Most of the hard work remains to be done. Can one succeed in finding the equivalent "σ" model for these fields, so that one can once again try to solve the current-algebra problem? What are the correct interactions to use? Are the perturbation sums for the interacting case finite? We hope that this formalism will be a help in answering some of these questions.

ACKNOWLEDGMENTS

One of us (F.C.) would like to thank the University of Pennsylvania Physics Department for its hospitality during the summer of 1970, when this work was begun. In addition, we wish to thank Professor D. H. Sharp, Professor M. L. Whippman, and Professor A. Komar for helpful discussions.

PHYSICAL REVIEW D VOLUME 3, NUMBER 10 15 MAY 1971

Some Constraints on Partial Waves of Helicity Amplitudes which Follow from Analyticity, Unitarity, and Crossing Symmetry*

William Case
Physics Department, Syracuse University, Syracuse, New York 13210
(Received 16 December 1970)

Some inequalities involving finite numbers of partial-wave helicity amplitudes are derived for the elastic scattering process $ab \to ab$ (arbitrary spins and masses). One set of inequalities involves algebraic combinations of t-channel ($a\bar{a} \to b\bar{b}$) partial-wave helicity amplitudes and holds for any value of t between 0 and $4\mu^2$ (μ is the lesser mass of the two particles involved in the scattering). A second set places restrictions on integrals over s-channel ($ab \to ab$) partial-wave helicity amplitudes. Finally, the above relations are applied to the particular case of π^0-nucleon elastic scattering, where inequalities among partial-wave helicity amplitudes are obtained.

I. INTRODUCTION

In recent years there has been a rebirth of interest in finding constraints on amplitudes which follow purely from analyticity and unitarity.[1] The present paper is an effort to bring together some[2] of these results with the work of Balachandran and co-workers[3] on crossing properties of partial waves and in particular the work by Balachandran, Modjtehedzadeh, and myself[3e] on constraints on partial waves of helicity amplitudes which follow from crossing symmetry. For the elastic scattering process $ab \to ab$ (s channel), inequalities are found for algebraic combinations of partial waves of t-channel helicity amplitudes for $0 \leqslant t < 4\mu^2$. This is done in Sec. II and the inequalities are given in Eqs. (6) and (10). In Sec. III inequalities for integrals over partial waves of s-channel helicity amplitudes are found for the same process [result given in Eqs. (13), (17), and (18)]. The main features of these constraints are as follows: (1) They follow from analyticity, unitarity, and crossing symmetry; (2) they involve only a finite number of partial waves in each inequality: and (3) they are constraints in the unphysical region.

In Sec. IV the results of Secs. II and III are applied to the special case of π^0-nucleon elastic scattering.

II. t-CHANNEL CONSTRAINTS

We begin by introducing various definitions and conventions. For the scattering process 1, 2 $\to$ 3, 4, we define

$$s \equiv (p_1+p_2)^2\,,$$
$$t \equiv (p_1-p_3)^2\,,$$
$$u \equiv (p_1-p_4)^2\,.$$

We will be considering the elastic scattering $ab \to ab$ where both particle a and b may have spin. Particles 1 and 3 are taken to be of type a with spin σ and mass m while particles 2 and 4 are of type b with spin σ' and mass μ. We also assume, without losing generality, that $m \geqslant \mu$. Physical processes in the various channels are

$$s\text{ channel}, \quad ab \to ab\,,$$
$$t\text{ channel}, \quad a\bar{a} \to b\bar{b}\,,$$
$$u\text{ channel}, \quad a\bar{b} \to a\bar{b}\,.$$

We express, for the case of elastic scattering, the Kibble

* Supported by the U. S. Atomic Energy Commission.

[1] (a) Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964); **135**, B1375 (1964); (b) A. Martin, Nuovo Cimento **47A**, 265 (1967); (c) **63A**, 167 (1969).

[2] Similar approaches have been used for various processes: S. M. Roy, Phys. Rev. Letters **20**, 1016 (1968); A. P. Balachandran and Maurice L. Blackmon, Phys. Letters **31B**, 655 (1970); F. J. Yndurain, *ibid.* **31B**, 368 (1970); A. K. Common CERN Report No. 1145, 1970 (unpublished); A. K. Common and F. J. Yndurain, CERN Report No. 1185, 1970 (unpublished).

