Loop Diagrams in Dual-Resonance Theories with Nonlinear Trajectories*

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Rules are presented for calculating loop diagrams in a dual-resonance theory with nonlinear trajectories. Explicit expressions for the self-energy graph, the vacuum bubble graph, and the vertex graph are given. In the limit of linear trajectories the integrands of these graphs diverge. This is in contrast to the behavior of the tree graphs which approach the Veneziano tree graphs in the same linear limit. It is also shown that when the trajectories are nonlinear, the self-energy graph including the $\int d^4k$ is finite. It should be emphasized that these rules have been developed without consideration of the factorization properties of the nonlinear N-point Born term. In this sense they are analogous to the original Veneziano-loop-diagram rules of Kikkawa, Sakita, and Virasoro, which were subsequently modified by the requirements of factorization. However, although our rules may likewise be modified, they present a simplified context for developing techniques which are of general value for the study of the nonlinear theory.

I. INTRODUCTION

In a previous paper[1] we have given rules for constructing generalizations of the Veneziano N-point functions[2–5] which have nonlinear trajectories. The pole structure of these generalizations is that of the N-point planar tree graphs. In a certain limit the trajectories become linear and the functions become the Veneziano N-point functions. This limit and other properties will be investigated in another paper.[6]

Our rules for constructing planar tree graphs are simple and suggest simple rules for loop diagrams. In this paper we assume these simple rules and investigate the properties of vacuum polarization, self-energy, and triangle graphs.

It is possible that a study of factorization might modify the rules set forth here just as the work of Fubini and Veneziano[7] modified[8] the loop-diagram rules of Kikkawa, Sakita, and Virasoro.[9] However, we have not yet studied in detail the problem of factorization at the high-spin poles. This may be more difficult than it was in the case of linear trajectories.[7] The results of the present paper provide considerable motivation to pursue that problem. Briefly, our results are as follows. (i) In the limit where trajectories become linear, our rules lead to expressions which are different from those obtained from the rules of Kikkawa, Sakita, and Virasoro. (ii) In the same linear limit our expressions for the integrands of loop diagrams generally diverge. (iii) The self-energy bubble with nonlinear trajectories has been shown to be finite. The finiteness of other graphs has not been investigated.

The self-energy graph is treated in the most detail because it is the simplest example of techniques which apply equally well to other graphs.

II. RULES FOR LOOP DIAGRAMS

First we will state the rules found in Ref. 1 for constructing a dual N-point Born term for spinless external particles corresponding to a given set of tree graphs.

(a) Introduce a quantity $\sigma_i=ap_i^2+b$ for each internal line L_i which carries distinct momentum p_i in the given set of graphs.

(b) Write a multiple product over L_i including for each L_i the factor

$$\sum_{n_i=0}^{\infty}\frac{(\sigma_i)^{n_i}}{f_{n_i}}, \tag{1}$$

where $f_0=1$,

$$f_{n_i}=(1-q)(1-q^2)\cdots(1-q^{n_i}),\quad n_i>0 \tag{2}$$

and q is a parameter, $0<q<1$. The factor (1) has an infinite number of poles at $\sigma_i=q^{-l}$, where $l=0, 1, 2, \ldots$. The product defined by rule (b) yields a multiple power series in the variables σ_i with simultaneous poles in all the variables.

(c) Under the resultant multiple sum, introduce a factor $q^{n_in_j}$ for each pair of variables p_i and p_j which are dual to each other. (p_i and p_j are dual if there is no Feynman diagram among the given set of tree graphs which contains both lines L_i and L_j.) The $q^{n_in_j}$ factors guarantee the absence of simultaneous poles in the "dual" variables σ_i and σ_j. They produce a residue at

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[1] M. Baker and D. D. Coon, Phys. Rev. D **2**, 2349 (1970).
[2] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[3] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968); Phys. Rev. **181**, 1884 (1969).
[4] Chan Hong-Mo, Phys. Letters **28B**, 425 (1969); Chan Hong-Mo and Tsou Sheung Tsun, *ibid.* **28B**, 485 (1969).
[5] C. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969); M. Virasoro, *ibid.* **22**, 37 (1969); Z. Koba and H. Nielsen, Nucl. Phys. **B10**, 633 (1969); **B12**, 517 (1969).
[6] M. Baker and D. D. Coon (unpublished).
[7] S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 811 (1969).
[8] K. Bardakci, M. B. Halpern, and J. A. Shapiro, Phys. Rev. **185**, 1910 (1969).
[9] K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969).

each pole in σ_i which is a polynomial in the dual variables.[1]

We now use the same rules (a)–(c) in order to construct the integrands corresponding to planar loop diagrams such as those depicted in Figs. 1, 3, and 4. This guarantees that the residues of the spin-zero poles of these integrands will be products of amplitudes corresponding to planar tree graphs. (See Secs. IV and VII.) We are unable to say anything about factorization properties at the higher-spin poles. To complete the rules for loop diagrams we add:

(d) Include $\int d^4k$ for each loop, where k is the loop momentum. (In Figs. 1 and 4 the loop momentum is p_0, while in Fig. 3 the loop momenta are p_1 and p_2.)

The first three rules define the loop-diagram integrands within the domain of convergence of the prescribed multiple power series. Using formulas given in Sec. III one can perform summations and thereby obtain expressions which permit analytic continuation to all values of momenta except for neighborhoods of the propagator poles. This procedure provides useful representations of integrands and is explicitly carried out for each of the examples discussed in this paper.

