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New Lagrangian Formalism for Infinite-Component Field Theories*

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The infinite-component wave equation $(\partial_\mu L^\mu - M)\varphi = 0$, whose fields are an infinite sum of $(\frac{1}{2}k,\frac{1}{2}k)$ representations of the Lorentz group, is shown to be invariant under a set of gauge transformations of the second kind. This invariance leads us to consider a new class of second-order Lagrangians which are invariant under these gauge transformations. The Lagrangians are a generalization to the infinite component field of the ordinary Stueckelberg formalism for a spin-1 field. This new formalism allows us to use ordinary field-theory techniques to discuss canonical quantization and to introduce local interactions. We explore two separate theories, one with full invariance under the gauge transformation, and one with invariance only under a restricted set of transformations $[(\Box^2+\bar{m}^2)\Lambda(x)=0]$. The former theory presents a natural framework for discussing the ρ trajectory and conserved vector currents with an infinite number of poles. The gauge current in that theory is conserved, contains the ρ and ρ daughters, and has finite c-number Schwinger terms (as in the algebra of fields). The latter theory seems a possible framework for the π-A_1 trajectory, the axial-vector obeying a generalized PCAC (partial conservation of axial-vector current) relation.

I. INTRODUCTION

A CERTAIN amount of progress has been achieved[1] in understanding the mathematical structure of infinite-component wave equations, and in investigating the question of which physical systems they may be applied to. One result which has been periodically reinforced[2] is that the use of infinite-dimensional irreducible representations of the Lorentz group in these theories is likely to lead to some kind of disaster: noncausality, completely degenerate mass spectra, particles with spacelike four-momenta, and similar undesirable results. Consequently, the alternative of using infinitely many finite-dimensional representations remains an attractive one. However, while theories with infinite-dimensional irreducible representations are unduly restrictive, those with only finite-dimensional representations are in a sense not restrictive enough: Any mass spectrum can be obtained by appropriately choosing the parameters of the wave equation.

In two previous papers,[3,4] the wave equation

$$(\partial_\mu L^\mu - M)\varphi(x) = 0, \tag{1.1}$$

which involves only finite-dimensional irreducible representations of the Lorentz group, has been explored in detail. It was shown that the matrix M acts as a potential in a Schrödinger-like equation for the infinite-component field φ. The choice of M determines whether the spectrum contains only bound states, bound states plus a continuum (scattering states), or only scattering states. Also, the bound-state spectrum may be of any shape (linear if one wishes), and hence can be made to agree with current ideas about meson Regge trajectories.

Since the mass spectrum is not specified *a priori* by this type of theory, it is of especial interest to investigate some of its other physical properties. In this paper we construct a Lagrangian formalism for infinite-component fields based on an extension of the well-known Stueckelberg theory[5] of vector fields. Fujii and Kamefuchi[6] showed how to generalize Stueckelberg's treatment to the case where the spin-0 and spin-1 pieces have different masses. In the present work, we extend the Stueckelberg formalism to the case of an infinite number of fields. We consider two theories. The first possesses gauge invariance of the second kind, restricted in the sense that the gauge function $\Lambda(x)$ must satisfy

$$(\Box^2+m^2)\Lambda(x)=0 \tag{1.2}$$

and yields an equation of motion equivalent to the original infinite-component wave equation (1.1). The second possesses unrestricted gauge invariance, and is equivalent to a new type of wave equation not previously considered.

In Sec. II we perform the transformations necessary to take us from the language of infinite-component wave equations to that of Stueckelberg fields, and present a brief review of the Stueckelberg formalism. In Sec. III, we postulate two Lagrangians, and determine the infinite-component wave equations that follow from

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[1] Y. Nambu, Progr. Theoret. Phys. (Kyoto) Suppl. **37–38**, 368 (1966); Phys. Rev. **160**, 1171 (1967); C. Fronsdal, *ibid.* **156**, 1653 (1967); **156**, 1665 (1967); **171**, 1811 (1968). See also a recent review article by A. O. Barut (unpublished), and references contained therein.

[2] E. Abers, I. T. Grodsky, and R. E. Norton, Phys. Rev. **159**, 1222 (1967); H. D. I. Abarbanel and Y. Frishman, *ibid.* **171**, 1442 (1968); I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968); A. I. Oksak and I. T. Todorov, Phys. Rev. D **1**, 3511 (1970).

[3] A. Chodos, Phys. Rev. D **1**, 2937 (1970).

[4] A. Chodos and R. W. Haymaker, Phys. Rev. D **2**, 793 (1970).

[5] E. C. G. Stueckelberg, Helv. Phys. Acta **11**, 299 (1938).

[6] Y. Fujii and S. Kamefuchi, Nuovo Cimento **33**, 1639 (1964).

them. Section IV is devoted to the canonical quantization of these theories, and to a discussion of the currents associated with the gauge transformations. In Sec. V we look ahead to the possibility of applying formalism we have developed to the case of interacting hadronic systems.

II. GAUGE INVARIANCE OF WAVE EQUATION

We begin with the infinite-component wave equation

$$(\partial_\mu L^\mu - M)\varphi(x)=0\,, \tag{2.1}$$

which has been studied in previous work. We recall[3] that $\varphi(x)$ is taken to transform according to the infinitely reducible representation

$$R=\sum_{k=0}^{\infty}\oplus(\tfrac{1}{2}k,\tfrac{1}{2}k) \tag{2.2}$$

of the homogeneous Lorentz group. That is, we label $\varphi(x)$ with the index $kj\sigma$, $k=0,\,1,\,2,\,\ldots,\,0\le j\le k$, $-\sigma\le j\le\sigma$, and require that

$$U(\Lambda)\varphi_{kj\sigma}(x)U^{-1}(\Lambda) = D_{j\sigma,j'\sigma'}{}^{(k/2,k/2)}(\Lambda^{-1})\varphi_{kj'\sigma'}(\Lambda x)\,. \tag{2.3}$$

Although we have previously worked exclusively in the "$kj\sigma$" basis, in this and succeeding sections we shall be interested in the form of equation (2.1) where we replace $\varphi_{kj\sigma}$ by $\varphi_{\mu_1\cdots\mu_k}$, where $\{\mu_1\cdots\mu_k\}$ is a traceless and fully symmetric set of 4-vector indices.

To obtain the matrix which transforms the "$kj\sigma$" basis to the "$\mu_1\cdots\mu_k$" basis, we study the vector L_μ that appears in (2.1). From Ref. 3, we know that

$$(L_\mu L^\mu)_{kj\sigma,k'j'\sigma'}=-[2+k(k+2)]\delta_{kk'}\delta_{jj'}\delta_{\sigma\sigma'} \tag{2.4}$$

and also that

$$[L_\mu,L_\nu]=iM_{\mu\nu}\,,$$

where $M_{\mu\nu}$ are the six generators of the Lorentz group. Therefore,

$$[L_\mu,L_\nu]_{kj\sigma,k'j'\sigma'}\propto\delta_{kk'}\,. \tag{2.5}$$

Furthermore, because L_μ transforms as a 4-vector under Lorentz transformation, it can connect only neighboring irreducible representations. That is,

$$(L_\mu)_{kj\sigma,k'j'\sigma'}=[A_{j\sigma,j'\sigma'}{}^{(-)}(k,\mu)\delta_{k,k'+1} + A_{j\sigma,j'\sigma'}{}^{(+)}(k,\mu)\delta_{k',k+1}]\,, \tag{2.6}$$

where $A^{(\pm)}$ are matrices whose exact form need not concern us here. It will be convenient, however, to define two new sets of matrices by

$$(L_\mu{}^{(\pm)})_{kj\sigma,k'j'\sigma'}=A_{j\sigma,j'\sigma'}{}^{(\pm)}(k,\mu)\delta_{k',k\pm1}\,, \tag{2.7}$$

so that

$$L_\mu=L_\mu{}^{(+)}+L_\mu{}^{(-)}\,. \tag{2.8}$$

By an extension of the techniques of Ref. 3, we can derive the following useful formulas:

$$(L_\mu{}^{(-)}L^{\mu(+)})_{kj\sigma,k'j'\sigma'}=-\tfrac{1}{2}k^2\delta_{kk'}\delta_{jj'}\delta_{\sigma\sigma'} \tag{2.9}$$

and

$$(L_\mu{}^{(+)}L^{\mu(-)})_{kj\sigma,k'j'\sigma'}=-\tfrac{1}{2}(k+2)^2\delta_{kk'}\delta_{jj'}\delta_{\sigma\sigma'}\,. \tag{2.10}$$

Notice that by adding (2.9) and (2.10) we recover (2.4).

We now define

$$\varphi_{\mu_1\cdots\mu_k}(x)=\sum_{j\sigma}T_{\mu_1\cdots\mu_k,j\sigma}\varphi_{kj\sigma}(x)\,, \tag{2.11}$$

where

$$T_{\mu_1\cdots\mu_k,j\sigma}=(L_{\mu_1}\cdots L_{\mu_k})_{0,kj\sigma}\,. \tag{2.12}$$

Observe that (2.4) and (2.5) guarantee that (2.11) will be traceless and symmetric, as required. Furthermore, since (2.12) requires us to go from zero to k in exactly k steps, we can replace L_{μ_j} by $L_{\mu_j}{}^{(+)}$ in (2.12) if we so desire.