[3] (a) A. P. Balachandran and J. Nuyts, Phys. Rev. **172**, 1821 (1968); (b) A. P. Balachandran, W. J. Meggs, and P. Ramond, *ibid.* **175**, 1974 (1968); (c) A. P. Balachandran, W. J. Meggs, J. Nuyts, and P. Ramond, *ibid.* **176**, 1700 (1968); (d) **187**, 2080 (1969); (e) A. P. Balachandran, W. Case, and M. Modjtehedzadeh, Phys. Rev. D **1**, 1773 (1970); (f) A. P. Balachandran and M. Blackmon, Syracuse University Report No. 223, 1970 (unpublished).

function:

$$\phi = st[2(m^2+\mu^2)-s-t]-t(m^2-\mu^2)^2.$$

We further express the cosines of scattering angles in the c.m. system for various channels:

$$z_s = 2st/\mathcal{S}^2+1,$$
$$z_t = \frac{2s+t-2(m^2+\mu^2)}{(t-4m^2)^{1/2}(t-4\mu^2)^{1/2}},$$
$$z_u = 2ut/\mathcal{U}^2+1,$$

where

$$\mathcal{S}^2 \equiv [s-(m-\mu)^2][s-(m+\mu)^2],$$
$$\mathcal{U}^2 \equiv [u-(m-\mu)^2][u-(m+\mu)^2]. \quad (1)$$

Note that z_s and z_u are defined as the cosines of the angles between particles of the same type. Using the above expressions for z_s, z_t, and z_u, the Kibble function ϕ may be written as

$$\phi = (1/4s)\mathcal{S}^4(1-z_s^2)$$
$$= \tfrac{1}{4}t(t-4m^2)(t-4\mu^2)(1-z_t^2)$$
$$= (1/4u)\mathcal{U}^4(1-z_u^2). \quad (2)$$

Mahous and Martin[4] have shown that the combination of s-channel helicity amplitudes

$$M = \mathcal{S}^{2(\sigma+\sigma')}\sum_{\lambda,\mu} M_{\lambda\mu,\lambda\mu}{}^s \quad \text{for } \sigma+\sigma' \text{ an integer},$$

$$M = \left(\frac{t}{-\phi}\right)^{1/2}\mathcal{S}^{2(\sigma+\sigma')}\sum_{\lambda,\mu} M_{\lambda\mu,\lambda\mu}{}^s \quad (3)$$
$$\text{for } \sigma+\sigma' \text{ a half-integer},$$

is free of kinematical singularities and the absorptive part obeys the inequalities[5]

$$\text{Abs}M(s,z_s) \geqslant 0 \quad \text{for} \quad s \geqslant (m+\mu)^2$$
$$\text{and} \quad 0 \leqslant t < 4\mu^2 \quad (\text{or } 1 \leqslant z_s < 8s\mu^2/\mathcal{S}^2+1),$$
$$\text{Abs}M(u,z_u) \geqslant 0 \quad \text{for} \quad u \geqslant (m+\mu)^2 \quad (4)$$
$$(\text{and} \quad 0 \leqslant t < 4\mu^2 \quad \text{or } 1 \leqslant z_u < 8u\mu^2/\mathcal{U}^2+1).$$

We now write a fixed-t dispersion relation for M in the region where $0 \leqslant t < 4\mu^2$. If M contains dynamical poles, we must subtract them out first and write the dispersion relation for the result. $\bar{M}$ will be used to denote M after these poles have been removed and in general a bar over a quantity will denote the corresponding quantity with the poles removed. The dispersion relation for $\bar{M}$ is

$$\bar{M} = \sum_{n=0}^{N-1} a_n z_t^n - \frac{z_t^N}{\pi}\int_{-\infty}^{-Z(t;m,\mu)} \frac{dz_t' \,\text{Abs}\bar{M}}{z_t'^N(z_t'-z_t)} + \frac{z_t^N}{\pi}\int_{Z(t;m,\mu)}^{\infty} \frac{dz_t'\,\text{Abs}\bar{M}}{z_t'^N(z_t'-z_t)},$$

where N is the number of subtractions required, and

$$Z(t;\, m,\mu) \equiv (4m\mu+t)[(t-4m^2)(t-4\mu^2)]^{-1/2}.$$

We have assumed that there are no two- or higher-body thresholds open for $s<(m+\mu)^2$ and likewise for $u<(m+\mu)^2$. We now project partial waves using the Jacobi polynomials $P_L{}^{\sigma_T\sigma_T}(z_t)$ $[\sigma_T \equiv 2(\sigma+\sigma')]$ with the corresponding measure $(1-z_t^2)^{\sigma_T}$ and require that $L \geqslant N$. This leads to a Froissart-Gribov relation:

