New Quantum Electrodynamics for Vector Mesons*

R. H. Tucker and C. L. Hammer
Institute for Atomic Research and Department of Physics, Iowa State University, Ames, Iowa 50010
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A quantum electrodynamics for vector mesons with arbitrary magnetic dipole and electric quadrupole moments is constructed in which the vector meson is described by a six-component column matrix satisfying a single equation of motion with no auxiliary condition. To avoid an interaction Hamiltonian which has an infinite number of surface terms, the S matrix is derived directly from Green-function solutions of the equations of motion. In the reduction of the S matrix, terms appear which do not correspond to Feynman-type terms but which vanish if only regularized integrals are used. The Feynman rules are then identical in form to the rules for scalar electrodynamics. A distinct calculational advantage of this theory is that all components of the Fock-space operators are treated on an equal footing and create and destroy particles in definite energy and helicity states. Trace theorems for the covariantly defined spin-1 matrices are given to further facilitate calculations. The same techniques are applied to the electrodynamics of arbitrary-spin particles. A discussion of the renormalization is given: All of the theories are found to be nonrenormalizable.

I. INTRODUCTION

The discovery of vector mesons in high-energy particle collisions has prompted increasing interest in the theory of vector mesons interacting with the electromagnetic field. To date no satisfactory theory analogous to either spin-$\frac{1}{2}$ or scalar electrodynamics exists. This paper represents another attempt to construct a renormalizable spin-1 electrodynamics. Although the theory developed is nonrenormalizable, it has the following advantages over other formalism. (1) The vector meson is described by a six-component column matrix satisfying a single equation of motion with no auxiliary condition. Therefore all components of Fock-space operators are treated on an equal footing and create and destroy particles in definite energy and helicity states. (2) The perturbation expansion contains no surface terms. (3) Trace theorems for the covariantly defined spin-1 matrices are developed which are analogs to the trace theorems for spin-$\frac{1}{2}$ γ matrices. Thus all calculations will be as straightforward as in the spin-$\frac{1}{2}$ case.

The vector electrodynamics existing in the literature has been based primarily on the β formalism (Kemmer theory[1]) and the canonical formalism (Proca theory[2]). For the vector electrodynamics in the β formalism, the Feynman rules[3,4] are of the same form as spin-$\frac{1}{2}$ electrodynamics with 10×10 β matrices replacing the spin-$\frac{1}{2}$ matrices. Kinoshita and Nambu[5] have investigated the divergences to second order in the coupling constant for vector electrodynamics in the β formalism. They found that charge, wave function, and mass renormalization did not remove all of the ultraviolet divergences, implying that the theory was nonrenormalizable.

Lee and Yang[6] have done a thorough study of the S matrix and its renormalizability for vector electrodynamics in the canonical formalism. They found that the canonical formalism led to a nonrenormalizable theory, and for a nonzero electric quadrupole moment, the Feynman rules contained surface terms. Aronson[7] has pointed out that these surface terms can be removed by an infinite series of counter terms in the original Lagrangian. Lee and Yang circumvented the problem by introducing one extra term with dimensionless coefficient ξ in the equations of motion. In addition, this led to a renormalizable theory. However it also introduced scalar mesons with negative energy. This new problem had to be remedied by the use of an indefinite metric.

Sheinblatt and Arnowitt[8] have discussed the interaction of a quantized vector meson field in the canonical formalism with an external electromagnetic field. They also found that the theory could be renormalized if a small spacelike distance was introduced in the current operator to separate the points at which the field operators act.

Formulations for a free, massive spin-1 particle which use a six-component column matrix have been given by several authors.[9–12] The six components are sufficient to describe the three spin states of a massive spin-1 particle and antiparticle which puts the description on the same footing as the Dirac theory for spin-$\frac{1}{2}$ particles. The equation found by Hammer *et al.*[12] is of particular interest since this equation is manifestly covariant and requires no auxiliary conditions.

The wave equation for a massive spin-1 particle found by Hammer *et al.*[12] was originally found by Shay and Good.[13] Shay and Good studied the unquantized spin-1 field coupled to an external electromagnetic field and also added magnetic-dipole and electric-quadrupole

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terms with arbitrary coefficients to the equation of motion. They found that the six-component description of a spin-1 particle in an electromagnetic field differs from the other formalisms in the way in which the dipole and quadrupole terms appear. Hence it may be said that although the various free-particle descriptions are all equivalent, in the presence of an electromagnetic field each formalism leads to a different prediction about the behavior of vector mesons. Since at present there is no experimental evidence to single out the correct theory, the authors feel that a study of vector electrodynamics in the six-component formalism deserves attention.