We also define

$$\tilde{T}^{\mu_1\cdots\mu_k}{}_{j\sigma}=(L^{\mu_1}\cdots L^{\mu_k})_{kj\sigma,0}\,. \tag{2.13}$$

$\tilde{T}$ is symmetric and traceless for the same reasons that T is, and in (2.13) we may, if desired, replace L^{μ_j} by $L^{\mu_j(-)}$ without changing the value of $\tilde{T}$. Using (2.9), we can derive the following relation:

$$\tilde{T}^{\mu_1\cdots\mu_k}{}_{j\sigma}T_{\mu_1\cdots\mu_k,j'\sigma'}=[(-1)^k/2^k](k!)^2\delta_{jj'}\delta_{\sigma\sigma'}\,, \tag{2.14}$$

and therefore, defining

$$S^{\mu_1\cdots\mu_k}{}_{j\sigma}=\frac{(-1)^k2^k}{(k!)^2}\tilde{T}^{\mu_1\cdots\mu_k}{}_{j\sigma}\,, \tag{2.15}$$

we have

$$\varphi_{kj\sigma}(x)=S^{\mu_1\cdots\mu_k}{}_{j\sigma}\varphi_{\mu_1\cdots\mu_k}(x)\,. \tag{2.16}$$

From this it follows that

$$\varphi_{\mu_1\cdots\mu_k}(x)=\sum_{j\sigma}T_{\mu_1\cdots\mu_k,j\sigma}S^{\nu_1\cdots\nu_k}{}_{j\sigma}\varphi_{\nu_1\cdots\nu_k}(x)\,, \tag{2.17}$$

so we conclude that

$$\sum_{j\sigma}T_{\mu_1\cdots\mu_k,j\sigma}S^{\nu_1\cdots\nu_k}{}_{j\sigma}=\delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k}\,. \tag{2.18}$$

Here $\delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k}$ is the unit matrix in the $(\frac{1}{2}k,\frac{1}{2}k)$ representation of the homogeneous Lorentz group. It is symmetric and traceless separately in its upper and lower indices; when acting on an arbitrary tensor with k 4-vector indices, it projects out the symmetric and traceless part. For example,

$$\delta_{\mu_1\mu_2}{}^{\nu_1\nu_2}=\tfrac{1}{2}(\delta_{\mu_1}{}^{\nu_1}\delta_{\mu_2}{}^{\nu_2}+\delta_{\mu_1}{}^{\nu_2}\delta_{\mu_2}{}^{\nu_1}-\tfrac{1}{2}g_{\mu_1\mu_2}g^{\nu_1\nu_2})\,, \tag{2.19}$$

and so forth.

We can now proceed to the task of transforming (2.1). We multiply (2.1) by $T_{\nu_1\cdots\nu_k}$ to obtain

$$\sum_{j\sigma}T_{\nu_1\cdots\nu_k,j\sigma}[\partial^\nu(L_\nu{}^{(+)}+L_\nu{}^{(-)})-M]_{kj\sigma,k'j'\sigma'} \times\varphi_{k'j'\sigma'}(x)=0\,. \tag{2.20}$$

From (2.12) we have

$$\sum_{j\sigma}T_{\nu_1\cdots\nu_k,j\sigma}(L_\nu{}^{(+)})_{kj\sigma,k+1j'\sigma'}=T_{\nu_1\cdots\nu_k\nu,j'\sigma'}\,, \tag{2.21}$$

while the $L^{(-)}$ term can be written

$$\sum_{j\sigma}\sum_{j'\sigma'} T_{\nu_1\cdots\nu_k,j\sigma}\partial_\mu(L^{\mu(-)})_{kj\sigma,k-1j'\sigma'} \times S^{\mu_1\cdots\mu_{k-1}}{}_{j'\sigma'}\varphi_{\mu_1\cdots\mu_{k-1}}(x)=\sum_{j\sigma} T_{\nu_1\cdots\nu_k,j\sigma} \times\partial_\mu(-\tfrac{1}{2}k^2)S^{\mu\mu_1\cdots\mu_{k-1}}{}_{j\sigma}\varphi_{\mu_1\cdots\mu_{k-1}}(x)\,,$$

which from (2.13) and (2.15) is

$$-\tfrac{1}{2}k^2\delta_{\nu_1\cdots\nu_k}{}^{\mu\mu_1\cdots\mu_{k-1}}\partial_\mu\varphi_{\mu_1\cdots\mu_{k-1}}(x)\,, \tag{2.22}$$

where we have used (2.18).

Since M is a matrix of the form

$$M_{kj\sigma,k'j'\sigma'}=m_k\delta_{kk'}\delta_{jj'}\delta_{\sigma\sigma'}\,, \tag{2.23}$$

the M term in (2.20) just becomes $m_k\varphi_{\nu_1\cdots\nu_k}(x)$. Thus (2.1) becomes, in our new basis,

$$\partial^\nu\varphi_{\nu\nu_1\cdots\nu_k}=m_k\varphi_{\nu_1\cdots\nu_k} +\tfrac{1}{2}k^2\delta_{\nu_1\cdots\nu_k}{}^{\mu\mu_1\cdots\mu_{k-1}}\partial_\mu\varphi_{\mu_1\cdots\mu_{k-1}}\,. \tag{2.24}$$

For brevity, we shall sometimes write

$$\partial\cdot\varphi_{k+1}=m_k\varphi_k+\tfrac{1}{2}k^2\delta\partial\varphi_{k-1}\,, \tag{2.24$'$}$$

which is to be understood as meaning (2.24).

In Ref. 4, it was shown how (2.1) led to a second-order difference equation that determined the mass spectrum. The following question arises: Can we see directly that (2.24) contains the same information, i.e., does the difference equation follow from (2.24) in a natural way?

We proceed by considering the effect on (2.24) of a countable set of gauge transformations of the following form. Let

$$\varphi_{\mu_1\cdots\mu_k}\rightarrow\varphi_{\mu_1\cdots\mu_k}+\gamma_k{}^{(j)}\delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k}\partial_{\nu_{j+1}\cdots\nu_k}V_{\nu_1\cdots\nu_j}\,,$$
$$j=0,1,2,\ldots,\quad \gamma_k{}^{(j)}=0\quad\text{for}\quad k<j\,, \tag{2.25}$$

where $V_{\nu_1\cdots\nu_j}$ is taken (i) to be symmetric and traceless, (ii) to be divergenceless,

$$\partial^{\mu_1}V_{\mu_1\mu_2\cdots\mu_j}=0\quad(j>0)\,, \tag{2.26}$$

(iii) to satisfy the Klein-Gordon equation with some mass $\bar{m}_j$,

$$(\Box^2+\bar{m}_j{}^2)V_{\mu_1\cdots\mu_j}=0\,. \tag{2.27}$$

Then the changes in (2.24) are given by

(Change in left-hand side)

$$=\gamma_{k+1}{}^{(j)}\partial^\nu\delta_{\nu_1\cdots\nu_k\nu}{}^{\mu_1\cdots\mu_{k+1}}\partial_{\mu_{j+1}}\cdots\partial_{\mu_{k+1}}V_{\mu_1\cdots\mu_j}\,, \tag{2.28a}$$

(Change in right-hand side)

$$=[m_k\gamma_k{}^{(j)}+\tfrac{1}{2}k^2\gamma_{k-1}{}^{(j)}]\delta_{\nu_1\cdots\nu_k}{}^{\mu_1\cdots\mu_k} \times\partial_{\mu_{j+1}}\cdots\partial_{\mu_k}V_{\mu_1\cdots\mu_j}\,. \tag{2.28b}$$

We state without proof (the proof depends only on careful counting) that

$$\partial^\nu\delta_{\nu_1\cdots\nu_k\nu}{}^{\mu_1\cdots\mu_{k+1}}\partial_{\mu_{j+1}}\cdots\partial_{\mu_{k+1}}V_{\mu_1\cdots\mu_j} =\Box^2\frac{(k-j+1)(k+j+2)}{2(k+1)^2}\delta_{\nu_1\cdots\nu_k}{}^{\mu_1\cdots\mu_k} \times\partial_{\mu_{j+1}}\cdots\partial_{\mu_k}V_{\mu_1\cdots\mu_j} \tag{2.29}$$

for any V that satisfies conditions (i) and (ii). Therefore, (2.24) will be invariant under (2.25) provided that

$$-\tfrac{1}{2}\bar{m}_j{}^2\frac{(k-j+1)(k+j+2)}{(k+1)^2}\gamma_{k+1}{}^{(j)} =m_k\gamma_k{}^{(j)}+\tfrac{1}{2}k^2\gamma_{k-1}{}^{(j)}\,. \tag{2.30}$$

Changing variable to $n=k-j$, and letting

$$\gamma_{n+j}{}^{(j)}=\frac{[\Gamma(n+j+1)]^2}{\Gamma(n+2j+2)\Gamma(n+1)}\left(\frac{-2}{\bar{m}_j{}^2}\right)^n D_n{}^{(j)}\,, \tag{2.31}$$

we have

$$D_{n+1}{}^{(j)}=m_{n+j}D_n{}^{(j)}-\tfrac{1}{4}\bar{m}_j{}^2n(n+2j+1)D_{n-1}{}^{(j)}\,. \tag{2.32}$$

This is to be compared with Eq. (3.6) of Ref. 4, with $x=\bar{m}_j{}^2$, $r_n{}^2=\frac{1}{4}$, and $\lambda=2j+1$. We see that (2.32), is in fact, the equation for determining the allowed masses $\bar{m}_j$, and that the parameters of the gauge transformations (2.25), i.e., the $\gamma_k{}^{(j)}$, are related to the eigenvectors that solve (2.32).

The fact that the mass conditions embodied in (2.24) are most readily expressed as an invariance requirement under the gauge transformations (2.25) leads us to explore more fully theories that involve the set of fields $\{\varphi_{\nu_1\cdots\nu_k}\}$, $k=0, 1, 2, \ldots$, and that are invariant under gauge transformations such as (2.25). An extension of the Stueckelberg formalism would seem a natural place to begin.

The Stueckelberg formalism[5,7] is a way of describing a massive vector-meson field by five variables A_μ and B satisfying some subsidiary conditions. The formalism has the nice properties that the vector field A_μ has the same properties under gauge transformations $A_\mu\rightarrow A_\mu+\partial_\mu\Lambda(x)$ as the electromagnetic field, and the renormalizability of the interacting theory depends only on whether the field B can be transformed away by a gauge transformation.