$$\int_{-1}^{+1} \bar{M} P_L{}^{\sigma_T\sigma_T}(z_t)(1-z_t^2)^{\sigma_T} dz_t$$
$$= -\frac{2}{\pi}\int_{-\infty}^{-Z(t;m,\mu)} \text{Abs}\bar{M}(t,z_t') \times (z_t'^2-1)^{\sigma_T} Q_L{}^{\sigma_T\sigma_T}(z_t')dz_t'$$
$$+\frac{2}{\pi}\int_{Z(t;m,\mu)}^{\infty} \text{Abs}\bar{M}(t,z_t') \times (z_t'^2-1)^{\sigma_T} Q_L{}^{\sigma_T\sigma_T}(z_t')dz_t', \quad (5)$$

where the following relations for the $Q_L{}^{\sigma_T\sigma_T}$ functions[6] are used:

$$Q_L{}^{\sigma_T\sigma_T}(z') = \tfrac{1}{2}(z'^2-1)^{-\sigma_T}\int_{-1}^{+1}\frac{(1-z^2)^{\sigma_T}P_L{}^{\sigma_T\sigma_T}(z)}{z'-z}dz.$$

The range of integration of the second integral in Eq. (5) corresponds to $s \geqslant (m+\mu)^2$ with t fixed at some value between 0 and $4\mu^2$. Similarly, the first integral is over the region $u \geqslant (m+\mu)^2$ with t fixed at the same value. By the inequalities given in Eq. (4), we have that the Abs$\bar{M}$ factor in the integrand is positive.[7] In the Appendix the following inequalities are shown:

$$\text{For } z'>1, \quad (z'^2-1)^{\sigma_T}Q_L{}^{\sigma_T\sigma_T}(z')>0 \quad \text{for all } L;$$
$$\text{for } z'<-1, \quad (z'^2-1)^{\sigma_T}Q_L{}^{\sigma_T\sigma_T}(z')<0 \quad \text{for } L \text{ even},$$
$$>0 \quad \text{for } L \text{ odd}.$$

From these inequalities we see that the integrand is positive at every point of the integration for L even,

[4] G. Mahoux and A. Martin, Phys. Rev. **174**, 2140 (1968). The expression given here for $\sigma+\sigma'$ half-integer is slightly different from that used by Mahoux and Martin. The treatment in their paper is sufficient to show that the expression given in Eq. (3) is of the form $\sum_\nu C_\nu \cos\nu\theta_s = \sum_\nu C_\nu' \cos\nu\theta_u$, where C_ν and C_ν' are all positive. This condition alone is enough to guarantee the inequalities given in Eq. (4). This is not sufficient for the inequalities used by Mahoux and Martin [Eqs. (4) and (5) in their paper] and hence they are forced to use combinations of amplitudes multiplied by higher powers of s and u.

[5] The upper limit on t is due to the presence of the t-channel cut. Also see Ref. 4.

[6] Bateman Manuscript Project, edited by A. Erdélyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill, New York, 1954), Vol. 2, p. 171.

[7] Note that Abs$\bar{M}$ = Abs$M$ since subtracting out the poles does not alter the discontinuity.

and we have

$$\int_{-1}^{+1}\bar{M}(t,z_t)P_L{}^{\sigma_T\sigma_T}(1-z_t{}^2)^{\sigma_T}dz_t\geqslant 0 \quad \text{for } L \text{ even and } 0\leqslant t<4\mu^2. \tag{6}$$

Martin[1c] has obtained a similar inequality for $\pi\pi$ scattering by the same argument.

The integration may be carried out when the combination of helicity amplitudes denoted by $\bar{M}$ are replaced by their t-channel Jacob-Wick expansions.[8] The regularized helicity amplitudes which are summed over in Eq. (3),

$$\begin{aligned}\bar{F}_{\lambda\mu\lambda\mu}{}^s &\equiv \mathcal{S}^{2(\sigma+\sigma')}\bar{M}_{\lambda\mu\lambda\mu}{}^s, \qquad 2(\sigma+\sigma') \text{ even},\\ \bar{F}_{\lambda\mu\lambda\mu}{}^s &\equiv\left(\frac{t}{-\phi}\right)^{1/2}\mathcal{S}^{2(\sigma+\sigma')}\bar{M}_{\lambda\mu\lambda\mu}{}^s, \quad 2(\sigma+\sigma') \text{ odd},\end{aligned} \tag{7}$$

may be written as a linear combination of the t-channel regularized helicity amplitudes (RHA) given in Ref. 8:

$$\bar{F}_{\lambda\mu\lambda\mu}{}^s=\sum_{\alpha\beta\gamma\delta}C_{\lambda\mu\lambda\mu}{}^{\alpha\beta\gamma\delta}\bar{F}_{\alpha\beta\gamma\delta}{}^t. \tag{8}$$