III. NOTATION AND USEFUL FORMULAS

In Ref. 1, some formulas are derived which are useful in performing the sums we will encounter. They involve the infinite product

$$G(z)\equiv\prod_{n=0}^{\infty}(1-zq^n). \qquad (3)$$

$G(az)/G(z)$ possesses the power-series expansion[1]

$$\frac{G(az)}{G(z)}=\sum_{n=0}^{\infty}\frac{(a)_{q,n}}{f_n}z^n, \qquad (4)$$

where

$$(a)_{q,n}\equiv[G(a)/G(aq^n)] = (1-a)(1-aq)\cdots(1-aq^{n-1}) \qquad (5)$$

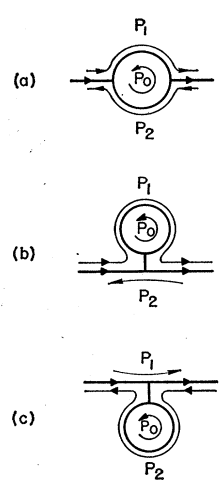

FIG. 1. Self-energy Feynman-like diagram (a) and the two tadpole Feynman-like diagrams (b) and (c) which are related to (a) by duality. The external lines have momentum p_1-p_2. The various internal lines have momenta p_0-p_1, p_0-p_2, or 0. The terminology used in describing the diagrams and the momentum labeling follow Kikkawa, Sakita, and Virasoro, Ref. 9.

and

$$f_n\equiv(q)_{q,n}=(1-q)(1-q^2)\cdots(1-q^n). \qquad (6)$$

The expansion (4) converges for $|z|<1$. When $a=0$, Eq. (4) becomes

$$\frac{1}{G(z)}=\sum_{n=0}^{\infty}\frac{z^n}{f_n}. \qquad (7)$$

In this paper we will use formulas from the mathematical literature dealing with basic hypergeometric series[10–13] which are defined by

$$ {}_A\Phi_B(a_1,a_2,\ldots,a_A;\, b_1,b_2,\ldots,b_B;\, z) =\sum_{n=0}^{\infty}\frac{(a_1)_{q,n}\cdots(a_A)_{q,n}}{(b_1)_{q,n}\cdots(b_B)_{q,n}}\frac{z^n}{f_n}. \qquad (8)$$

The series in Eqs. (4) and (7) are special cases of this series. There exist tables[13] of identities involving these functions with which one can study various asymptotic limits.

As the simplest example of the use of the rules of Sec. II and formulas (4) and (5) of this section, let us construct the four-point-function Born term $B_4(s,t)$ in the case where u-channel exchanges are forbidden. The rules of Sec. II then give the following expression for $B_4(s,t)$:

$$B_4(s,t)=\sum_{n,m=0}^{\infty}\frac{\sigma^n}{f_n}q^{nm}\frac{\tau^m}{f_m},$$

where $\sigma=as+b$ and $\tau=at+b$.

Using Eqs. (4) and (5), we can carry out the above double summation. This yields the result

$$B_4(st)=\sum_{n,m=0}^{\infty}\frac{\sigma^n}{f_n}q^{nm}\frac{\tau^m}{f_m}=\frac{G(\sigma\tau)}{G(\sigma)G(\tau)}. \qquad (9)$$

The right-hand side of Eq. (9) is just the form of the four-point function originally proposed by Coon.[14] The poles of B_4 arise from the zeros of $G(\sigma)$ and $G(\tau)$. The pole at $\sigma\equiv as+b=1$ has spin zero because the residue obtained from (9) is independent of τ and hence also t.

IV. SELF-ENERGY GRAPH

The simplest loop diagram is the spin-zero, self-energy graph of Fig. 1(a). This graph is dual to the two tadpole graphs of Figs. 1(b) and 1(c). These graphs are analogous to those of Kikkawa, Sakita, and Virasoro[9]

[10] W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge U. P., London, 1935).

[11] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, p. 195.

[12] L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge U. P., London, 1966).

[13] D. B. Sears, Proc. London Math. Soc. **53**, 158 (1951); **53**, 181 (1951).

[14] D. Coon, Phys. Letters **29B**, 669 (1969).

and form the basis for the construction of the self-energy graph according to the rules of Sec. II.

The momentum of the external line in these graphs is p_1-p_2. The momenta of the propagators in the loops are p_1-p_0 and p_2-p_0 and the tadpole tails have zero momentum. According to the rules of Sec. III we introduce

$$\sigma_{01}=a_{01}(p_1-p_0)^2+b_{01}, \tag{10}$$

$$\sigma_{02}=a_{02}(p_2-p_0)^2+b_{02}, \tag{11}$$

$$\sigma_{11}=a_{11}(p_1-p_1)^2+b_{11}=b_{11}, \tag{12}$$

$$\sigma_{22}=a_{22}(p_2-p_2)^2+b_{22}=b_{22}. \tag{13}$$

We must write a quadruple sum [rule (b)] which has poles in each of the σ_{ij}. From Fig. 1 we see that the propagator corresponding to σ_{11} never occurs with the σ_{02} propagator (i.e., the p_2-p_0 internal line) or with the σ_{22} propagator. Similarly the propagators corresponding to σ_{22} and σ_{01} do not occur in the same diagram. The duality of these propagators is incorporated in the multiple sum by introducing the appropriate $q^{n_i n_j}$ factors according to rule (c). Including one integral over the loop four-momentum $k=p_0$ [rule (d)], we obtain the expression

$$M((p_1-p_2)^2)=\int d^4k\, E((p_1-k)^2,(p_2-k)^2), \tag{14}$$

where E is given by the analytic continuation of

$$E=\sum_{\text{all } n_{ij}=0}^{\infty}\frac{\sigma_{01}{}^{n_{01}}}{f_{n_{01}}}\frac{\sigma_{02}{}^{n_{02}}}{f_{n_{02}}}\frac{\sigma_{11}{}^{n_{11}}}{f_{n_{11}}}\frac{\sigma_{22}{}^{n_{22}}}{f_{n_{22}}}\times q^{n_{01}n_{22}+n_{02}n_{11}+n_{11}n_{22}}, \tag{15}$$

which has four propagator terms and three duality factors and which converges for all $|\sigma_{ij}|<1$. The $|\sigma|<1$ restrictions on the constants σ_{11} and σ_{22} are just restrictions on the parameters b_{11} and b_{22} whereas the other restrictions can be met by choosing appropriate ranges for the values of the momenta. Analytic continuation to all momenta is trivial after we perform some of the sums in Eq. (15). If we should want $|b_{11}|$ or $|b_{22}|$ to be greater than 1, we can define E by analytic continuation of E in b_{11} and b_{22}.