The free-field Lagrangian in the Stueckelberg formalism is given by

$$\mathfrak{L}=-\tfrac{1}{2}(\partial_\mu A_\nu\partial^\mu A^\nu-m^2A_\nu A^\nu) -\tfrac{1}{2}(\partial_\mu B\partial^\mu B-m^2B^2)\,, \tag{2.33}$$

which leads to the equations of motion

$$(\Box^2+m^2)A_\nu=0\,,\quad(\Box^2+m^2)B=0\,. \tag{2.34}$$

We notice that under the transformation

$$A_\mu\rightarrow A_\mu+\partial_\mu\Lambda\,,\quad B\rightarrow B+m\Lambda\,, \tag{2.35}$$

[7] S. Kamefuchi, "Lectures on the Stueckelberg Formalism," Matscience Report No. 14, 1963 (unpublished).

the above Lagrangian is invariant if Λ is restricted to obey

$$(\Box^2+m^2)\Lambda=0. \qquad (2.36)$$

The subsidiary conditions $(\partial_\mu A^\mu+mB)=0$ make this theory identical with the usual formalism for massive spin-1 fields.[8] The usual Lagrangian for spin 1 is

$$\mathcal{L}=-\tfrac{1}{2}G_{\mu\nu}(\partial^\mu\varphi^\nu-\partial^\nu\varphi^\mu)+\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}+\tfrac{1}{2}m^2\varphi_\mu\varphi^\mu, \qquad (2.37)$$

which leads to the field equations

$$G_{\mu\nu}=\partial_\mu\varphi_\nu-\partial_\nu\varphi_\mu, \quad \partial_\mu G^{\mu\nu}+m^2\varphi^\nu=0. \qquad (2.38a)$$

Taking the derivative of (2.38a) we get

$$\partial_\nu\varphi^\nu=0. \qquad (2.38b)$$

If we now let $\varphi^\mu=A^\mu-(1/m)\partial^\mu B$, we notice that φ_μ and $G_{\mu\nu}$ are fully gauge invariant under (2.35). We get

$$\Box^2A^\nu-\partial^\nu\partial_\mu A^\mu+m^2A^\nu-m\partial^\nu B=0, \qquad (2.38a')$$

$$\Box^2B-m\partial_\mu A^\mu=0. \qquad (2.38b')$$

But the Lagrangian

$$\mathcal{L}=-\tfrac{1}{2}G_{\mu\nu}(\partial^\mu A^\nu-\partial^\nu A^\mu)+\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}+\tfrac{1}{2}m^2[A^\mu-(1/m)\partial^\mu B]^2 \qquad (2.37')$$

is invariant under (2.35). Thus we can always choose a gauge in which $\partial_\mu A^\mu+mB=0$ ("Lorentz" gauge).

In that gauge, Eqs. (2.38) reduce to

$$(\Box^2+m^2)B=0, \quad (\Box^2+m^2)A^\mu=0. \qquad (2.39)$$

The theory in the "Lorentz" gauge is still invariant under the restricted gauge transformation with $(\Box^2+m^2)\Lambda(x)=0$. One can describe a spin-1 particle and a spin-0 particle with different masses with a slight modification of the above formalism. One merely adds to (2.37′) the quantity $\frac{1}{2}(\partial_\mu A^\mu+mB)^2=\frac{1}{2}\chi^2$ to obtain

$$\mathcal{L}=-\tfrac{1}{2}G^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}+\tfrac{1}{2}m^2[A^\mu-(1/m)\partial^\mu B]^2+\tfrac{1}{2}(\partial_\mu A^\mu+mB)^2. \qquad (2.40)$$

The last term breaks the total gauge invariance under $A_\mu\to A_\mu+\partial_\mu\Lambda$, $B\to B+m\Lambda$, but leaves the Lagrangian invariant under the restricted transformation with $(\Box^2+m^2)\Lambda=0$. This theory is discussed in detail by Fujii and Kamefuchi.[6]

III. CONNECTION BETWEEN WAVE EQUATION AND STUECKELBERG FIELDS

To generalize the Stueckelberg formalism discussed in Sec. II, we define quantities

$$G_{\nu_1\cdots\nu_k}{}^{(\mathrm{I})}=\delta_{\nu_1\cdots\nu_k}{}^{\mu_1\cdots\mu_k}\partial_{\mu_1}\tilde{\varphi}_{\mu_2\cdots\mu_k}+\alpha_k\tilde{\varphi}_{\nu_1\cdots\nu_k}+\beta_k\partial^\nu\tilde{\varphi}_{\nu\nu_1\cdots\nu_k}, \quad k=0,1,2,\ldots. \qquad (3.1)$$

Here the $\tilde{\varphi}_k$ are symmetric, traceless fields, which should not be confused with the φ_k of the previous section.

[8] J. Schwinger, in *Lectures on Particles and Field Theory* (Prentice-Hall, Englewood Cliffs, N. J., 1965), pp. 147–288.

Equation (3.1) can be written, in obvious shorthand, as

$$G_k{}^{(\mathrm{I})}=\delta\partial\tilde{\varphi}_{k-1}+\alpha_k\tilde{\varphi}_k+\beta_k\partial\cdot\tilde{\varphi}_{k+1}. \qquad (3.1')$$

If we make the gauge transformations

$$\tilde{\varphi}_k\to\tilde{\varphi}_k+\tilde{\gamma}_k{}^{(j)}\delta(\partial)^{k-j}V_j, \quad j=0,1,2,\ldots, \qquad (3.2)$$

then $G_k{}^{(\mathrm{I})}$ will remain unchanged provided

$$\left[\beta_k\frac{(k-j+1)(k+j+2)}{2(k+1)^2}\tilde{\gamma}_{k+1}{}^{(j)}\Box^2+\alpha_k\tilde{\gamma}_k{}^{(j)}+\tilde{\gamma}_{k-1}{}^{(j)}\right]\times\delta(\partial)^{k-j}V_j=0. \qquad (3.3)$$

Assuming $(\Box^2+\bar{m}_j{}^2)V_j=0$, we have

$$\left[-\tfrac{1}{2}\bar{m}_j{}^2\frac{(k-j+1)(k+j+2)}{(k+1)^2}\beta_k\tilde{\gamma}_{k+1}{}^{(j)}+\alpha_k\tilde{\gamma}_k{}^{(j)}+\tilde{\gamma}_{k-1}{}^{(j)}\right]=0. \qquad (3.4)$$

Using these G_k's written in terms of the fields φ, we can construct the Lagrangian density:

$$\mathcal{L}^{(\mathrm{I})}=\tfrac{1}{2}\sum_{k=0}^{\infty}\eta_kG_k{}^{(\mathrm{I})2}(\tilde{\varphi},\partial_\mu\tilde{\varphi}), \qquad (3.5)$$

from which follow the equations of motion (see Sec. IV for details)

$$\eta_{k+1}\partial\cdot G_{k+1}{}^{(\mathrm{I})}+\eta_{k-1}\beta_{k-1}\delta\partial G_{k-1}{}^{(\mathrm{I})}=\eta_k\alpha_kG_k{}^{(\mathrm{I})}, \qquad (3.6)$$

which are manifestly invariant under the gauge transformations (3.2). These transformations, however, are really only restricted gauge transformations, because the gauge functions V_j are required to satisfy the Klein-Gordon equation. We can obtain a fully gauge-invariant theory by setting all the $\beta_k=0$. We then have

$$G_k{}^{(\mathrm{II})}=\delta\partial\tilde{\varphi}_{k-1}+\alpha_k\tilde{\varphi}_k, \quad k=1,2,\ldots \qquad (3.7)$$

which are invariant under

$$\tilde{\varphi}_k\to\tilde{\varphi}_k+\tilde{\gamma}_k\delta(\partial)^k\Lambda \qquad (3.8)$$

for any scalar function Λ, provided we choose

$$\alpha_k=-\tilde{\gamma}_{k-1}/\tilde{\gamma}_k. \qquad (3.9)$$

However, rather than a series of gauge transformations, there is now only one that leaves $G_k{}^{(\mathrm{II})}$ invariant. Defining the Lagrangian density

$$\mathcal{L}^{(\mathrm{II})}=\tfrac{1}{2}\sum_{k=1}^{\infty}\eta_kG_k{}^{(\mathrm{II})2}(\tilde{\varphi},\partial_\mu\tilde{\varphi}), \qquad (3.10)$$

we have the equations of motion

$$\partial\cdot G_{k+1}{}^{(\mathrm{II})}=\zeta_k\alpha_kG_k{}^{(\mathrm{II})}, \qquad (3.11)$$

where $\zeta_k=\eta_k/\eta_{k+1}$.

We now wish to establish a precise connection between the equations of motion (2.1) and (3.6), and to express (3.11) in difference-equation form.

We turn first to the expression (3.1) for $G_k^{(\mathrm{I})}$, and let $\tilde{\varphi}_k = A_k \chi_k$, where A_k is to be chosen. We have

$$G_k^{(\mathrm{I})} = \beta_k A_{k+1}\left(\frac{A_{k-1}}{\beta_k A_{k+1}} \delta\partial\chi_{k-1} + \frac{\alpha_k A_k}{\beta_k A_{k+1}}\chi_k + \partial\cdot\chi_{k+1}\right). \quad (3.12)$$

Choosing A_k to satisfy

$$A_{k-1}/\beta_k A_{k+1} = -\tfrac{1}{2}k^2 \quad (3.13)$$

and defining diagonal matrices A, $\tilde{A}$, and β by

$$A_{kk'} = A_k \delta_{kk'}, \quad \tilde{A}_{kk'} = A_{k+1}\delta_{kk'}, \quad \beta_{kk'} = \beta_k \delta_{kk'}, \quad (3.14)$$

we see that (3.12) can be rewritten

$$G^{(\mathrm{I})} = \beta \tilde{A}(\partial\cdot L - M_1) A^{-1}\tilde{\varphi}, \quad (3.15)$$

with

$$(M_1)_{kk'} = -(\alpha_k A_k / \beta_k A_{k+1})\delta_{kk'}. \quad (3.16)$$

Now in (3.6) we let $\eta_k G_k^{(\mathrm{I})} = B_k \mathcal{K}_k$, with B_k to be appropriately chosen. Then (3.6) becomes

$$\partial\cdot\mathcal{K}_{k+1} + \frac{\beta_{k-1}B_{k-1}}{B_{k+1}}\delta\partial\mathcal{K}_{k-1} - \frac{\alpha_k B_k}{B_{k+1}}\mathcal{K}_k = 0. \quad (3.17)$$