$\bar{F}_{\alpha\beta\gamma\delta}{}^t$ is of the form

$$\bar{F}_{\alpha\beta\gamma\delta}{}^t=g_{\alpha\beta\gamma\delta}(t)\sum_j\left[M_{\alpha\beta\gamma\delta}{}^{tj}P_{j-n}{}^{|\alpha-\beta-(\gamma-\delta)|,|\alpha-\beta+\gamma-\delta|}(z_t)+\eta\bar{M}_{\alpha\beta-\gamma-\delta}{}^{tj}P_{j-n}{}^{|\alpha-\beta+\gamma-\delta|,|\alpha-\beta-(\gamma-\delta)|}(z_t)\right], \tag{9}$$

where $g_{\alpha\beta\gamma\delta}$ is a function of t,

$$n\equiv\max(|\alpha-\beta|,|\gamma-\delta|),$$

η is a constant (see Table IX of CMN for the detailed expression), and some factors involving only j have been suppressed. From the crossing relations for ordinary helicity amplitudes and the definition given above, one sees that $C_{\lambda\mu\lambda\mu}{}^{\alpha\beta\gamma\delta}$ is a polynomial in s and hence in z_t. Collecting the above expressions, we see that $\bar{M}$ is a sum of nonterminating sums of Jacobi polynomials where each nonterminated sum is multiplied by a polynomial in z_t. When this is substituted in Eq. (6) the sum over j will be terminated because of the orthogonality of the Jacobi polynomial of Eq. (9) where the measure is formed from part of the $(1-z_t{}^2)^{\sigma_T}$ factor of Eq. (6). A similar argument for terminating such summations is used in Ref. 3e. This will result in a series of inequalities (a series for each even L) of the form

$$\sum_{j=0}^{M}\sum_{\alpha,\beta,\gamma,\delta}^{\sigma',\sigma',\sigma,\sigma}K_{\alpha\beta\gamma\delta}(t)\bar{M}_{\alpha\beta\gamma\delta}{}^{tj}(t)\geqslant 0 \quad \text{for} \quad 0\leqslant t<4\mu^2, \tag{10}$$

each involving a finite number of partial waves of helicity amplitudes. The $K_{\alpha\beta\gamma\delta}{}^{(t)}$, of course, are all completely determined, but to obtain them will require a liberal use of identities relating Jacobi polynomials.

[8] The conventions used here for Jacob-Wick expansions are those of G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) 46, 239 (1968), hereafter referred to as CMN.

III. s-CHANNEL CONSTRAINTS

Inequalities involving integrals of s-channel partial waves may be obtained by multiplying Eq. (6) by judiciously chosen functions and integrating over t from 0 to $4\mu^2$. The functions used are[9]

$$\tfrac{1}{2}[(t-4m^2)(t-4\mu^2)]^{(L+1)/2}\frac{\phi^{\sigma_T}}{(1-z_t{}^2)^{\sigma_T}}T(t),$$

where $T(t)$ is any polynomial in t which is positive for t between 0 and $4\mu^2$. By Eq. (2) we see that $\phi^{\sigma_T}/(1-z_t{}^2)^{\sigma_T}$ is dependent only on t and hence this entire expression is purely a function of t. Finally we note that when the positive determination of the root is taken this expression is positive for t between 0 and $4\mu^2$ and leads to the inequality

$$\int_0^{4\mu^2}\left[\int_{-1}^{+1}\bar{M}P_L{}^{\sigma_T\sigma_T}(z_t)(1-z_t{}^2)^{\sigma_T}dz_t\right]\times[(t-4m^2)(t-4\mu^2)]^{L/2}\frac{\phi^{\sigma_T}}{(1-z_t{}^2)^{\sigma_T}}T(t)\times\tfrac{1}{2}[(t-4m^2)(t-4\mu^2)]^{1/2}dt\geqslant 0. \tag{11}$$

We now change variables from t, z_t to s, z_s so that we may project s-channel partial waves:

$$\int_{(m-\mu)^2}^{(m+\mu)^2}\int_{-1}^{+1}\bar{M}P_L{}^{\sigma_T\sigma_T}(z_t)[(t-4m^2)(t-4\mu^2)]^{L/2}\times\phi^{\sigma_T}T(t)dz_s\frac{\mathcal{S}^2}{2s}ds\geqslant 0. \tag{12}$$

$\bar{M}$ is now written as a combination of s-channel helicity amplitudes which are in turn expressed in their Jacob-Wick expansion.