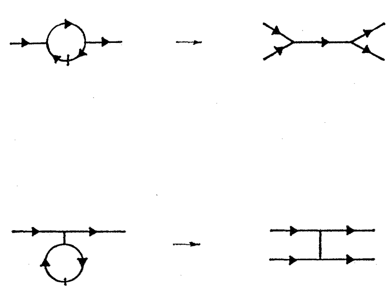

FIG. 2. Loop diagrams and the tree graphs which are obtained from them by factorization.

We now illustrate the use of the formulas of Sec. III in connection with our rules by deriving the correct factorization properties of Eq. (15) at spin-zero poles. Using Eq. (7) we can carry out the summation over n_{02}. This yields the factor $G^{-1}(\sigma_{02}q^{n_{11}})$, which from Eq. (3) has poles at $\sigma_{02}=q^{-(n_{11}+l)}$, where $l=0, 1, 2, \ldots$. The spin-zero pole at $\sigma_{02}=1$ arises only from the term $n_{11}=0$. The residue of σ_{02} at this pole is then proportional to the remaining sums over n_{22} and n_{01}, i.e.,

$$\sum_{n_{22},n_{01}}^{\infty}\frac{(\sigma_{01})^{n_{01}}}{f_{n_{01}}}\frac{(\sigma_{22})^{n_{22}}}{f_{n_{22}}}q^{n_{01}n_{22}},$$

which is just the four-point function Eq. (9) for the forward scattering of particles of momenta p_0-p_2 and p_1-p_2. (See Figs. 1 and 2.)

We can use Eqs. (9), (7), and (5) to obtain

$$\begin{aligned}E&=\sum_{n_{11},n_{02}=0}^{\infty}\frac{G(\sigma_{01}\sigma_{22}q^{n_{11}})}{G(\sigma_{01})G(\sigma_{22}q^{n_{11}})}\frac{\sigma_{02}{}^{n_{02}}}{f_{n_{02}}}\frac{\sigma_{11}{}^{n_{11}}}{f_{n_{11}}}q^{n_{02}n_{11}}\\&=\sum_{n_{11}=0}^{\infty}\frac{G(\sigma_{01}\sigma_{22}q^{n_{11}})}{G(\sigma_{01})G(\sigma_{22}q^{n_{11}})}\frac{1}{G(\sigma_{02}q^{n_{11}})}\frac{\sigma_{11}{}^{n_{11}}}{f_{n_{11}}}\\&=\frac{G(\sigma_{01}\sigma_{22})}{G(\sigma_{01})G(\sigma_{22})}\frac{1}{G(\sigma_{02})}\\&\qquad\times\sum_{n_{11}=0}^{\infty}\frac{(\sigma_{22})_{q,n_{11}}(\sigma_{02})_{q,n_{11}}}{(\sigma_{01}\sigma_{22})_{q,n_{11}}}\frac{\sigma_{11}{}^{n_{11}}}{f_{n_{11}}}.\end{aligned} \tag{16}$$

By the ratio test, the sum in Eq. (16) is seen to converge because $|\sigma_{11}|<1$ has already been assumed. Thus, Eq. (16) provides a continuation of E which is suitable for use under the integral in Eq. (14). In the notation of Eq. (8),

$$E=\frac{G(\sigma_{01}\sigma_{22})}{G(\sigma_{01})G(\sigma_{22})}\frac{1}{G(\sigma_{02})}\,{}_2\Phi_1(\sigma_{22},\sigma_{02};\sigma_{01}\sigma_{22};\sigma_{11}). \tag{17}$$

We will now investigate the asymptotic behavior of E in the k_0 component of the loop momentum k because of its relevance to the convergence of the self-energy loop integration. As $|k_0|\to\infty$, $|\sigma_{01}|\to\infty$ and $|\sigma_{02}|\to\infty$. To determine the behavior of E in this asymptotic region we note the following particularly useful identity[15]:

$$\begin{aligned}&G\left(\frac{q}{b}\right)G\left(\frac{qa}{e}\right)G(e)\,{}_2\Phi_1(a,b;e;x)\\&=\frac{S(bx)}{S(e/bx)}G(a)G\left(\frac{e}{b}\right)G\left(\frac{q^2}{e}\right){}_2\Phi_1\left(\frac{qa}{e},\frac{qb}{e};\frac{q^2}{e};x\right)\\&\quad+\frac{S(e)}{S(bx/e)}G\left(\frac{abx}{e}\right)G(a)G\left(\frac{q^2}{bx}\right){}_2\Phi_1\left(\frac{qe}{abx},\frac{q}{b};\frac{q^2}{bx};a\right),\end{aligned} \tag{18}$$

[15] The particular identity we use is designated III(c) on p. 178 of Ref. 13.