We let B_k satisfy

$$(\beta_{k-1}B_{k-1}/B_{k+1}) = -\tfrac{1}{2}k^2 \quad (3.18)$$

and (3.17) is therefore

$$[\partial\cdot L - M_2] B^{-1}\eta G^{(\mathrm{I})} = 0, \quad (3.19)$$

with

$$B_{kk'} = B_k \delta_{kk'}, \quad \eta_{kk'} = \eta_k \delta_{kk'},$$

and

$$(M_2)_{kk'} = (\alpha_k B_k / B_{k+1})\delta_{kk'}. \quad (3.20)$$

Putting (3.19) and (3.15) together, we write

$$(\partial\cdot L - M_2) D (\partial\cdot L - M_1) A^{-1}\tilde{\varphi} = 0, \quad (3.21)$$

where D is the diagonal matrix

$$D = B^{-1}\eta\beta\tilde{A}. \quad (3.22)$$

Thus (3.21) looks like the product of two first-order wave operators of the form (2.1). At first sight, it might appear that two different mass spectra are allowed by (3.21); we can either choose $A^{-1}\tilde{\varphi}$ to satisfy

$$(\partial\cdot L - M_1) A^{-1}\tilde{\varphi} = 0$$

or we can pick $D[\partial\cdot L - M_1]A^{-1}\tilde{\varphi} \equiv \psi$ to be a nonzero solution of

$$(\partial\cdot L - M_2)\psi = 0.$$

However, we show that the two procedures are equivalent, in the sense that the same mass spectrum is obtained either way. This is most easily done by remembering from, e.g., Eq. (3.8) of Ref. 4, that the allowed masses are determined by the combination $m_k m_{k-1}$. We calculate from (3.16), (3.13), (3.20), and (3.18)

$$(M_1)_k (M_1)_{k-1} = -(\alpha_k \alpha_{k-1}/\beta_{k-1})\tfrac{1}{2}k^2 = (M_2)_k (M_2)_{k-1}. \quad (3.23)$$

We add a final remark about (3.21). The presence of $(A^{-1}\tilde{\varphi})$ suggests that for the purposes of applying the gauge transformation (3.2), we identify $A^{-1}\tilde{\varphi}$ with the φ of Sec. II. Here A is given by (3.13). In fact, making the replacement $\tilde{\gamma}_k^{(j)} = A_k \gamma_k^{(j)}$ in (3.4), and putting $m_k = -\alpha_k A_k / \beta_k A_{k+1}$, we recover (2.30).

The same general procedure can be applied to theory II, where now we make the correspondences

$$\partial^\mu \tilde{\varphi}_{\mu\mu_1\cdots\mu_k} \leftrightarrow (\partial\cdot L^{(+)})_{kj\sigma,k'j'\sigma'}\tilde{\varphi}_{k'j'\sigma'}$$

and

$$-\tfrac{1}{2}k^2 \delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k}\partial_{\nu_1}\tilde{\varphi}_{\nu_2\cdots\nu_k} \leftrightarrow (\partial\cdot L^{(-)})_{kj\sigma,k'j'\sigma'}\tilde{\varphi}_{k'j'\sigma'}, \quad (3.24)$$

as deduced in Sec. II.

In (3.11) we let

$$G_k^{(\mathrm{II})} = \hat{B}_k \mathcal{K}_k \quad (3.25)$$

and choose

$$\hat{B}_{k+1} = g^{-1}(\alpha_k \zeta_k)\hat{B}_k, \quad (3.26)$$

so that (3.11) becomes

$$[\partial\cdot L^{(+)} - g]\mathcal{K} = 0. \quad (3.27)$$

Here g is a constant inserted for dimensional reasons. The mass spectrum will of course be independent of the value of g. In (3.7), we let $\tilde{\varphi} = \hat{A}\chi$, and choose $\hat{A}$ so that

$$(g\hat{A}_{k-1}/\alpha_k \hat{A}_k) = -\tfrac{1}{2}k^2, \quad (3.28)$$

which yields

$$G^{(\mathrm{II})} = (\alpha\hat{A}/g)[\partial\cdot L^{(-)} + g]\chi. \quad (3.29)$$

Combining (3.27) and (3.29) gives

$$[\partial\cdot L^{(+)} - g](\hat{B}^{-1}\alpha\hat{A}/g)[\partial\cdot L^{(-)} + g]\hat{A}^{-1}\tilde{\varphi} = 0. \quad (3.30)$$

The advantage of having proceeded in this particular way is that we have shown, in (3.30), that the properties of the system are governed by a single function of k, namely,

$$\alpha_k \hat{A}_k / g\hat{B}_k \equiv d_k, \quad (3.31)$$

with $\hat{A}$ and $\hat{B}$ determined by (3.28) and (3.26). To reduce (3.30) further, we use the standard technique of considering $\chi_k(p)$, the Fourier transform of $(A^{-1}\tilde{\varphi})_k$, and using the manifest covariance of (3.30) to go to the rest frame. Then $\partial\cdot L^{(\pm)} \to -ip_0 L_0^{(\pm)}$, and we have the explicit expressions

$$(L_0^{(+)})_{kk'} = -a_{k+1}^{(j)}\delta_{k'k+1}, \quad (L_0^{(-)})_{kk'} = a_k^{(j)}\delta_{kk'+1}, \quad (3.32)$$

with

$$a_k^{(j)} = \tfrac{1}{2}[(k-j)(k+j+1)]^{1/2}.$$

Using this in (3.30) gives

$$[p_0^2 a_{k+1}^{(j)2} d_{k+1} - g^2 d_k]\chi_k^{(j)} + i p_0 g a_{k+1}^{j} d_{k+1}\chi_{k+1}^{(j)} + i p_0 g a_k^{(j)} d_k \chi_{k-1}^{(j)} = 0. \quad (3.33)$$

We make the following changes in (3.33): We let

$$D_k = \left(\frac{g p_0}{i}\right)^k \left[\prod_{l=j+1}^{k} a_l^{(j)}\right]\chi_k^{(j)}, \quad (\text{and } D_0 = \chi_0) \quad (3.34)$$

and define

$$e_k = g^2 d_k / d_{k+1}, \quad (3.35)$$

to obtain the equation

$$D_{k+1} = [p_0^2 a_{k+1}^{(j)2} - e_k] D_k + p_0^2 e_k a_k^{(j)2} D_{k-1}, \quad (3.36)$$

which is the analog for theory II of the difference equation (2.32) in the theory determined by (2.1). In a succeeding paper we shall study the solutions of (3.36) and elaborate on the physical properties of the system it describes.

IV. QUANTIZATION OF FIELD THEORY

A. Derivation of Equations of Motion

Let us now consider the field-theoretic aspects of the foregoing theories. The most elegant formalism for discussing both the equations of motion and the canonical commutation relations is Schwinger's quantum action principle.[8] By varying the action with fixed end points, one obtains the Euler-Lagrange equations of motion. Quantization follows by identifying the generator $\bar{G}$, associated with the boundary variation, with the infinitesimal generator of unitary transformations on a quantum-mechanical system.

We modify the action (3.5) to yield not only the equation of motion for $G^{\nu_1\cdots\nu_k}$ but also its definition in terms of φ_k. That is,

$$W = \int d^4x \sum_{k=0}^{\infty} [\eta_k G^{\nu_1\cdots\nu_k}(\delta_{\nu_1\cdots\nu_k}{}^{\mu_1\cdots\mu_k}\partial_{\mu_k}\varphi_{\mu_1\cdots\mu_{k-1}} + \alpha_k \varphi_{\nu_1\cdots\nu_k} + \beta_k \partial^\nu \varphi_{\nu\nu_1\cdots\nu_k}) - \tfrac{1}{2}\eta_k G^{\nu_1\cdots\nu_k} G_{\nu_1\cdots\nu_k}]. \quad (4.1)$$

(In this section for simplicity we replace $\tilde{\varphi}$ everywhere by φ.) Leaving the end points fixed, we obtain from $\delta W = 0$ the usual Euler-Lagrange equations:

$$\frac{\delta \mathfrak{L}}{\delta G^{\nu_1\cdots\nu_k}} = 0, \quad \frac{\delta \mathfrak{L}}{\partial \varphi_{\nu_1\cdots\nu_k}} = \partial_\mu \frac{\delta \mathfrak{L}}{\delta \partial_\mu \varphi_{\nu_1\cdots\nu_k}}. \quad (4.2)$$

We thus find for the equations of motion

$$\begin{aligned} G_{\nu_1\cdots\nu_k} &= \delta_{\nu_1\cdots\nu_k}{}^{\mu_1\cdots\mu_k}\partial_{\mu_k}\varphi_{\mu_1\cdots\mu_{k-1}} + \alpha_k \varphi_{\nu_1\cdots\nu_k} + \beta_k \partial^\nu \varphi_{\nu\nu_1\cdots\nu_k}, \\ \eta_k \alpha_k G_{\nu_1\cdots\nu_k} &= \eta_{k-1}\beta_{k-1}\delta_{\nu_1\cdots\nu_k}{}^{\mu_1\cdots\mu_k}\partial_{\mu_k} G_{\mu_1\cdots\mu_{k-1}} + \beta_k \partial^\nu G_{\nu\nu_1\cdots\nu_k}. \end{aligned} \quad (4.3)$$

Displaying the first three cases, we have

$$\begin{aligned} G &= \alpha_0\varphi + \beta_0 \partial_\nu \varphi^\nu, \\ G^\mu &= \partial^\mu \varphi + \alpha_1 \varphi^\mu + \beta_1 \partial_\nu \varphi^{\mu\nu}, \\ G^{\mu\nu} &= \tfrac{1}{2}(\partial^\mu \varphi^\nu + \partial^\nu \varphi^\mu - \tfrac{1}{2} g^{\mu\nu}\partial^\lambda \varphi_\lambda) + \alpha_2 \varphi^{\mu\nu} + \beta_2 \partial_\lambda \varphi^{\mu\nu\lambda}, \end{aligned} \quad (4.4a)$$

$$\begin{aligned} &\eta_1 \partial_\nu G^\nu = \alpha_0 \eta_0 G, \\ &\eta_2 \partial_\mu G^{\mu\nu} + \eta_0 \beta_0 \partial^\nu G = \alpha_1 \eta_1 G^\nu, \\ &\eta_3 \partial_\lambda G^{\mu\nu\lambda} + \tfrac{1}{2}\eta_1\beta_1(\partial^\mu G^\nu + \partial^\nu G^\mu - \tfrac{1}{2} g^{\mu\nu}\partial_\lambda G^\lambda) = \alpha_2 \eta_2 G^{\mu\nu}. \end{aligned} \quad (4.4b)$$

As stated before, the Lagrangian is invariant under the following set of gauge transformations of the second kind:

$$\varphi_{\nu_1\cdots\nu_k} \to \varphi_{\nu_1\cdots\nu_k} + \gamma_k \delta_{\nu_1\cdots\nu_k}{}^{\mu_1\cdots\mu_k}\partial_{\mu_1}\cdots\partial_{\mu_k}\Lambda(x), \quad (4.5)$$

provided that $\Lambda(x)$ is constrained to satisfy

$$(\Box^2 + \bar{m}^2)\Lambda(x) = 0$$

and (4.6)

$$\beta_k \gamma_{k+1}[(k+2)/2(k+1)]\bar{m}^2 = \gamma_{k-1} + \alpha_k \gamma_k.$$

If we consider a particular $\mathfrak{L}^k$ with

$$\mathfrak{L}^k \equiv \tfrac{1}{2}\eta_k G^{\nu_1\cdots\nu_k}(\varphi) G_{\nu_1\cdots\nu_k}(\varphi), \quad (4.7)$$

then it is invariant under a gauge transformation on the φ's of the first kind. For example,

$$\mathfrak{L}^0 = \tfrac{1}{2}\eta_0(\alpha_0\varphi + \beta_0\partial_\nu\varphi^\nu)^2$$

is invariant under

$$\varphi^\nu \to \varphi^\nu + \Lambda^\nu, \quad (4.8)$$

with Λ^ν a constant. However, the entire Lagrangian is not invariant under (4.8) since G^μ contains φ^μ. But we can use the Gell-Mann–Lévy equation[9] to determine the current that generates this gauge transformation, and its divergence.