σ_T even:

$$\begin{aligned}\bar{M}&=\mathcal{S}^{\sigma_T}\sum_{\lambda\mu}\bar{M}_{\lambda\mu\lambda\mu}{}^s\\ &=\mathcal{S}^{\sigma_T}\sum_{\lambda\mu}\left(\frac{1+z_s}{2}\right)^{|\lambda-\mu|}\times\sum_j(j+\tfrac{1}{2})\bar{M}_{\lambda\mu\lambda\mu}{}^{sj}P_{j-|\lambda-\mu|}{}^{0,2|\lambda-\mu|}(z_s),\end{aligned}$$

σ_T odd:

$$\begin{aligned}\bar{M}&=\frac{\mathcal{S}^{\sigma_T-1}}{\cos\frac{1}{2}\theta_s}\sum_{\lambda\mu}\bar{M}_{\lambda\mu\lambda\mu}{}^s\\ &=\mathcal{S}^{\sigma_T-1}\sum_{\lambda\mu}\left(\frac{1+z_s}{2}\right)^{|\lambda-\mu|-1/2}\times\sum_j(j+\tfrac{1}{2})\bar{M}_{\lambda\mu\lambda\mu}{}^{sj}P_{j-|\lambda-\mu|}{}^{0,2|\lambda-\mu|}(z_s).\end{aligned}$$

[9] The factor $[(t-4m^2)(t-4\mu^2)]^{L/2}$ is introduced to remove the singularity in the variable t from $P_L{}^{\sigma_T\sigma_T}(z_t)$. $\frac{1}{2}[(t-4m^2)(t-4u^2)]^{1/2}$ is included with an eye on the coming change of variables:

$$\tfrac{1}{2}[(t-4m^2)(t-4u^2)]^{1/2}dz_tdt=dsdt=(\mathcal{S}^2/2s)dsdz_s.$$

$\phi^{\sigma_T}/(1-z_t{}^2)^{\sigma_T}$ will supply the measure required in the s channel. Approaches similar to this may be found in the papers of Ref. 3.

With this substitution, Eq. (12) becomes

σ_T even:

$$\sum_j (j+\tfrac{1}{2}) \int_{(m-\mu)^2}^{(m+\mu)^2} ds \int_{-1}^{+1} \frac{\mathcal{S}^{5\sigma_T+2}}{2s(4s)^{\sigma_T}} \sum_{\lambda\mu} \bar{M}_{\lambda\mu\lambda\mu}{}^{sj}(s)\left(\frac{1+z_s}{2}\right)^{|\lambda-\mu|}$$
$$\times P_L{}^{\sigma_T\sigma_T}(z_t)[(t-4m^2)(t-4\mu^2)]^{L/2} T P_{j-|\lambda-\mu|}{}^{0,2|\lambda-\mu|}(z_s)(1-z_s{}^2)^{\sigma_T} dz_s \geqslant 0, \quad (13a)$$

σ_T odd:

$$\sum_j (j+\tfrac{1}{2}) \int_{(m-\mu)^2}^{(m+\mu)^2} ds \int_{-1}^{+1} \frac{\mathcal{S}^{5\sigma_T+1}}{2s(4s)^{\sigma_T}} \sum_{\lambda\mu} \bar{M}_{\lambda\mu\lambda\mu}{}^{sj}(s)\left(\frac{1+z_s}{2}\right)^{|\lambda-\mu|-1/2}$$
$$\times P_L{}^{\sigma_T\sigma_T}(z_t)[(t-4m^2)(t-4\mu^2)]^{L/2} T P_{j-|\lambda-\mu|}{}^{0,2|\lambda-\mu|}(z_s)(1-z_s{}^2)^{\sigma_T} dz_s \geqslant 0. \quad (13b)$$

Since $P_L{}^{\sigma_T\sigma_T}(z_t)[(t-4m^2)(t-4\mu^2)]^{L/2}$ is a polynomial of degree L in t and s, we see that the entire coefficient of $P_{j-|\lambda-\mu|}{}^{0,2|\lambda-\mu|}(z_s)(1+z_s)^{2|\lambda-\mu|}$ is a polynomial in z_s. Using the orthogonality of the Jacobi polynomials $P_{j-|\lambda-\mu|}{}^{0,2|\lambda-\mu|}(z_s)$, we see that the sum over j will vanish for j greater than $2\sigma_T-3|\lambda-\mu|+L+q$, where q is the order of the polynomial $T(t)$. Hence the inequalities given in Eq. (13) involve integrals over finite numbers of s-channel partial waves.

