naive Feynman rules, provided the interaction Hamiltonian is chosen to be

$$\mathcal{H}_I{}^{(\mathrm{IP})}=-\tfrac{1}{2}\varphi^\mu{}_a\bar{G}_{ab}\varphi_{\mu b}+\tfrac{1}{2}i\delta^4(0)\mathrm{Tr}\,\ln G. \quad \text{(A15)}$$

APPENDIX B

In this appendix we consider the correction of order λ^2 to the pion propagator, using the noncovariant perturbation theory developed in Sec. II. We expand Eq. (2.22) as a power series in λ^2 and obtain

$$\mathcal{H}_I{}^{(\mathrm{IP})}=-\tfrac{1}{2}\alpha_1\lambda^2\varphi^2\partial_\mu\varphi^a\partial^\mu\varphi^a$$
$$-\tfrac{1}{2}\beta_1\lambda^2\varphi^a\partial_\mu\varphi^a\varphi^b\partial^\mu\varphi^b+O(\lambda^4). \quad \text{(B1)}$$

We take note of the fact that the expansion of the noncovariant piece of Eq. (2.22) starts off as λ^4 and thus does not contribute to this calculation. The constants α_1 and β_1 are those defined by Charap.

Now since, to this order, $\mathcal{H}_I{}^{(\mathrm{IP})}=-\mathcal{L}_I$, the calculation proceeds just as in Refs. 4 and 5 *except* that we use the noncovariant propagator of Eq. (2.26c). The Feynman diagram is shown in Fig. 1. We note that the most divergent contribution, the term which apparently gives the pion its mass, occurs when both derivatives act on the lines of the closed loop. Working this out, we find it is proportional to the trace (in μ and ν) of $\Delta_{\mu\nu}$ which, from Eq. (2.26c), vanishes. Thus there is no term which breaks chiral invariance coming from loops which begin and end at the same point. The total contribution of this diagram to the pion propagator is

$$\frac{1}{q^2}\delta_{ab}\lambda^2(3\alpha_1+\beta_1)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon}. \quad \text{(B2)}$$

Thus we see that the pion remains massless to this order independent of the choice of the pion field.

The point we wish to emphasize is that the noncovariant piece of $\mathcal{H}_I$ does not enter the calculation; the chiral-invariant result follows simply by taking the noncovariant character of $\Delta_{\mu\nu}$ into account. From this we see immediately and directly that the two noncovariant pieces do *not* simply cancel; they cannot be dropped.