Specifically, we consider the single transformation

$$\varphi^{\nu_1\cdots\nu_k} \to \varphi^{\nu_1\cdots\nu_k} + \Lambda^{\nu_1\cdots\nu_k}, \quad (4.9)$$

where $\Lambda^{\nu_1\cdots\nu_k}$ is an arbitrary symmetric and traceless tensor. The Gell-Mann–Lévy equations are

$$\frac{\delta \mathfrak{L}}{\delta \partial_\mu \Lambda^{\nu_1\cdots\nu_k}(x)} = j_{\nu_1\cdots\nu_k}{}^\mu(x), \quad (4.10a)$$

where $\int j_{\nu_1\cdots\nu_k}{}^0(x) d^3x$ is the generator of the above transformation and

$$\frac{\delta \mathfrak{L}}{\delta \Lambda^{\nu_1\cdots\nu_k}} = \partial_\mu j^\mu{}_{\nu_1\cdots\nu_k}. \quad (4.10b)$$

These equations yield

$$j^{\mu\nu_1\cdots\nu_k}(x) = \eta_{k-1}\beta_{k-1}\delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k} g^{\mu\mu_k} G^{\mu_1\cdots\mu_{k-1}} + \eta_{k+1} G^{\mu\nu_1\cdots\nu_k} \quad (4.11)$$

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[9] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

and

$$\partial_\mu j^{\mu\nu_1\cdots\nu_k}=\eta_k\alpha_k G^{\nu_1\cdots\nu_k}. \tag{4.12}$$

These generalized partially conserved current equations are identical to the equations of motion for the $G^{\nu_1\cdots\nu_k}$ [Eq. (4.3)]. Thus we see that the G's play the role of the gauge currents, while the φ's are the gauge fields.

B. Canonical Quantization

If we allow the boundary to be varied, we obtain from the Schwinger action principle

$$\delta W=\sum_k \int d\sigma_{\mu_k}\eta_k G^{\nu_1\cdots\nu_k}\delta_{\nu_1\cdots\nu_k}{}^{\mu_1\cdots\mu_k}\delta\varphi_{\mu_1\cdots\mu_{k-1}}\Big|_{\sigma_1}^{\sigma_2} +\sum_k \int d\sigma^\nu \eta_k\beta_k G^{\nu_1\cdots\nu_k}\delta\varphi_{\nu_1\cdots\nu_k\nu}\Big|_{\sigma_1}^{\sigma_2}$$
$$=\int d^3x\{\sum_k(\eta_k G^{\nu_1\cdots\nu_{k-1}0}+\eta_{k-2}\beta_{k-2}\delta_{\mu_1\cdots\mu_{k-1}}{}^{\nu_1\cdots\nu_{k-1}} \times g^{\mu_{k-1}0}G^{\mu_1\cdots\mu_{k-2}})\delta\varphi_{\nu_1\cdots\nu_{k-1}}\}\Big|_{t_1}^{t_2}. \tag{4.13}$$

Thus if all the $\varphi_{\nu_1\cdots\nu_{k-1}}$ are independent (aside from being symmetric and traceless) we have the identification that

$$\bar{G}[\varphi_{\nu_1\cdots\nu_{k-1}}]=\int d^3x(\eta_k G^{\nu_1\cdots\nu_{k-1}0}+\eta_{k-2}\beta_{k-2} \times\delta_{\mu_1\cdots\mu_{k-1}}{}^{\nu_1\cdots\nu_{k-1}}g^{\mu_{k-1}0}G^{\mu_1\cdots\mu_{k-2}})\delta\varphi_{\nu_1\cdots\nu_{k-1}}, \tag{4.14}$$

where $\bar{G}$ is the generator of the displacement $\delta\varphi$. This implies the commutation relations

$$[\varphi^{\nu_1\cdots\nu_{k-1}}(x),\ \eta_k G_{\mu_1\cdots\mu_{k-1}0}(x')+\eta_{k-2}\beta_{k-2}\delta_{\mu_1\cdots\mu_{k-1}}{}^{\lambda_1\cdots\lambda_{k-1}} \times g_{\lambda_{k-1}0}G_{\lambda_1\cdots\lambda_{k-2}}(x')]_{x_0=x_0'} =i\delta_{\mu_1\cdots\mu_{k-1}}{}^{\nu_1\cdots\nu_{k-1}}\delta^3(x-x') \tag{4.15}$$

$$[\varphi^N(x),\varphi^M(x')]_{x_0=x_0'}=0;\qquad [\pi^N(x),\pi^M(x')]_{x_0=x_0'}=0,$$

where φ^M is any of the canonical fields $\varphi^{\nu_1\cdots\nu_M}$ and π^N is any of the canonical momenta,

$$\pi^{N-1}=\eta_N G_{\mu_1\cdots\mu_{N-1}0}+\eta_{N-2}\beta_{N-2}\delta_{\mu_1\cdots\mu_{N-1}}{}^{\lambda_1\cdots\lambda_{N-1}} \times g_{\lambda_{N-1}0}G_{\lambda_1\cdots\lambda_{N-2}}. \tag{4.16}$$

We can derive these commutation rules again from the gauge invariance properties of the theory. The charge

$$\int j^{0\nu_1\cdots\nu_k}(x)d^3x\equiv Q^{\nu_1\cdots\nu_k} \tag{4.17}$$

is the generator of the constant gauge transformations. That is,

$$\exp(iQ^{\nu_1\cdots\nu_k}\Lambda_{\nu_1\cdots\nu_k})\varphi_{\mu_1\cdots\mu_k}(x')\exp(-iQ^{\nu_1\cdots\nu_k}\Lambda_{\nu_1\cdots\nu_k}) =\varphi_{\mu_1\cdots\mu_k}(x')+\Lambda_{\mu_1\cdots\mu_k}. \tag{4.18}$$

This tells us that

$$i\int[j^{0\nu_1\cdots\nu_k}(x),\varphi_{\mu_1\cdots\mu_k}(x')]d^3x\,\Lambda_{\nu_1\cdots\nu_k}=\Lambda_{\mu_1\cdots\mu_k} \tag{4.19}$$

or

$$i[j^{0\nu_1\cdots\nu_k}(x),\varphi_{\mu_1\cdots\mu_k}(x')]_{x_0=x_0'} =\delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k}\delta^3(x-x'), \tag{4.20}$$

which is equivalent to (4.15) and (4.11).

C. Gauge-Invariant Theory

The previous theory becomes invariant under gauge transformations of the second kind if we set $\beta_k=0$ and leave out the term with $\mathcal{L}^{(0)}$. The equations of motion (4.4) now become equations of motion and equations of constraint:

$$\eta_1\partial_\nu G^\nu=0,\qquad G^\mu=\partial^\mu\varphi+\alpha_1\varphi^\mu$$
$$\eta_2\partial_\mu G^{\mu\nu}=\alpha_1\eta_1 G^\nu,\quad G^{\mu\nu}=\tfrac{1}{2}(\partial^\mu\varphi^\nu+\partial^\nu\varphi^\mu-\tfrac{1}{2}g^{\mu\nu}\partial^\lambda\varphi_\lambda)+\alpha_2\varphi^{\mu\nu}. \tag{4.21}$$

At first glance it looks like G^0, $G^{0\nu}$, ..., φ, φ^ν, ... all obey equations of motion, whereas G^k, G^{kl}, ... obey equations of constraint. However, the Lagrangian is invariant under the gauge transformations

$$\varphi_{\mu_1\cdots\mu_k}\to\varphi_{\mu_1\cdots\mu_k}+\gamma_k\delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k}\partial_{\nu_1}\cdots\partial_{\nu_k}\Lambda(x), \tag{4.22a}$$

provided

$$\gamma_{k-1}+\alpha_k\gamma_k=0. \tag{4.22b}$$

In particular we can choose a gauge where

$$\sum_{i=0}^{3}\partial_i\varphi^i=0\quad\text{(radiation gauge)}. \tag{4.23}$$

For if $\partial_i\varphi^i\neq 0$, we can make a gauge transformation

$$\varphi_i'=\varphi_i+\gamma_1\partial_i\Lambda. \tag{4.24}$$

Choosing

$$\gamma_1\nabla^2\Lambda=-\partial_i\varphi^i \tag{4.25}$$

or

$$\Lambda(x)=\frac{1}{4\pi\gamma_1}\int\frac{\partial_i\varphi^i d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

ensures that φ_i' is divergenceless. Thus $\varphi_i=\varphi_i^T$. Consider G^{0k}. We can always break a vector into its transverse and longitudinal components:

$$G^{0k}=G^{0kT}+G^{0kL}. \tag{4.26}$$

The longitudinal piece can be written in the form

$$G^{0kL}=\nabla^k g(x),$$

with

$$\partial_k G^{0k}=\nabla^2 g(x).$$

Thus

$$G^{0kL}(x)=-\frac{1}{4\pi}\nabla^k\int\frac{\partial_l G^{0l}(x')d^3x'}{|\mathbf{x}'-\mathbf{x}|}. \tag{4.27}$$