These inequalities may be studied more easily if we have a more symmetric way of writing the polynomial T. It can be shown[10] that any polynomial in t which is positive for $0<t<4\mu^2$ can be written in the form

$$\text{odd order:}\quad T(t)=t[A(t)]^2+(4\mu^2-t)[B(t)]^2,$$
$$\text{even order:}\quad T(t)=[C(t)]^2+t(4\mu^2-t)[D(t)]^2,$$

where A, B, C, and D are real polynomials in t. The converse of this theorem is obviously true as well.

To simplify the expressions, we write the inequalities of Eq. (13) as

$$\int_{(m-\mu)^2}^{(m+\mu)^2} ds \int_{-1}^{+1} \mathcal{G}_L(s,z_s) T dz_s \geqslant 0, \quad (14)$$

where $\mathcal{G}_L(s,z_s)$ is defined by equating the above integral with those of Eq. (13). When the above expressions for T are used in Eq. (14), we see that we have four inequalities:

$$T\text{ of odd order:}\begin{cases} 0\leqslant \int\int \mathcal{G}_L t(\sum_{i=0}^{r} a_i t^i)^2 dz_s ds = \sum_{ij}^{r} a_i a_j \int\int \mathcal{G}_L t^{i+j+1} dz_s ds, \\ 0\leqslant \int\int \mathcal{G}_L (4\mu^2-t)(\sum_{i}^{r} b_i t^i)^2 dz_s ds = \sum_{ij}^{r} b_i b_i \int\int \mathcal{G}_L (4\mu^2-t) t^{i+j} dz_s ds, \end{cases}$$
$$T\text{ of even order:}\begin{cases} 0\leqslant \int\int \mathcal{G}_L (\sum_{i}^{r} c_i t^i)^2 dz_s ds = \sum_{ij}^{r} c_i c_j \int\int \mathcal{G}_L t^{i+j} dz_s ds, \\ 0\leqslant \int\int \mathcal{G}_L t(4\mu^2-t)(\sum_{i}^{r-1} d_i t^i)^2) dz_s ds = \sum_{ij}^{r-1} d_i d_j \int\int \mathcal{G}_L (4\mu^2-t) t^{i+j+1} dz_s ds, \end{cases} \quad (15)$$

where $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, and $\{d_i\}$ are sets of arbitrary real constants. From the above theorem, its converse, and the linearity of integration, we see that these inequalities are equivalent to those of Eq. (13) or (14).

Given a quadratic form,

$$\sum_{ij} \omega_i \omega_j W^{ij}, \quad \{\omega_i\} \text{ real}$$

a necessary and sufficient condition that the form be positively definite is[11]

$$W^{00}\geqslant 0,$$
$$\begin{vmatrix} W^{00} & W^{01} \\ W^{10} & W^{11} \end{vmatrix} \geqslant 0,$$
$$\begin{vmatrix} W^{00} & W^{01} & W^{02} \\ W^{10} & W^{11} & W^{12} \\ W^{20} & W^{21} & W^{22} \end{vmatrix} \geqslant 0, \quad (16)$$
$$\vdots$$

[10] N. I. Akhiezer, *The Classical Moment Problem* (Oliver and Boyd, London, 1965), p. 74.
[11] L. M. Blumenthal, Am. Math. Monthly **35**, 551 (1928).

When applied to any of our inequalities,

$$W_{ij}{}^L \equiv \begin{cases} \displaystyle\int\int \mathcal{G}_L t^{i+j+1} dz_s ds\,, \\ \displaystyle\int\int \mathcal{G}_L (4\mu^2 - t) t^{i+j} dz_s ds\,, \\ \displaystyle\int\int \mathcal{G}_L t^{i+j} dz_s ds\,, \\ \displaystyle\int\int \mathcal{G}_L (4\mu^2 - t) t^{i+j+1} dz_s ds\,, \end{cases} \qquad (17)$$

it will give a set of inequalities for each even L:

$$\begin{gathered} W_{00}{}^L \geqslant 0\,, \\ \begin{vmatrix} W_{00}{}^L & W_{01}{}^L \\ W_{10}{}^L & W_{11} \end{vmatrix} \geqslant 0\,, \\ \begin{vmatrix} W_{00}{}^L & W_{01}{}^L & W_{02}{}^L \\ W_{10}{}^L & W_{11}{}^L & W_{12}{}^L \\ W_{20}{}^L & W_{21}{}^L & W_{22}{}^L \end{vmatrix} \geqslant 0\,, \\ \vdots \end{gathered} \qquad (18)$$