In this paper we will primarily be interested in studying a quantized spin-1 field satisfying a manifestly covariant equation[12,13] when the radiation field is introduced through minimal electromagnetic coupling. The vector meson described in this manner has an intrinsic dipole moment of $e/4M$ and an intrinsic quadrupole moment of $-e/2M^2$. However the results are later extended to a vector meson with arbitrary dipole and quadrupole moments (Appendix B).

In Sec. II a free-field theory is constructed for particles with arbitrary spin satisfying manifestly covariant equations with no auxiliary conditions.[12] Since these equations all have essentially the same form, it is just as easy to discuss the arbitrary-spin case as the spin-1 case alone. The equations for the electrodynamics of arbitrary-spin particles are then introduced. In Sec. III we discuss the difficulties associated with the usual S-matrix treatment in the interaction representation. For spins greater than or equal to 1, it is seen that the interaction Hamiltonian in the interaction representation has an infinite series of noncovariant terms. In Sec. IV we specialize to spin 1 and derive the S matrix directly from Green-function solutions of the equations of motion thereby bypassing the difficulties associated with interaction Hamiltonians. In Sec. V we return to the equations for an electrodynamics of arbitrary spin and discuss their renormalizability. All equations are found to be nonrenormalizable except for the spin-$\frac{1}{2}$ case. In Sec. VI there is a general discussion of the results and suggestions for further study. Appendix A contains properties of the covariantly defined spin-1 matrices and trace theorems. In Appendix B the Feynman rules for a vector meson with arbitrary dipole and quadrupole moments are given.

II. FREE-FIELD THEORY AND EQUATIONS FOR QUANTUM ELECTRODYNAMICS OF ARBITRARY SPIN

This section begins with a discussion of quantized fields for arbitrary spin and (nonzero) mass. The field commutators, Fock-space operators, and the commutators between the fields and the Fock-space operators are given. Although equivalent discussions have been given by Weinberg[10] and by Nelson and Good,[14] the treatment given here is of interest since it is based entirely on conserved currents derived from the field equations. Thus a Lagrangian can be specifically avoided and a detailed discussion of plane-wave expansions in not found necessary.

The equations to be considered are those found by Hammer *et al.*[12] The integral- and half-integral-spin cases are treated separately. For integral spin, the equation used is Eq. (98) of Ref. 12:

$$[\gamma_{[\mu]}p_{[\mu]}+(p^2)^{s-1}(p^2+2M^2)]\psi^{(s)}(x)=0\,,\qquad s=1,2,3,\ldots \quad (2.1)$$

where $\gamma_{[\mu]}\equiv\gamma_{\mu_1\cdots\mu_{2s}}$ are generalized $2(2s+1)$-dimensional Dirac matrices and $p_{[\mu]}\equiv p_{\mu_1}\cdots p_{\mu_{2s}}$. The four-vector p_μ has components

$$p_\mu=(-i\partial/\partial x_i,\ -\partial/\partial t)\,.$$

The adjoint $\bar{\psi}^{(s)}(x)$ of $\psi^{(s)}(x)$ is defined by

$$\bar{\psi}^{(s)}(x)=[\gamma_{[4]}\psi^{(s)}(x)]^\dagger$$

and it can be shown that $\bar{\psi}^{(s)}(x)$ satisfies

$$p_{[\mu]}\bar{\psi}^{(s)}(x)\gamma_{[\mu]}+(p^2)^{s-1}(p^2+2M^2)\bar{\psi}^{(s)}(x)=0\,. \quad (2.2)$$

An important property of $\gamma_{[\mu]}$ is

$$\gamma_{[\mu]}p_{[\mu]}\gamma_{[\nu]}p_{[\nu]}=(p^2)^{2s}\,. \quad (2.3)$$

This means that solutions of Eq. (2.1) have the correct relativistic dispersion, since by Eq. (2.3)

$$\begin{aligned}[\gamma_{[\nu]}p_{[\nu]}-(p^2)^{s-1}(p^2+2M^2)]&\\ \times[\gamma_{[\mu]}p_{[\mu]}+(p^2)^{s-1}(p^2+2M^2)]\psi^{(s)}(x)&\\ =-4M^2(p^2)^{2s-2}(p^2+M^2)\psi^{(s)}(x)&=0\,. \quad (2.4)\end{aligned}$$

Thus the $2(2s+1)$ independent solutions of Eq. (2.1) are sufficient to describe all of the possible spin-s, free-particle and -antiparticle states with no auxiliary conditions needed.