But

$$G^{0k}=\tfrac{1}{2}(\partial^0\varphi^k+\partial^k\varphi^0)+\alpha_2\varphi^{0k}. \tag{4.28}$$

If we now take the 3-divergence in the radiation gauge, we find

$$\partial_k G^{0k}=\tfrac{1}{2}\nabla^2\varphi^0+\alpha_2\partial_k\varphi^{0k}. \tag{4.29}$$

This is an equation of constraint hidden in the equations of motion which suggests that the independent variables are $\varphi_k{}^T$ and $G_{0k}{}^T$. This alters the canonical commutation rules from the previous case. In this gauge-invariant theory the term obtained by varying the boundary is

$$\delta W=\int d^3x\sum_k \eta_{k+1}G^{\nu_1\cdots\nu_k 0}\delta\varphi_{\nu_1\cdots\nu_k}\Big|_{t_1}^{t_2}. \tag{4.30}$$

Considering the $k=1$ term, we have

$$\int d^3x\;\eta_2 G^{\nu 0}\delta\varphi_\nu=\int d^3x\;\eta_2 G^{00}\delta\varphi_0+\int d^3x\;\eta_2(G^{k0T}+G^{k0L})\delta\varphi^{kT}. \tag{4.31}$$

Now

$$\int d^3x\;\eta_2 G^{k0L}\delta\varphi^{kT}=0, \tag{4.32}$$

since $G^{k0L}=\nabla^k g$. Thus only G^{0kT} enters as a generator of the φ variation. This leads to the equal-time commutation relation

$$[\varphi^k(x),\eta_2 G_{0i}{}^T(x')]=i\delta_i{}^k\delta^3(x-x')^T \equiv i(\delta_i{}^k-\partial^k\partial_i/\nabla^2)\delta^3(x-x'), \tag{4.33}$$

which is consistent with $\partial_k\varphi^k=0$.

To find the commutation relations satisfied by $G_{0k}{}^L$, one must use the fact that

$$G^{0kL}(x)=-\frac{1}{4\pi}\nabla^k\int\frac{\partial_l G^{0l}(x')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \doteq \frac{-1}{8\pi}\nabla^k\int\frac{\nabla^2\varphi^0(x')d^3x'}{|\mathbf{x}-\mathbf{x}'|} \tag{4.34}$$

(where $\doteq$ here means that we have dropped the $\alpha_2\partial_k\varphi^{0k}$ term in (4.29) since it will commute with G^{00}) and the known commutation relations of φ^0

$$[\varphi^0(x),\eta_2 G^{00}(x')]_{x_0=x_0'}=i\delta^3(x-x') \tag{4.35}$$

to obtain

$$[G^{0k}(x),G^{00}(y)]_{x_0=y_0}=\frac{i}{8\pi\eta_2}\nabla_x{}^k\int\frac{\nabla^2\delta^3(y-x')}{|\mathbf{x}-\mathbf{x}'|}d^3x' =\frac{i}{2\eta_2}\nabla^k\delta^3(x-y). \tag{4.36}$$

This can be rewritten as

$$[j_0{}^k(x),j_0{}^0(y)]_{x_0=y_0}=\tfrac{1}{2}i\eta_2\nabla^k\delta^3(x-y), \tag{4.37}$$

where, following (4.11) with $\beta_k=0$,

$$j^{\mu\nu_1\cdots\nu_k}(x)=\eta_{k+1}G^{\mu\nu_1\cdots\nu_k} \tag{4.11$'$}$$

and the $\int j^{0k}d^3x$ are the generators of the constant gauge transformations. Thus the "currents" in this theory seem to have c-number Schwinger terms. Since $\partial_\mu j^\mu(x)=0$ with $j^\mu=\eta_1 G^\mu$, it is tempting to identify j^μ as the field containing the ρ^0 and all the 1^- satellites of the ρ^0 trajectory (and eventually with the entire isotopic triplet of ρ mesons when we introduce non-Abelian gauge transformations). Using the constraint quation $G^k=\partial^k\varphi+\alpha_1\varphi^k$ and

$$[\varphi(x),\eta_1 G^0(x')]=i\delta^3(x-x'),$$

we indeed find

$$[G_k(x),G_0(x')]_{x_0=x_0'}=i\frac{1}{\eta_1}\partial_k\delta^3(x-x') \tag{4.38a}$$

or

$$[j_k(x),j_0(x')]_{x_0=x_0'}=i\eta_1\partial_k\delta^3(x-x'). \tag{4.38b}$$

These commutation relations are reminiscent of the algebra of fields, the main difference being that G^μ (j^μ) describes the ρ meson and all the 1^- satellites of the ρ trajectory. It thus seems that this gauge-invariant theory might be the appropriate vehicle for approximating the vector current.

D. Comparison with First-Order Lagrangian

It is of some interest to compare the field theory we have generated from an extension of the Stueckelberg formalism with the theory we would have obtained had we postulated the Lagrangian density

$$\mathfrak{L}(x)=\sum_{kj\sigma;k'j'\sigma'}\bar\psi_{kj\sigma}(x)[\partial_\mu L^\mu-M]_{kj\sigma,k'j'\sigma'}\times\psi_{k'j'\sigma'}(x). \tag{4.39}$$

Here $\bar\psi=\psi^\dagger Q$, where

$$Q_{kj\sigma,k'j'\sigma'}=(-1)^j\delta_{kk'}\delta_{jj'}\delta_{\sigma\sigma'} \tag{4.40}$$

is a matrix inserted to ensure the Lorentz invariance of $\mathfrak{L}(x)$. Notice that Q has the property

$$QL^\mu=-(L^\mu)^\dagger Q, \tag{4.41}$$

as it must if $\mathfrak{L}(x)$ is to be Hermitian. By treating $\psi(x)$ and $\bar\psi(x)$ as independent fields, we obtain, in the standard way, the equation of motion

$$(\partial_\mu L^\mu-M)\psi=0 \tag{2.1$'$}$$

and the canonical commutation rules

$$[(\bar\psi L^0)_{kj\sigma}(x),\psi_{k'j'\sigma'}(x')]_{x_0=x_0'}=-i\delta^3(x-x')\delta_{kk'}\delta_{jj'}\delta_{\sigma\sigma'}. \tag{4.42}$$

We have, of course, chosen $\mathfrak{L}(x)$ to give equation (2.1′). What interests us here is the commutation relation (4.42). We multiply (4.42) by $\tilde T_{kj\sigma}T_{k'j'\sigma'}$ and sum on ($j\sigma$) and ($j'\sigma'$). Employing essentially the same techniques used in Sec. II to convert (2.1) into (2.24), we arrive at the following commutation rule:

$$[\psi_{\mu_1\cdots\mu_k0}(x) -\tfrac{1}{2}k^2\delta_{\mu_1\cdots\mu_k}{}^{\rho_1\cdots\rho_{k-1}0}\psi_{\rho_1\cdots\rho_{k-1}}(x),\psi^{\nu_1\cdots\nu_{k'}}(x')]_{x_0=x_0'} = i\delta^3(x-x')\delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k}[(k!)^2/2^k]\delta_{kk'}. \quad (4.43)$$

Here we have dropped the distinction between $\psi_{\mu_1\cdots\mu_k}$ and $\psi_{\mu_1\cdots\mu_k}{}^\dagger$ in order to compare (4.15) with (4.42). By looking at (2.1′) and (3.19), we conclude that the simplest correspondence we can make is

$$\psi(x) \leftrightarrow B^{-1}\eta G^{(\mathrm{I})}(x). \quad (4.44)$$

If this is true, then (4.43) becomes

$$[\eta_{k+1}G_{\mu_1\cdots\mu_k0}{}^{(\mathrm{I})}(x)+\eta_{k-1}\beta_{k-1}\delta_{\mu_1\cdots\mu_k}{}^{\rho_1\cdots\rho_{k-1}0} \times G_{\rho_1\cdots\rho_{k-1}}{}^{(\mathrm{I})}(x),G^{\nu_1\cdots\nu_{k'}(\mathrm{I})}]_{x_0=x_0'} = i\delta^3(x-x')\delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k}\frac{(k!)^2}{2^k}\frac{B_kB_{k+1}}{\eta_k}\delta_{kk'}, \quad (4.45\text{a})$$

where we have used (3.18); i.e.,

$$[\pi_{\mu_1\cdots\mu_k}(x),G^{\nu_1\cdots\nu_{k'}}(x')]_{x_0=x_0'} = i\delta^3(x-x')\delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k}\frac{(k!)^2}{2^k}\frac{B_kB_{k+1}}{\eta_k}\delta_{kk'}. \quad (4.45\text{b})$$

One might wonder if this is possibly a consequence of the commutation rules already derived, viz.,

$$[\pi_{\mu_1\cdots\mu_k}(x),\varphi^{\nu_1\cdots\nu_{k'}}(x')]_{x_0=x_0'} = i\delta^3(x-x')\delta_{\mu_1\cdots\mu_k}{}^{\nu_1\cdots\nu_k}\delta_{kk'}, \quad (4.15'\text{a})$$

$$[\pi_{\mu_1\cdots\mu_k}(x),\pi^{\nu_1\cdots\nu_{k'}}(x')]_{x_0=x_0'}=0, \quad (4.15'\text{b})$$

$$[\varphi_{\mu_1\cdots\mu_k}(x),\varphi^{\nu_1\cdots\nu_{k'}}(x')]_{x_0=x_0'}=0. \quad (4.15'\text{c})$$

Although (4.45b) has certain similarities with (4.15′a), it turns out that one can derive a contradiction between (4.45b) and (4.15′b). The momentum $\pi(x)$ conjugate to the scalar field $\varphi(x)$ is just $G^0(x)$, and therefore from (4.15′b)

$$[\pi_{\mu_1\cdots\mu_k}(x),G^0(x')]_{x_0=x_0'}=0. \quad (4.46)$$

Choosing $k=1$ and $\mu_1=0$, we have

$$[\pi_0(x),G^0(x')]_{x_0=x_0'}=0.$$

But (4.45b) tells us that

$$[\pi_0(x),G^0(x')]_{x_0=x_0'}=i\delta^3(x-x')(\tfrac{1}{2})B_1B_2/\eta_1.$$

Since the B's are assumed not to vanish, we have arrived at a contradiction. This means that, assuming the consistency of Eqs. (4.15′a)–(4.15′c), the Lagrangian (3.5) and (4.39) generate canonically inequivalent theories, even though the fields obey the same equations of motion. The fact that functions obeying the same equations of motion can have different canonical commutation relations is known even on the classical level in terms of Poisson brackets.[10]

[10] H. Primakoff (private communication).