IV. π^0-NUCLEON ELASTIC SCATTERING

In this section we consider π^0-nucleon elastic scattering:

$$p\pi^0 \to p\pi^0$$

or equivalently,

$$n\pi^0 \to n\pi^0\,.$$

This particular choice is made because it has only even partial waves in the t channel. The same approach is applicable to a somewhat wider class of pion-nucleon elastic scattering processes. The diagonal helicity amplitude for this process is given by

$$M_{\frac{1}{2}0\frac{1}{2}0} = M_{-\frac{1}{2}0-\frac{1}{2}0} = \cos\tfrac{1}{2}\theta_s [A + \tfrac{1}{2}B(s - m^2 - \mu^2)]\,,$$

where A and B have the standard definitions.[12] For this special case Eq. (3) becomes

$$M = 2(t/-\phi)^{1/2} \mathcal{S} M_{\frac{1}{2}0\frac{1}{2}0} = 2[A + \tfrac{1}{2}B(s - m^2 - \mu^2)]\,.$$

This amplitude contains a dynamical pole in the B term[12] which must be subtracted out:

$$\bar{M} = 2[A + \tfrac{1}{2}\bar{B}(s - m^2 - \mu^2)]\,, \qquad (19)$$

where

$$\bar{B} = B - G^2\left(\frac{1}{m^2 - s} - \frac{1}{m^2 - \mu}\right).$$

Substituting this into Eq. (6), we get

$$2\int_{-1}^{+1} [A + \tfrac{1}{2}\bar{B}(s - m^2 - \mu^2)] P_L{}^{1,1}(z_t)(1 - z_t{}^2) dz_t \geqslant 0\,, \quad (20)$$

where we must require $L \geqslant 2$ since there are two subtractions.[12]

This may be written as an algebraic condition on the t-channel partial waves by using

$$\bar{M} = \frac{4m}{(t - 4m^2)^{1/2}} \bar{M}_{\frac{1}{2}\frac{1}{2}00}{}^t + \frac{2\bar{M}_{\frac{1}{2}-\frac{1}{2}00}{}^t}{\sin\theta_t} + t^{1/2}\left[\frac{\cos\theta_t}{(t - 4m^2)^{1/2}} - \frac{1}{(t - 4\mu^2)^{1/2}}\right]$$

and

$$\bar{M}_{\frac{1}{2}\frac{1}{2}00}{}^t = \sum_j (j + \tfrac{1}{2}) \bar{M}_{\frac{1}{2}\frac{1}{2}00}{}^{tj}(t) P_j{}^{00}(z_t)\,,$$

$$\frac{\bar{M}_{\frac{1}{2}-\frac{1}{2}00}{}^t}{\sin\theta_t} = -\tfrac{1}{2} \sum_j (j + \tfrac{1}{2}) \left(\frac{j+1}{j}\right)^{1/2} \bar{M}_{\frac{1}{2}-\frac{1}{2}00}{}^{tj}(t) P_{j-1}{}^{1,1}(z_t)\,.$$

When this is substituted into Eq. (20) and use is made of relations among the Jacobi polynomials, we obtain

$$\begin{aligned} &\frac{2m}{(t - 4m^2)^{1/2}}\left[\frac{L+1}{2L+3} \bar{M}_{\frac{1}{2}\frac{1}{2}00}{}^{tL}(t) + \frac{(L+1)(2L+5)}{(2L+3)^2} \bar{M}_{\frac{1}{2}-\frac{1}{2}00}{}^{t,L+2}(t)\right] \\ &- \frac{t^{1/2}}{(t - 4m^2)^{1/2}}\left[\frac{L^{1/2}(L+1)^{1/2}}{2L+3} \bar{M}_{\frac{1}{2}-\frac{1}{2}00}{}^{tL}(t) + \frac{(L+1)}{(2L+3)}\left(\frac{L+3}{L+2}\right)^{1/2} \bar{M}_{\frac{1}{2}-\frac{1}{2}00}{}^{t,L+2}(t)\right] \\ &+ \frac{t^{1/2}}{(t - 4\mu^2)^{1/2}}\left(\frac{L+1}{L+2}\right)^{1/2} \bar{M}_{\frac{1}{2}-\frac{1}{2}00}{}^{t,L+1}(t) \geqslant 0, \end{aligned}$$

$$\text{where} \quad L \geqslant 2, \quad 0 \leqslant t < 4\mu^2. \qquad (21)$$

The inequality given in Eq. (21) is a special case of that given in Eq. (10).