where

$$S(z)\equiv G(z)G(q/z). \tag{19}$$

The $1/e$ and $1/b$ factors in the Φ's will lead to simple expressions as $b\to\infty$ and $e\to\infty$. Straightforward substitution of this identity in Eq. (17) gives

$$E=\frac{1}{G(\sigma_{01})G(\sigma_{02})G(q/\sigma_{01})G(q/\sigma_{02})}\times\Bigg[\frac{S(\sigma_{02}\sigma_{11})}{S(\sigma_{01}\sigma_{22}/\sigma_{02}\sigma_{11})}G\bigg(\frac{\sigma_{01}\sigma_{22}}{\sigma_{02}}\bigg)G\bigg(\frac{q^2}{\sigma_{01}\sigma_{22}}\bigg)\times{}_2\Phi_1\bigg(\frac{q}{\sigma_{01}},\frac{q\sigma_{02}}{\sigma_{01}\sigma_{22}};\frac{q^2}{\sigma_{01}\sigma_{22}};\sigma_{11}\bigg)+\frac{S(\sigma_{01}\sigma_{22})}{S(\sigma_{02}\sigma_{11}/\sigma_{01}\sigma_{22})}G\bigg(\frac{\sigma_{02}\sigma_{11}}{\sigma_{01}}\bigg)G\bigg(\frac{q^2}{\sigma_{02}\sigma_{11}}\bigg)\times{}_2\Phi_1\bigg(\frac{q}{\sigma_{02}},\frac{q\sigma_{01}}{\sigma_{02}\sigma_{11}};\frac{q^2}{\sigma_{02}\sigma_{11}};\sigma_{22}\bigg)\Bigg], \tag{20}$$

which exhibits the $\sigma_{01}\leftrightarrow\sigma_{02}$, $\sigma_{11}\leftrightarrow\sigma_{22}$ symmetry of the defining representation (15). We note that

$$G(0)=1 \tag{21}$$

and

$$\lim_{|k_0|\to\infty}\frac{\sigma_{01}}{\sigma_{02}}=\frac{a_{01}}{a_{02}}, \tag{22}$$

and from Eqs. (9), (5), and (4),

$${}_2\Phi_1(0,a;0;z)={}_1\Phi_0(a;z)=G(az)/G(z). \tag{23}$$

Using Eqs. (19)–(23), we easily find that

$$E\sim\frac{1}{G(\sigma_{01})G(\sigma_{02})}\bigg[\frac{G(\sigma_{02}\sigma_{11})G(a_{01}\sigma_{22}/a_{02})}{G(a_{01}\sigma_{22}/a_{02}\sigma_{11})G(\sigma_{11})}+\frac{G(\sigma_{01}\sigma_{22})G(a_{02}\sigma_{11}/a_{01})}{G(a_{02}\sigma_{11}/a_{01}\sigma_{22})G(\sigma_{22})}\bigg] \tag{24}$$

as $|k_0|\to\infty$. The leading terms of the asymptotic series of $\ln G(z)$ are given by[1,14,16]

$$\ln G(z)\sim-\frac{\ln^2(-z)}{2\ln q}+\tfrac{1}{2}\ln(-z) \tag{25}$$

as $|z|\to\infty$, as long as we remain a finite distance away from the zeros of $G(z)$. In the Appendix we derive the asymptotic behavior of $G(z)$ on large circles in order to provide a clear proof that the asymptotic behavior (25) holds everywhere on the circles as long as these pass between the zeros. The next leading terms in the asymptotic series (25) for $\ln G(z)$ all contribute complicated but harmless multiplicative, nondecreasing factors to $G(z)$. Using Eqs. (24) and (25), we find that as $|k_0|\to\infty$,

$$E\sim c_1\exp\bigg[\frac{\ln^2(-a_{01}k_0^2)}{2\ln q}-\tfrac{1}{2}\ln(-a_{01}k_0^2)-\frac{\ln(-a_{02}k_0^2)\ln\sigma_{11}}{\ln q}\bigg]+c_2\exp\bigg[\frac{\ln^2(-a_{02}k_0^2)}{2\ln q}-\tfrac{1}{2}\ln(-a_{02}k_0^2)-\frac{\ln(-a_{01}k_0^2)\ln\sigma_{22}}{\ln q}\bigg], \tag{26}$$

where the constants c_1 and c_2 depend on a_{01}, a_{02}, σ_{11}, and σ_{22}. Since $\ln q$ is negative, it is clear that E decreases faster than any power of $|k_0|$ as $|k_0|\to\infty$ along any path which avoids the poles of E coming from $1/G(\sigma_{01})$ and $1/G(\sigma_{02})$. The usual $i\epsilon$ prescription displaces the poles into the second and fourth quadrants of the k_0 plane and the asymptotic behavior (26) allows us to rotate the k_0 contour from the real to the imaginary k_0 axis. In the resulting four-dimensional Euclidean integration we take p_1-p_2 spacelike, let $\mathbf{p}_1-\mathbf{p}_2=\mathbf{Q}$, $i(p_1-p_2)_0=Q_4$, shift the origin and change to spherical coordinates to obtain

$$M(-Q^2)=-4\pi i\int_0^\infty K^3dK\int_0^\pi\sin^2\theta d\theta\times E(-K^2,-K^2+2KQ\cos\theta-Q^2). \tag{27}$$

Inspection of Eq. (26) shows that the asymptotic falloff of E is independent of p_1 or p_2. Thus, without repeating any arguments we can see that independent of Q and θ the $K\to\infty$ behavior of E is given by (26) with the replacement $k_0^2\to-K^2$. Therefore, the K integration in Eq. (27) is obviously convergent for $0\leq\theta\leq\pi$ and a region of Q (including the spacelike p_1-p_2 region) controlled by singularities of the integrand. We conclude that the self-energy graph is finite in the nonlinear $q<1$ theory.