E. Relativistic Invariance

Schwinger has shown[8] that the necessary and sufficient conditions for a system to be local and have Lorentz invariance is that the equal-time commutation relation

$$(1/i)[T^{00}(x),T^{00}(x')] = -\partial_k\delta^3(x-x')[T^{0k}(x)+T^{0k}(x')] \quad (4.47)$$

is obeyed with just a gradient of a δ function. He finds spin 0, $\frac{1}{2}$, and 1 have this property, whereas higher spins have extra terms that do not contribute to the commutation relations between $P_{\mu\nu}$ and $J_{\mu\nu}$ where $P^\mu=\int T^{0\mu}d^3x$, etc. We find that $T^{00(1)}(x)$ does obey this condition, and the $T^{00(2)}$ commutation relation contains extra terms which are not local (in the Schwinger sense, since they are third derivatives of δ functions).

The stress tensor $T^{\mu\nu}$ is defined as the measure of the response of the system to the space-time displacement $x^\mu\to x^\mu+\delta x^\mu$.

$$\delta W=\int d^4x\,\partial_\mu\delta x_\nu T^{\mu\nu}(x). \quad (4.48)$$

We find

$$T^{\mu\nu}=\sum_k\left(\frac{\delta\mathfrak{L}}{\delta\partial_\mu\varphi_{\mu_1\cdots\mu_k}}\partial^\nu\varphi_{\mu_1\cdots\mu_k}\right)-g^{\mu\nu}\mathfrak{L}. \quad (4.49)$$

Using

$$\mathfrak{L}=\tfrac{1}{2}\sum_k\eta_kG^{\nu_1\cdots\nu_k}G_{\nu_1\cdots\nu_k}=\sum_k\mathfrak{L}^k$$

we have for the spin-1 piece of the stress tensor

$$T^{\mu\nu(1)}\equiv\frac{\delta\mathfrak{L}}{\delta\partial_\mu\varphi}\partial^\nu\varphi-g^{\mu\nu}\mathfrak{L}^{(1)}. \quad (4.50)$$

Thus for the gauge-invariant theory,

$$T^{00(1)}=\eta_1[G^0\partial_0\varphi-\tfrac{1}{2}(G^0)^2+\tfrac{1}{2}(G^k)^2] =\eta_1[\tfrac{1}{2}(G^0)^2+\tfrac{1}{2}(G^k)^2-\alpha_1\varphi_0G^0], \quad (4.51)$$

wher

$$T^{k0(1)}=\eta_1G^k\partial^0\varphi. \quad (4.52)$$

Usin{

$$[G_k(x),G_0(x')]=(i/\eta_1)\partial_k\delta^3(x-x') \quad (4.53)$$

and

$$\partial_k\delta^3(x-x')[f(x)g(x')+f(x')g(x)] =\partial_k\delta^3(x-x')[f(x)g(x)+f(x')g(x')], \quad (4.54)$$

we find

$$(1/i)[T^{00(1)}(x),T^{00(1)}(x')]_{x_0=x_0'} =-\partial_k\delta^3(x-x')[T^{0k(1)}(x)+T^{0k(1)}(x')]. \quad (4.55)$$

Thus the spin-one piece is local.

We can rewrite $T^{00(1)}$ in terms of the independent variables. In the radiation gauge $\varphi^k=\varphi^{kT}$, and thus

$$T^{00(1)}=\eta_1\tfrac{1}{2}[(G^0)^2+(\partial_k\varphi)^2+\alpha_1{}^2(\varphi^{kT})^2] -\eta_1\alpha_1\varphi_0G^0+(\text{3-dimensional divergence}). \quad (4.56)$$

In calculating $[T^{00(2)}(x),T^{00(2)}(x')]$, one needs the commutator

$$[G^{kl}(x),G_{0i}{}^{T}(y)]=\frac{i}{\eta_2}\Big[\tfrac{1}{2}\partial^l\Big(\delta_i{}^k-\frac{\partial^k\partial_i}{\nabla^2}\Big)\delta^3(x-y)+\tfrac{1}{2}\partial^k\Big(\delta_i{}^l-\frac{\partial^l\partial_i}{\nabla^2}\Big)\delta^3(x-y)\Big]. \quad (4.57)$$

Thus at the spin-2 level we obtain nonlocal terms, which are third derivatives of δ functions. We have not attempted to verify the commutation relations to all orders owing to the increasing difficulty of the task.

To obtain the stress tensor in the theory with $\beta\neq 0$, we first have to reexpress $\partial_0\varphi^k$ in terms of φ^M, $\partial_k\varphi^M$, and π^M. The problem is soluble in principle, but we have not found a simple expression for the solution. Briefly we shall sketch how to make the inversion.

Suppose we want to find $\partial_0\varphi=f(\pi^M,\varphi^M,\partial_k\varphi^M)$. One needs to look at π, π^{00}, π^{0000}, etc.:

$$\pi=\eta_1G^0=\eta_1(\partial_0\varphi+\beta_1\partial_0\varphi^{00}+\alpha_1\varphi^0+\beta_1\partial_k\varphi^{0k})$$
$$\pi^{00}=\eta_3G^{000}+\tfrac{3}{4}\eta_1\beta_1G^0=\eta_3[\tfrac{2}{3}\partial_0\varphi^{00}+\beta_3\partial_0\varphi^{0000}+\beta_3\partial_k\varphi^{000k}-\tfrac{1}{3}\partial_k\varphi^{0k}+\alpha_3\varphi^{000}]+\tfrac{3}{4}\beta_1\pi. \quad (4.58)$$

In general we get the infinite set of equations

$$\partial_0\varphi+\beta_1\partial_0\varphi^{00}=f_0(\pi,\partial_k\varphi^M,\varphi^M),$$
$$c_2\partial_0\varphi^{00}+\beta_3\partial_0\varphi^{0000}=f_2(\pi,\pi^{00},\partial_k\varphi^M,\varphi^M),$$
$$\vdots$$
$$c_N\partial_0\varphi^{0\cdots 0\ (N\text{ indices})}+\beta_{N+1}\partial_0\varphi^{0\cdots 0\ (N+2\text{ indices})}=f_N(\pi,\ldots,\pi^N,\ldots). \quad (4.59)$$

So we can solve in principle for all the $\partial_0\varphi^N$. Since we have not found any simple way of expressing these equations we have as yet not been able to verify that T^{00} is positive definite, and that $(1/i)[T^{00}(x),T^{00}(x')]=-\partial_k\delta^3(x-x')[T^{0k}(x)+T^{0k}(x')]$+terms not contributing to the $[J^{\mu\nu},J^{\lambda\sigma}]$ commutation rules.

It has come to our attention[11] that it has never been shown even for a massive spin-2 field that, when one turns on interactions, T^{00} is positive definite or satisfies simple commutation relations, because of the overwhelming amount of work involved. Thus it seems a sufficient achievement to have been able to verify the stress-tensor relations for the spin-1 part of the gauge-invariant theory.

V. SPECULATIONS AND CONCLUSIONS

In the previous sections of this paper, we developed a Lagrangian formalism for fields obeying an infinite-component wave equation. The fields in the theory, say, φ^μ or G^μ, had the interesting property of having an infinite number of particles associated with them. The theory which was gauge invariant ($\beta=0$) contained a conserved gauge current G_μ, whereas the theory with the β's contained a partially conserved current $\partial_\mu G^\mu=cG$. We would like to point out in this section that the conserved G^μ suggests itself as a generalized ρ field, whereas the partially conserved G^μ suggests itself as a generalized axial-vector field, the divergence being the generalized π field with an infinite number of pseudoscalar mesons. Thus we have the skeletal beginnings of a connection with current algebra. In this section we will explore this connection and also briefly discuss how to introduce interactions so as to reproduce Veneziano-type scattering amplitudes. In the gauge-invariant theory, $\partial_\mu G^\mu=0$ and

$$[G^0(x),G^k(y)]=-(i/\eta_1)\partial_k\delta^3(x-y),\qquad G^\mu\equiv\alpha_1\varphi^\mu+\partial^\mu\varphi. \quad (5.1)$$

These two equations are identical to those obeyed by the usual free ρ^0 fields, except here G^μ contains an infinite number of vector mesons, with the masses determined by the eigenvalues of the infinite-component wave equation. The G^μ suggests itself as the generalized ρ field (and therefore the "free" gauge-invariant Lagrangian as the Lagrangian for the ρ trajectory). We can therefore use field-current-identity methods[12] to obtain nucleon form factors which will be infinite-pole dominated at the first order of perturbation theory. The current then will also have finite c-number Schwinger terms. In what follows we will introduce a naive field-current identity (which is not gauge invariant) which is suggestive of models which we eventually hope to produce.

We introduce the following free Maxwell Lagrangian with an interaction with $G^\mu=(\alpha_1\varphi^\mu+\partial^\mu\varphi)$:

$$\mathcal{L}^{(\mathrm{em})}=-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-\gamma_1eA^\mu(\alpha_1\varphi_\mu+\partial_\mu\varphi). \quad (5.2)$$

This leads to the following equations:

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu,\qquad \partial_\mu F^{\mu\nu}=\gamma_1e(\alpha_1\varphi^\nu+\partial^\nu\varphi), \quad (5.3)$$

which implies

$$\partial_\mu G^\mu=0=\partial_\mu J^{(\mathrm{em})\mu}$$
$$J^{(\mathrm{em})\mu}=\gamma_1G^\mu.$$

This interaction is not gauge invariant with respect to the electromagnetic field, since under

$$A^\mu(x)\to A^\mu(x)+\partial^\mu\Lambda(x),\qquad \delta\mathcal{L}=-\gamma_1G^\mu\partial_\mu\Lambda\neq 0. \quad (5.4)$$

[11] Tung-Mow Yan (private communication).

[12] T. D. Lee, in *Theory and Phenomenology in Particle Physics*, edited by A. Zichichi (Academic, New York, 1969), Part A.

[13] See J. J. Sakurai [*Currents and Mesons* (Chicago U. P., Chicago, 1969)] for a discussion of this point.