A convenient method for quantizing fields satisfying equations like Eq. (2.1) has been given by Hammer and Tucker.[15] From Eq. (2.1) one constructs a conserved current, $j_\mu(\bar{\psi}_1^{(s)},\psi_2^{(s)})$,

$$\begin{aligned}p_\mu j_\mu(\bar{\psi}_1^{(s)}(x),\psi_2^{(s)}(x))&\\ =\bar{\psi}_1^{(s)}(x)[\gamma_{[\mu]}\vec{p}_{[\mu]}+(\vec{p}^2)^{s-1}(\vec{p}^2+2M^2)]\psi_2^{(s)}(x)&\\ -\bar{\psi}_1^{(s)}(x)[\gamma_{[\mu]}\overleftarrow{p}_{[\mu]}+(\overleftarrow{p}^2)^{s-1}(\overleftarrow{p}^2+2M^2)]\psi_2^{(s)}(x)& \quad (2.5)\end{aligned}$$

for any solutions ψ_1 and ψ_2 of Eq. (2.1). The arrow indicates the direction of differential operation. Fock-space creation and annihilation operators are then defined:

$$a_k^{(s)\dagger}(\mathbf{p})=\int d\sigma_\mu(x)\,j_\mu(\bar{\psi}^{(s)}(x),u_k^{(s)}(\mathbf{p},x)) \quad (2.6a)$$

and

$$a_k^{(s)}(\mathbf{p})=\int d\sigma_\mu(x)\,j_\mu(\bar{u}_k^{(s)}(\mathbf{p},x),\psi^{(s)}(x))\,, \quad (2.6b)$$

[14] T. J. Nelson and R. H. Good, Jr., Rev. Mod. Phys. **40**, 508 (1968).

[15] C. L. Hammer and R. H. Tucker, J. Math. Phys. (to be published).

respectively, where σ is a spacelike hypersurface with unit normal η, i.e., $d\sigma_\mu=\eta_\mu d\sigma$; $u_k^{(s)}(\mathbf{p},x)$ is any c-number solution of Eq. (2.1) with discrete eigenvalue k and continuous eigenvalue $\mathbf{p}$. The basic quantization postulate is

$$[a_k^{(s)}(\mathbf{p}),a_l^{(s)}(\mathbf{q})]_-=0\,, \tag{2.7a}$$

$$[a_k^{(s)}(\mathbf{p}),a_l^{(s)\dagger}(\mathbf{q})]_- =\int d\sigma_\mu(x)j_\mu(\bar{u}_k^{(s)}(\mathbf{p},x),u_l^{(s)}(\mathbf{q},x))\,. \tag{2.7b}$$

It can be shown[15] from Eqs. (2.5)–(2.7) that the commutation relations for the fields are

$$[\psi^{(s)}(x),\psi^{(s)}(y)]_-=0\,, \tag{2.8a}$$

$$[\psi^{(s)}(x),\bar{\psi}^{(s)}(y)]_-=iG^{(s)}(x-y)\,, \tag{2.8b}$$

where $G^{(s)}(x)$ is the homogeneous Green function for Eq. (2.1), i.e.,

$$\psi^{(s)}(x)=\int d\sigma_\mu(x)j_\mu(iG^{(s)}(x-y),\,\psi(y))\,. \tag{2.9}$$

This Green function can be constructed from advanced and retarded Green functions

$$G^{(s)}(x)=G_A^{(s)}(x)-G_R^{(s)}(x)\,, \tag{2.10}$$

where

$$G_{A,R}^{(s)}(x)=(-1/4M^2)[\gamma_{[\mu]}p_{[\mu]}-(p^2)^{s-1}(p^2+2M^2)] \times\int_{\mathcal{C}_A,\mathcal{C}_R}\frac{d^4k\,e^{ikx}}{(2\pi)^4(k^2)^{2s-2}(k^2+M^2)}\,. \tag{2.11}$$

The contours $\mathcal{C}_A$ and $\mathcal{C}_R$ run parallel to the real k_0 axis at a distance $-\epsilon$ and $+\epsilon$ from the real k_0 axis, respectively.[16] Clearly $G_A^{(s)}(x)$ $[G_R^{(s)}(x)]$ vanishes for $x_0>0$ $[x_0<0]$, and also

$$[\gamma_{[\mu]}p_{[\mu]}+(p^2)^{s-1}(p^2+2M^2)]G_{A,R}^{(s)}(x)=\delta(x)\,. \tag{2.12}$$

A second consequence of Eqs. (2.6) and (2.7) is the commutation relations between the field and the Fock-space operator $a_k(\mathbf{p})$.

$$[\psi^{(s)}(x),a_k^{(s)}(\mathbf{p})]_-=0\,, \tag{2.13a}$$

$$[\psi^{(s)}(x),a_k^{(s)\dagger}(\mathbf{p})]_-=u_k^{(s)}(\mathbf{p},x)\,. \tag{2.13b}$$