V. SELF-ENERGY GRAPH IN THE LIMIT OF LINEAR TRAJECTORIES

We will now investigate the interesting question of the finiteness of the self-energy graph in the Veneziano limit $q\to1$. In the case of our four-point planar tree graph[1,14] it was necessary to include a multiplicative constant which vanishes as $q\to1$ in order to obtain a finite four-point function in the limit $q\to1$. In order to determine the corresponding q-dependent factor for the self-energy graph we will relate the self-energy graph to the four-point planar tree graph by factorization of a spin-zero pole.

In order for the four-point planar tree graph to be nontrivial in the limit $q\to1$, the σ_{ij}'s must have q

[16] J. E. Littlewood, Proc. London Math. Soc. 5, 361 (1907); G. N. Watson, Trans. Cambridge Phil. Soc. 11, 281 (1910).

dependence of the form[1,14]

$$\sigma=1+(1-q)\sigma', \tag{28}$$

where

$$\lim_{q\to 1}\sigma'\neq 0. \tag{29}$$

In the limit, we obtain[1,14] the Veneziano four-point function with ($\lim\sigma'$) as the linear trajectory function.

Using Eqs. (17) and (3), we find that the residue of the spin-zero pole of E at $\sigma_{02}'=0$ ($\sigma_{02}=1$) is

$$-\frac{1}{(1-q)G(q)}\frac{G(\sigma_{01}\sigma_{22})}{G(\sigma_{01})G(\sigma_{22})}\,{}_2\Phi_1(\sigma_{22},1;\sigma_{01}\sigma_{22};\sigma_{11}). \tag{30}$$

From Eqs. (8) and (5), we see that this ${}_2\Phi_1=1$. We can write the residue (30) in the form

$$\frac{1}{(1-q)^2G^2(q)}\left[-\frac{(1-q)G(q)G(\sigma_{01}\sigma_{22})}{G(\sigma_{01})G(\sigma_{22})}\right], \tag{31}$$

where the term in square brackets is just our four-point function[1,14] evaluated at zero-momentum transfer as it should be (see Fig. 2). The constant factor $(1-q)G(q)$ ensures[1,14] that the limit $q\to 1$ of the four-point function exists. Of course, we still have the arbitrariness of an over-all constant which does not vanish as $q\to 1$. The minus sign in the square brackets leads to a positive residue of the spin-zero pole at $\sigma_{01}=1$. To be consistent with the q dependence of the four-point function, we should therefore make the replacement

$$E\to E_1=(1-q)^2G^2(q)E.$$

We can then write E_1 in the form

$$E_1=\left[\frac{(1-q)G(q)G(\sigma_{01}\sigma_{22})}{G(\sigma_{01})G(\sigma_{22})}\right]\left[\frac{(1-q)G(q)}{G(\sigma_{02})}\right]\times{}_2\Phi_1(\sigma_{22},\sigma_{02};\sigma_{01}\sigma_{22};\sigma_{11}). \tag{32}$$

To see what happens to E_1 in the limit $q\to 1$, we examine each of the three factors in Eq. (32) separately and quote a few results which are easy to verify.

Using Eqs. (5), (6), (8), and (28), it can be shown that

$$\lim_{q\to 1}{}_2\Phi_1(\sigma_{22},\sigma_{02};\sigma_{01}\sigma_{22};\sigma_{11})={}_2F_1(-\sigma_{22}',-\sigma_{02}';-\sigma_{01}'-\sigma_{22}';1), \tag{33}$$

where

$${}_2F_1(a,b;c;z)=1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{c(c+1)}z^2+\cdots \tag{34}$$

is Gauss's hypergeometric series, which converges[17] for $\mathrm{Re}(c-a-b)>0$ when $|z|\le 1$. Hence for $\mathrm{Re}(\sigma_{02}'-\sigma_{01}')>0$ (and elsewhere by analytic continuation), we see that the limit of ${}_2\Phi_1$ exists.

[17] See Ref. 11, p. 57.

In Ref. 14 it was shown that

$$\lim_{q\to 0}\frac{(1-q)G(q)G(\sigma_{01}\sigma_{22})}{G(\sigma_{01})G(\sigma_{22})}=B(-\sigma_{01}',-\sigma_{22}'), \tag{35}$$

where $B(x,y)$ is the beta function which is finite everywhere except at its poles. We next consider

$$\lim_{q\to 1}\frac{(1-q)G(q)}{G(\sigma_{02})}. \tag{36}$$

Using Eqs. (3) and (28), we can write

$$\begin{aligned}\frac{(1-q)G(q)}{G(\sigma_{02})}&=\frac{1-q}{1-\sigma_{02}}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^n\sigma_{02}}\\&=-\frac{1}{\sigma_{02}'}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^n-(1-q)\sigma_{02}'q^n}\\&=-\frac{1}{\sigma_{02}'}\exp\left[\sigma_{02}'(1-q)\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right]\\&\times\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^n-(1-q)\sigma_{02}'q^n}\exp\left(\frac{1-q}{1-q^n}q^n\sigma_{02}'\right).\end{aligned} \tag{37}$$

In the limit $q\to 1$, the factors in this last infinite product are just those of the infinite-product representation[18] of the gamma function $\Gamma(-\sigma_{02}')$. The sum

$$\sum_{n=1}^{\infty}\frac{1-q}{1-q^n}q^n, \tag{38}$$

which occurs in the exponent in Eq. (37), logarithmically diverges as $q\to 1$. We have already seen that the other infinite series and products in Eqs. (33), (35), and (37) which enter expression (32) for E_1 all converge for $\mathrm{Re}(\sigma_{02}'-\sigma_{01}')>0$. Thus, we have isolated in rather simple form a divergence of the representation of E_1 in the limit $q\to 1$.