The integral inequality given in Eq. (13b) becomes

$$\begin{aligned} \sum_j (j + \tfrac{1}{2}) \int_{-1}^{+1} \int_{(m-\mu)^2}^{(m+\mu)^2} \frac{\mathcal{S}^6}{4s^2} ds \bar{M}_{\frac{1}{2}0\frac{1}{2}0}{}^{sj}(s) &\times P_L{}^{1,1}(z_t)[(t - 4m^2)(t - 4\mu^2)]^{L/2} \\ &\times T(1 - z_s) P_{j-1}{}^{0,1}(z_s)(1 + z_s) dz_s \geqslant 0\,. \quad (22) \end{aligned}$$

We now note that

$$\begin{aligned} \mathcal{G}_L = \sum_j (j + \tfrac{1}{2}) \frac{\mathcal{S}^6}{4s^2} \bar{M}_{\frac{1}{2}0\frac{1}{2}0}{}^{sj}(s) &\times P_L{}^{1,1}(z_t)[(t - 4m^2)(t - 4\mu^2)]^{L/2} \\ &\times (1 - z_s) P_{j-\frac{1}{2}}{}^{0,1}(z_s)(1 + z_s), \quad L \geqslant 2\,. \end{aligned}$$

With the definition of $W_{ij}{}^L$ given in Eq. (17), we now

[12] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963).

have the set of inequalities given in Eq. (18) which are equivalent to that given in Eq. (22).

ACKNOWLEDGMENT

I am most grateful to Professor A. P. Balachandran for his encouragement and recommendations during the course of this research.

APPENDIX

Proof[13] of the following inequalities:

$$\begin{aligned} &\text{For } z'>1, && (z'^2-1)^{\sigma T}Q_L{}^{\sigma T\sigma T}(z')\geqslant 0 \text{ for all } L, \\ &\text{for } z'<-1, && (z'^2-1)^{\sigma T}Q_L{}^{\sigma T\sigma T}(z')\leqslant 0 \text{ for } L \text{ even}, \\ & && \geqslant 0 \text{ for } L \text{ odd}. \end{aligned} \tag{A1}$$

Consider

$$\int_{-1}^{+1}\frac{[P_L{}^{\sigma T\sigma T}(z)]^2(1-z^2)^{\sigma T}}{z'-z}dz.$$

For $z'>+1$,

$$0<\int_{-1}^{+1}\frac{[P_L{}^{\sigma T\sigma T}(z)]^2(1-z^2)^{\sigma T}}{z'-z}dz. \tag{A2}$$

Since $P_L{}^{\sigma T\sigma T}(z)$ is a polynomial in z of order L,

$$P_L{}^{\sigma T\sigma T}(z)=\sum_{i=0}^{L} a_i(z'-z)^i. \tag{A3}$$

When this substitution is made, (A2) becomes

$$0<a_0\int_{-1}^{+1}\frac{P_L{}^{\sigma T\sigma T}(z)(1-z^2)^{\sigma T}}{z'-z}dz, \tag{A4}$$

where the orthogonality of the polynomials $P_L{}^{\sigma T\sigma T}$ has been used. From (A3) we see that $a_0=P_L{}^{\sigma T\sigma T}(z')$ and since

$$P_L{}^{\sigma T\sigma T}(z')>0 \quad \text{for} \quad z'>1,$$

we have

$$0<\int_{-1}^{+1}\frac{P_L{}^{\sigma T\sigma T}(z)(1-z^2)^{\sigma T}}{z'-z}dz=2(z'^2-1)^{\sigma T}Q_L{}^{\sigma T\sigma T}(z').$$

This is the first of the inequalities given in (A1). For $z'<-1$, a similar argument leads to

$$0>P_L{}^{\sigma T\sigma T}(z')\int_{-1}^{+1}\frac{P_L{}^{\sigma T\sigma T}(z)(1-z^2)^{\sigma T}}{z'-z}dz.$$

Since

$$\begin{aligned} P_L{}^{\sigma T\sigma T}(z')>0 \quad \text{for} \quad z'<-1, \quad L \text{ even} \\ P_L{}^{\sigma T\sigma T}(z')<0 \quad \text{for} \quad z'<-1, \quad L \text{ odd} \end{aligned}$$

we have

$$0>\int_{-1}^{+1}\frac{P_L{}^{\sigma T\sigma T}(z)(1-z^2)^{\sigma T}}{z'-z}dz=2(z'-1)^{\sigma T}Q_L{}^{\sigma T\sigma T}(z') \quad \text{for } z'<-1, \quad L \text{ even}$$

$$0<\int_{-1}^{+1}\frac{P_L{}^{\sigma T\sigma T}(z)(1-z^2)^{\sigma T}}{z'-z}dz=2(z'-1)^{\sigma T}Q_L{}^{\sigma T\sigma T}(z') \quad \text{for } z'<-1, \quad L \text{ odd}$$

which proves the second of the inequalities given in (A1). These inequalities may also be proven using generalized hypergeometric functions.

---

[13] This proof is due to A. P. Balachandran and M. L. Blackmon (private communication).