This causes problems[13] with the maintenance of the zero mass of the photon under renormalization. Ignoring this problem for the moment, (we have not yet found an appropriate gauge-invariant interaction), let us see what such an identity implies.

We can introduce an interaction with nucleons via the replacement

$$\partial_\mu \to \partial_\mu - igG_\mu \qquad (5.5)$$

in the free-nucleon Lagrangian. That is,

$$\mathcal{L}^{(\text{nucleon})} = i\bar{\psi}\gamma_\mu(\partial^\mu - igG^\mu)\psi - m\bar{\psi}\psi. \qquad (5.6)$$

The total Lagrangian then becomes

$$\mathcal{L} = \mathcal{L}^{(\rho)} + \mathcal{L}^{(\text{em})} + \mathcal{L}^{(\text{nucleon})}. \qquad (5.7)$$

We find that the interactions maintain the commutation relation Eq. (5.1). Thus the electromagnetic current has finite c-number Schwinger terms

$$[J_{(\text{em})}{}^0(x), J_{(\text{em})}{}^k(y)] = -(i\gamma_1/\eta_1)\partial_k\delta^3(x-y). \qquad (5.8)$$

The full Lagrangian [Eq. (5.7)] leads to a Born term for the nucleon form factor as illustrated in Fig. 1. As is well known, a sum of an infinite number of spin-1 poles leads to form factors which are hypergeometric functions, and can lead to any asymptotic power behavior for large negative q^2 that one desires. For example, in the Veneziano model, form factors have infinite pole dominance and turn out to be ratios of Γ functions, for example,[14]

$$\frac{\Gamma(1-\alpha_\rho(t))}{\Gamma(\frac{1}{2}\gamma-\alpha_\rho(t))}, \quad \alpha_\rho(t) = t + \tfrac{1}{2} \quad (\text{the } \rho \text{ trajectory})$$

which behaves like $t^{-\gamma/2+1}$.

It is tempting to consider the theory with $\beta\neq 0$ as the infinite-component field corresponding to the π-A_1 trajectory. One is then led to postulate a field-current identity for the axial-vector current

$$A^\mu = cG^\mu = c(\partial^\mu\varphi + \alpha_1\varphi^\mu + \beta_1\partial_\nu\varphi^{\mu\nu}). \qquad (5.9)$$

Consider, for example, the Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi + \gamma_2\bar{\psi}\gamma_\mu\gamma_5\psi(\partial^\mu\varphi + \alpha_1\varphi^\mu + \beta_1\partial_\nu\varphi^{\mu\nu}) + \mathcal{L}(\pi - A_1 \text{ system}). \qquad (5.10)$$

Now if we let $\psi \to e^{i\gamma_5\Lambda}\psi$, with Λ constant, $\mathcal{L}\to\mathcal{L}$ and thus $V_5{}^\mu \equiv \bar{\psi}\gamma_\mu\gamma_5\psi$ obeys $\partial_\mu V_5{}^\mu = 0$. With this interaction we find the equation for G^μ from

$$\delta\mathcal{L}/\delta\varphi = \partial_\mu(\delta\mathcal{L}/\delta\partial_\mu\varphi).$$

We get

$$\alpha_0\eta_0 G = \eta_1\partial_\nu G^\nu + \gamma_2\partial_\mu V_5{}^\mu = \eta_1\partial_\nu G^\nu. \qquad (5.11)$$

[14] F. Cooper, Phys. Rev. D 1, 1140 (1970).

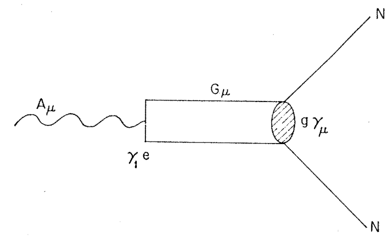

FIG. 1. Nucleon electromagnetic form factor which follows from the Lagrangian (5.7).

Thus with the identification of G as the field containing the pion and all the pion satellites, and with the full-current identity postulate, we find the generalized PCAC (partial conservation of axial-vector current) condition

$$\partial_\mu A^\mu = c\phi_\pi, \qquad (5.12)$$

where now both A^μ and ϕ_π contain an infinite number of particles.

We do not advocate taking the above Lagrangians too seriously. We are merely suggesting that once one has an ordinary Lagrangian formalism for describing infinite towers of particles, one can start using the techniques of Gell-Mann and Lévy and hopefully find the correct "σ" model. The new "σ" model (if found) will automatically be Regge behaved and hopefully will be free from the infinities inherent in ordinary field theories.

The problem of introducing general vertices in this type of theory has already been considered by Abarbanel.[15] He studied two possible trilinear couplings of fields

$$H_1(x) = \sum_{LMN} g_1(LMN)[A_{\mu_1\cdots\mu_L}(x)B_{\mu_{L+1}\cdots\mu_{L+N}} \times(\partial_{\mu_{L+N+1}}\cdots\partial_{\mu_M})C^{\mu_1\cdots\mu_M}] \qquad (5.13\text{a})$$

(where the derivatives will go on C, A, or B depending on whether $M>L+N$, $L>M+N$, or $N>L+M$, respectively). The second coupling we considered was

$$H_2(x) = \sum_{LMN} g_2(LMN)[A_{\mu_1\cdots\mu_N}(x)B_{\mu_{L+1}\cdots\mu_{L+N}}(x) \times(\overleftrightarrow{\partial}_{\mu_{L+N+1}}\cdots\overleftrightarrow{\partial}_{\mu_M})C^{\mu_1\cdots\mu_M}(x)], \qquad (5.13\text{b})$$

where

$$\overleftrightarrow{\partial} = \tfrac{1}{2}(\overrightarrow{\partial} - \overleftarrow{\partial}).$$

With trilinear couplings of this type he was able to reproduce the Veneziano amplitude (with or without odd daughters). However, Abarbanel dealt only with the interaction Lagrangian in the spirit of Weinberg.[16] Here we have instead a complete Lagrangian formulation of the problem.

[15] H. D. I. Abarbanel, Ann. Phys. (N. Y.) 57, 525 (1970).
[16] S. Weinberg, Phys. Rev. 133, B1318 (1964).

Most of the hard work remains to be done. Can one succeed in finding the equivalent "σ" model for these fields, so that one can once again try to solve the current-algebra problem? What are the correct interactions to use? Are the perturbation sums for the interacting case finite? We hope that this formalism will be a help in answering some of these questions.

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Some Constraints on Partial Waves of Helicity Amplitudes which Follow from Analyticity, Unitarity, and Crossing Symmetry*

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Some inequalities involving finite numbers of partial-wave helicity amplitudes are derived for the elastic scattering process $ab \to ab$ (arbitrary spins and masses). One set of inequalities involves algebraic combinations of t-channel ($a\bar{a} \to b\bar{b}$) partial-wave helicity amplitudes and holds for any value of t between 0 and $4\mu^2$ (μ is the lesser mass of the two particles involved in the scattering). A second set places restrictions on integrals over s-channel ($ab \to ab$) partial-wave helicity amplitudes. Finally, the above relations are applied to the particular case of π^0-nucleon elastic scattering, where inequalities among partial-wave helicity amplitudes are obtained.

I. INTRODUCTION

In recent years there has been a rebirth of interest in finding constraints on amplitudes which follow purely from analyticity and unitarity.[1] The present paper is an effort to bring together some[2] of these results with the work of Balachandran and co-workers[3] on crossing properties of partial waves and in particular the work by Balachandran, Modjtehedzadeh, and myself[3e] on constraints on partial waves of helicity amplitudes which follow from crossing symmetry. For the elastic scattering process $ab \to ab$ (s channel), inequalities are found for algebraic combinations of partial waves of t-channel helicity amplitudes for $0 \leqslant t < 4\mu^2$. This is done in Sec. II and the inequalities are given in Eqs. (6) and (10). In Sec. III inequalities for integrals over partial waves of s-channel helicity amplitudes are found for the same process [result given in Eqs. (13), (17), and (18)]. The main features of these constraints are as follows: (1) They follow from analyticity, unitarity, and crossing symmetry; (2) they involve only a finite number of partial waves in each inequality: and (3) they are constraints in the unphysical region.

In Sec. IV the results of Secs. II and III are applied to the special case of π^0-nucleon elastic scattering.

II. t-CHANNEL CONSTRAINTS

We begin by introducing various definitions and conventions. For the scattering process 1, 2 $\to$ 3, 4, we define

$$s \equiv (p_1+p_2)^2\,,$$
$$t \equiv (p_1-p_3)^2\,,$$
$$u \equiv (p_1-p_4)^2\,.$$

We will be considering the elastic scattering $ab \to ab$ where both particle a and b may have spin. Particles 1 and 3 are taken to be of type a with spin σ and mass m while particles 2 and 4 are of type b with spin σ' and mass μ. We also assume, without losing generality, that $m \geqslant \mu$. Physical processes in the various channels are

$$s \text{ channel}, \quad ab \to ab\,,$$
$$t \text{ channel}, \quad a\bar{a} \to b\bar{b}\,,$$
$$u \text{ channel}, \quad a\bar{b} \to a\bar{b}\,.$$

We express, for the case of elastic scattering, the Kibble

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[2] Similar approaches have been used for various processes: S. M. Roy, Phys. Rev. Letters **20**, 1016 (1968); A. P. Balachandran and Maurice L. Blackmon, Phys. Letters **31B**, 655 (1970); F. J. Yndurain, *ibid.* **31B**, 368 (1970); A. K. Common CERN Report No. 1145, 1970 (unpublished); A. K. Common and F. J. Yndurain, CERN Report No. 1185, 1970 (unpublished).

[3] (a) A. P. Balachandran and J. Nuyts, Phys. Rev. **172**, 1821 (1968); (b) A. P. Balachandran, W. J. Meggs, and P. Ramond, *ibid.* **175**, 1974 (1968); (c) A. P. Balachandran, W. J. Meggs, J. Nuyts, and P. Ramond, *ibid.* **176**, 1700 (1968); (d) **187**, 2080 (1969); (e) A. P. Balachandran, W. Case, and M. Modjtehedzadeh, Phys. Rev. D **1**, 1773 (1970); (f) A. P. Balachandran and M. Blackmon, Syracuse University Report No. 223, 1970 (unpublished).