Upon comparing the sum in Eq. (37) with an integral we find that

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}>\int_1^{\infty}dx\,\frac{q^x}{1-q^x}. \tag{39}$$

All terms in the sum are positive. We next evaluate the integral and obtain the lower bound

$$(1-q)\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}>\frac{1-q}{\ln q}\ln(1-q). \tag{40}$$

Since $q<1$ and

$$\lim_{q\to 1}\frac{1-q}{\ln q}=-1, \tag{41}$$

[18] See Ref. 11, p. 1.

we see that our lower bound approaches $+\infty$ as $q \to 1$. The sign of the real part of the exponent involving this sum in Eq. (37) is determined by σ_{02}'. If $\mathrm{Re}\sigma_{02}'$ is negative, then the limit of the expression in Eq. (37) is zero. This implies that

$$\lim_{q\to 1} E_1 = 0 \tag{42}$$

for $\mathrm{Re}\sigma_{02}' < 0$ which includes all or almost all of the spacelike momentum region of $p_2 - p_0 = p_2 - k$. If E_1 could be defined when $q=1$ by analytic continuation, then Eq. (42) would hold for all values of momentum. However, the crucial exponential factor in Eq. (37) becomes infinite as $q \to 1$ if $\mathrm{Re}\sigma_{02}' > 0$. Thus, in addition to finding that the limit of E_1 does not exist for $\mathrm{Re}\sigma_{02}' > 0$, we also find that the $q \to 1$ limit is not analytic. Actually we could have guessed that there was some subtlety in the linear limit on the basis of the previous results (33) and (35). In (33) we saw that all of the possible dependence on σ_{11}' dropped out whereas (35) would contribute poles in σ_{22}' to the final answer. These results conflict with the initial symmetry under $\sigma_{01} \leftrightarrow \sigma_{02}$ and $\sigma_{11} \leftrightarrow \sigma_{22}$ unless the limit $q \to 1$ is zero or does not exist.

The above conclusions regarding the limit of E_1 do not imply the nonexistence of

$$\lim_{q\to 1} \int d^4k\, E_1 . \tag{43}$$

If this limit (43) could be shown to exist, it could have very interesting consequences for attempts[8,9] to find expressions for loop diagrams consistent with the Veneziano N-point functions.

The nonexistence of the $q \to 1$ limit of E_1 shows that in the limit where trajectories become linear our rules for loop diagrams do not become the same as those of Kikkawa, Sakita, and Virasoro[9] which lead to a finite result for the integrand of the $\int d^4k$. However, their rules also lead to expressions with incorrect factorization properties.[8] This gives one additional motivation to investigate the problem of factorization for general $q \leq 1$ as was done[7] in the case of the Veneziano N-point functions.

VI. VACUUM BUBBLE GRAPH

We now apply our rules to the construction of an expression which represents all those two-loop planar graphs (Fig. 3) which are dual to the vacuum bubble graph. Six propagators appear in these graphs and so we introduce: σ_{01}, σ_{02}, σ_{12}, σ_{00}, σ_{11}, and σ_{22}. Inspection of the graphs in Fig. 3 tells us which propagators, such as those associated with $\sigma_{01}(p_0 - p_1)$ and $\sigma_{22}(p_2 - p_2) = \sigma_{22}(0)$, never occur together. We can then straightforwardly apply our rules and find that the graphs are represented by

$$R = \int d^4p_1 d^4p_2 H(p_0 - p_1, p_0 - p_2) , \tag{44}$$

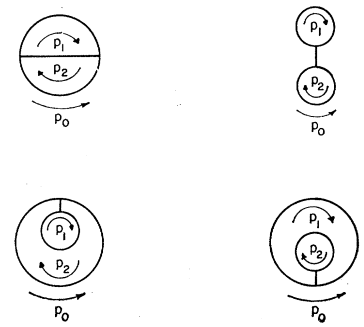

FIG. 3. Vacuum graphs related by duality.

where

$$H = \sum_{\text{all } n_{ij}=0}^{\infty} \frac{\sigma_{01}^{n_{01}}}{f_{n_{01}}} \frac{\sigma_{12}^{n_{12}}}{f_{n_{12}}} \frac{\sigma_{02}^{n_{02}}}{f_{n_{02}}} q^{n_{01}n_{22}+n_{12}n_{00}+n_{02}n_{11}} \times \frac{\sigma_{00}^{n_{00}}}{f_{n_{00}}} \frac{\sigma_{11}^{n_{11}}}{f_{n_{11}}} \frac{\sigma_{22}^{n_{22}}}{f_{n_{22}}} q^{n_{00}n_{11}+n_{11}n_{22}+n_{22}n_{00}} , \tag{45}$$

which converges for all $|\sigma_{ij}| < 1$.

Using Eq. (7) to do the n_{01} and n_{02} sums, Eq. (9) to do the n_{12} and n_{00} sums and using the notation (5), we obtain

$$H = \frac{1}{G(\sigma_{01})G(\sigma_{02})} \frac{G(\sigma_{12}\sigma_{00})}{G(\sigma_{12})G(\sigma_{00})} \times \sum_{n_{11},n_{22}=0}^{\infty} \frac{(\sigma_{01})_{q,n_{22}}(\sigma_{02})_{q,n_{11}}(\sigma_{00})_{q,n_{11}+n_{22}}}{(\sigma_{12}\sigma_{00})_{q,n_{11}+n_{22}}} \times \frac{\sigma_{11}^{n_{11}}}{f_{n_{11}}} q^{n_{11}n_{22}} \frac{\sigma_{22}^{n_{22}}}{f_{n_{22}}} . \tag{46}$$

We can repeat the spin-zero-factorization and linear-limit arguments made in the case of the self-energy graph. While examining factorization at any one of the spin-zero poles $\sigma_{01}=1$, $\sigma_{02}=1$, or $\sigma_{12}=1$, we can determine the q dependence of an over-all constant and thus obtain consistency with the self-energy graph and the four-point planar tree graph. Spin-zero factorization is consistent with the replacement

$$H \to H_1 = (1-q)^3 G^3(q) H . \tag{47}$$

To study the limit $q \to 1$, we apply the techniques of Sec. V. Thus, we find that in the linear limit the doubly infinite sum[19] in Eq. (46) converges, the four-point function structure involving σ_{12} and σ_{00} converges, and the remaining $G(\sigma_{01})$, $G(\sigma_{02})$ factors again lead to divergence problems.

[19] In the limit $q \to 1$ the double sum in Eq. (46) becomes one of Appell's hypergeometric functions (F_1) of two variables. See Ref. 10, p. 73 and Ref. 11, p. 222.

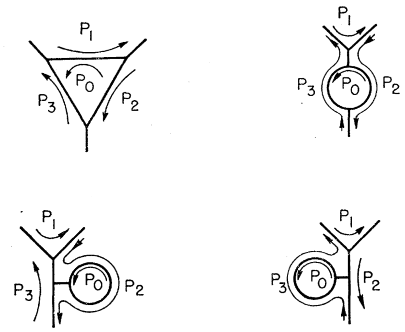

FIG. 4. Triangle graph and some representative graphs which are related to it by duality.

VII. TRIANGLE GRAPH

Three representative graphs related to the triangle graph by duality are shown in Fig. 4. Here we will simply write down the representation we obtain from our rules for the integrand T of the $\int d^4p_0 T$ expression:

$$T=\sum_{\text{all } n_{ij}=0}^{\infty}\frac{\sigma_{01}{}^{n_{01}}}{f_{n_{01}}}\frac{\sigma_{02}{}^{n_{02}}}{f_{n_{02}}}\frac{\sigma_{03}{}^{n_{03}}}{f_{n_{03}}}\frac{\sigma_{12}{}^{n_{12}}}{f_{n_{12}}}\frac{\sigma_{23}{}^{n_{23}}}{f_{n_{23}}}\frac{\sigma_{31}{}^{n_{31}}}{f_{n_{31}}}$$
$$\times\frac{\sigma_{11}{}^{n_{11}}}{f_{n_{11}}}\frac{\sigma_{22}{}^{n_{22}}}{f_{n_{22}}}\frac{\sigma_{33}{}^{n_{33}}}{f_{n_{33}}}q^{n_{12}n_{23}+n_{23}n_{31}+n_{31}n_{12}}$$
$$\times q^{n_{01}(n_{23}+n_{22}+n_{33})+n_{02}(n_{31}+n_{33}+n_{11})+n_{03}(n_{12}+n_{11}+n_{22})}$$
$$\times q^{n_{11}n_{23}+n_{22}n_{31}+n_{33}n_{12}+n_{11}n_{22}+n_{22}n_{33}+n_{33}n_{11}}, \quad (48)$$

which converges if all σ_{ij} satisfy $|\sigma_{ij}|<1$. Again, by factorization of a σ_{01}, σ_{02}, or σ_{03} spin-zero pole, we can obtain a five-point function,[1,6] or by factorization of two spin-zero poles, we can obtain a four-point function and find that the q-dependent normalizing factor is $(1-q)^3G^3(q)$. Using Eqs. (4), (5), (7), and (9), we can evaluate five of the sums in Eq. (48) and thus obtain

$$T=\frac{G(\sigma_{01}\sigma_{23})}{G(\sigma_{01})G(\sigma_{23})}\frac{G(\sigma_{03}\sigma_{12})}{G(\sigma_{03})G(\sigma_{12})}\frac{1}{G(\sigma_{02})}$$
$$\times\sum_{n_{11}=0}^{\infty}\sum_{n_{22}=0}^{\infty}\sum_{n_{33}=0}^{\infty}\sum_{n_{31}=0}^{\infty}\left[\frac{(\sigma_{01})_{q,n_{22}+n_{33}}(\sigma_{23})_{q,n_{31}+n_{11}}}{(\sigma_{01}\sigma_{23})_{q,n_{22}+n_{33}+n_{31}+n_{11}}}\right.$$
$$\times\frac{(\sigma_{03})_{q,n_{11}+n_{22}}(\sigma_{12})_{q,n_{31}+n_{33}}}{(\sigma_{03}\sigma_{12})_{q,n_{11}+n_{22}+n_{31}+n_{33}}}$$
$$\times\frac{(\sigma_{12}\sigma_{23}/\sigma_{22})_{q,n_{31}+n_{22}}(\sigma_{02})_{q,n_{33}+n_{31}+n_{11}}}{(\sigma_{12}\sigma_{23}/\sigma_{22})_{q,n_{31}}f_{n_{11}}f_{n_{22}}f_{n_{33}}f_{n_{31}}}$$
$$\left.\times q^{n_{11}n_{22}+n_{22}n_{33}+n_{33}n_{11}+n_{22}n_{31}}\sigma_{11}{}^{n_{11}}\sigma_{22}{}^{n_{22}}\sigma_{33}{}^{n_{33}}\sigma_{31}{}^{n_{31}}\right]. \quad (49)$$

This is one among many formulas which we can derive using our summation formulas. Such representations permit analytic continuation beyond the region of convergence of the power series (48) and allow us to investigate the linear limit.

As in Secs. V and VI, we combine the $(1-q)^3G^3(q)$ normalizing factor with the three factors in front of the multiple sum in Eq. (49). As before we find that the lone $1/G(\sigma_{02})$ factor leads to nonexistence of the integrand in the limit $q\rightarrow 1$.

VIII. CONCLUSION

(A) We have formulated a simple procedure for calculating loop diagrams in dual-resonance theories with nonlinear trajectories.

(B) We have investigated the properties of the integral for the self-energy loop and have shown that it converges.

(C) We have shown that the linear limit of the integrand for loop diagrams does not agree with the finite integrands arrived at by Kikkawa, Sakita, and Virasoro[9] in the case of linear trajectories. In fact, this linear limit does not exist at all values of the loop momentum for those diagrams which we have investigated. This is in contrast with the situation for tree diagrams where the $q\rightarrow 1$ limit of the nonlinear theory yields the Veneziano N-point functions.

Since the general situation with loop diagrams is still unclear even in the usual Veneziano theory, point (C) suggests the possibility of a new set of rules for linear-trajectory loop diagrams which may be consistent with the N-point Veneziano tree graphs. In this regard, the present paper leaves open the possibility that the linear limit may exist if it is carried out after the integrations over loop momenta have been performed.

Of course, we do not yet know whether a completely consistent dual-resonance theory of the linear and/or the nonlinear type can be constructed. It might turn out that only the nonlinear theory is of interest. However, even if only the linear theory turns out to be of interest it still may be important to regard it as the $q\rightarrow 1$ limit of a nonlinear theory in order to calculate higher-order effects.

(D) These results are preliminary since we have not investigated the problem of factorization at higher-spin poles. However, we hope that the simplicity of these first results will stimulate further study of the detailed properties of dual-resonance theories with nonlinear trajectories.

(E) A successful unitarization program via higher-order diagrams would indicate that the undesirable fixed-angle behavior[14,20] of the tree graphs of the nonlinear theory is only a characteristic of the Born series

[20] F. Capra, Phys. Letters **30B**, 53 (1969).

and not a difficulty of the theory itself. This problem is associated with the fact that the real part of the trajectory $\alpha(t)=-\ln(at+b)/\ln q$ rises as $t\to-\infty$. But the dual Born term is expected on physical grounds to be a good approximation only for peripheral processes with t fixed, $s\to\infty$ or s fixed, $t\to\infty$. Thus, we would expect that higher-order diagrams would modify $\alpha(t)$ as $t\to-\infty$ so that the resulting fixed-angle behavior of the full amplitude would be consistent with the Froissart bound. We are of course still a long way from seeing in detail how this would occur.

(F) Although the expression for the self-energy given in Sec. IV is finite, the bad fixed-angle behavior of the tree graphs might be expected to give rise to divergences in the integration over loop momenta in higher-order diagrams such as the box diagram. Some relevant information concerning the convergence of such loop-momentum integrals could be obtained from our conjectured rules because these rules correctly incorporate the lowest-mass spinless particles in intermediate states.

APPENDIX: LARGE-CIRCLE ASYMPTOTIC BEHAVIOR OF $G(z)$

The asymptotic behavior of the function

$$G(z)=\prod_{n=0}^{\infty}(1-zq^n) \tag{A1}$$

plays a key role in determining the convergence of loop diagrams discussed in this paper. Here we will give a simple derivation of the behavior of $G(z)$ on large circles in the complex z plane.

We consider the product

$$S(z)=G(z)G(q/z)=\prod_{n=0}^{\infty}(1-zq^n)\left(1-\frac{1}{z}q^{n+1}\right), \tag{A2}$$

which converges for all finite $|z|>0$. After rearranging terms in this product we find that

$$\begin{aligned}S(z)&=\frac{1-z}{1-1/z}\prod_{n=0}^{\infty}(1-zq^{n+1})\left(1-\frac{1}{z}q^n\right)\\&=-zS(qz).\end{aligned} \tag{A3}$$

Successive application of Eq. (A3) gives

$$S(z)=(-z)^m q^{m(m-1)/2}S(q^m z), \tag{A4}$$

where m is an integer.

We next consider a circle of radius $q^{-m+\epsilon}$ about the origin in the complex z plane. Any circle corresponds to some integer m and $0\leq\epsilon<1$. The circle is described by

$$z=-e^{i\phi}q^{-m+\epsilon}, \tag{A5}$$

with $-\pi\leq\phi\leq\pi$. Substituting (A5) in Eq. (A4) and using Eqs. (A1) and (A2) gives

$$G(z)=e^{im\phi}q^{m\epsilon}q^{-m(m+1)/2}\frac{S(-e^{i\phi}q^{\epsilon})}{G(-e^{-i\phi}q^{m+1-\epsilon})}. \tag{A6}$$

As $m\to\infty$, the factor $1/G\to1$. Thus, on large circles (large m) we find that

$$G(z)\sim e^{im\phi}q^{m\epsilon}q^{-m(m+1)/2}S(-e^{i\phi}q^{\epsilon}). \tag{A7}$$

Since $0<q<1$, $|G(z)|$ increases as we go to larger circles as long as we avoid the zeros of $G(z)$, i.e., as long as $|\phi|<\pi$ or $\epsilon\neq0$. The zeros appear on the right-hand side of (A7) in the factor $S(-e^{i\phi}q^{\epsilon})$.

It is a simple matter to show that the leading asymptotic behavior of $\ln G$ given in Eq. (25) agrees with $\ln G$ as computed from (A7). Thus we have established that (25) holds everywhere on circles which pass between the zeros of $G(z)$.

With regard to the $\phi=\arg(-z)$ dependence of (A7) it might be worthwhile to note that the function $S(z)$ which we have encountered in this Appendix is closely related to the elliptic theta functions.[21] Thus, (A7) involves only elementary functions and a well-known special function.

[21] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge U. P., London, 1963), p. 469.