One can also show from Eqs. (2.6) and (2.7) that the Fock-space operators

$$O_l^{(s)}\equiv\int d\sigma_\mu(x)j_\mu(\bar{\psi}^{(s)}(x),\mathcal{O}_l(\partial)\psi^{(s)}(x))\,, \tag{2.14}$$

where $\mathcal{O}_l(\partial)$ is a c-number tensor operator of rank l, satisfy

$$[\psi^{(s)}(x),O_l^{(s)}]_-=\mathcal{O}_l(\partial)\psi^{(s)}(x)\,. \tag{2.15}$$

Thus the operator

$$P_\mu^{(s)}\equiv\int d\sigma_\mu(x)j_\mu(\bar{\psi}^{(s)}(x),p_\mu\psi^{(s)}(x)) \tag{2.16}$$

satisfies the Heisenberg equation of motion

$$[\psi^{(s)}(x),P_\mu^{(s)}]_-=p_\mu\psi^{(s)}(x) \tag{2.17}$$

and is identified as the four-momentum operator in Fock space.

For half-integral spin, the equation to be considered is Eq. (66) of Ref. 12:

$$[m\gamma_{[\mu]}p_{[\mu]}+i(p^2)^{s+1/2}]\psi^{(s)}(x)=0\,,\quad s=\tfrac{1}{2},\tfrac{3}{2},\,\ldots\,. \tag{2.18}$$

The conserved current is defined here by

$$-\partial_\mu j_\mu[\bar{\psi}_1^{(s)},\psi_2^{(s)}] =\bar{\psi}_1^{(s)}(x)[m\gamma_{[\mu]}\overrightarrow{p}_{[\mu]}+i(\overrightarrow{p}^2)^{s+1/2}]\psi_2^{(s)}(x) +\bar{\psi}_1^{(s)}(x)[m\gamma_{[\mu]}\overleftarrow{p}_{[\mu]}-i(\overleftarrow{p}^2)^{s+1/2}]\psi_2^{(s)}(x)\,. \tag{2.19}$$

The quantization procedure parallels that for the integral-spin case. The advanced and retarded Green functions for Eq. (2.18) are

$$G_{A,R}^{(s)}(x)=[m\gamma_{[\mu]}p_{[\mu]}-i(p^2)^{s+1/2}] \times\int_{\mathcal{C}_A,\mathcal{C}_R}\frac{d^4k}{(2\pi)^4}\frac{e^{ikx}}{(k^2)^{2s}(k^2+M^2)}\,, \tag{2.20}$$

so that the anticommutation relations for the fields are

$$[\psi^{(s)}(x),\psi^{(s)}(y)]_+=0\,, \tag{2.21a}$$

$$[\psi^{(s)}(x),\bar{\psi}^{(s)}(y)]_+=G_A^{(s)}(x-y) -G_R^{(s)}(x-y)\equiv G^{(s)}(x-y)\,. \tag{2.21b}$$

To consider the interaction of particles with arbitrary spin with the electromagnetic field, the substitution $p_\mu\to p_\mu-eA_\mu$ is made in Eqs. (2.1) and (2.18):

$$\{\gamma_{[\mu]}(p_{[\mu]}-eA_{[\mu]})+(p-eA)^{2(s-1)} \times[(p-eA)^2+M^2]\}\psi^{(s)}(x)=0\,,\quad s=1,\,2,\,\ldots \tag{2.22a}$$

$$[M\gamma_{[\mu]}(p_{[\mu]}-eA_{[\mu]})+i(p-eA)^{2s+1}]\psi^{(s)}(x)=0\,, \quad s=\tfrac{1}{2},\tfrac{3}{2},\,\ldots\,. \tag{2.22b}$$

The four-potential $A_\mu(x)$ satisfies

$$\partial_\nu\partial_\nu A_\mu(x)=-eJ_\mu^{(s)}(x) \tag{2.23}$$

The conserved current $J_\mu^{(s)}(x)$ is found by replacing $p_\mu\psi^{(s)}(x)$ by $(p_\mu-eA_\mu)\psi^{(s)}(x)$ and $p_\mu\bar{\psi}^{(s)}(x)$ by $(p_\mu+eA_\mu)\bar{\psi}^{(s)}(x)$ in the conserved current $j_\mu(\bar{\psi}^{(s)}(x),\psi^{(s)}(x))$ defined by Eqs. (2.5) and (2.19) for the integral- and half-integral-spin cases, respectively.

In this work we will consider Eqs. (2.22) and (2.23) in some detail for $s=1$. These equations are

$$[(\gamma_{\mu\nu}+\delta_{\mu\nu})(p_\mu-eA_\mu) \times(p_\nu-eA_\nu)+2M^2]\psi^{(s=1)}(x)=0\,, \tag{2.24}$$

$$\partial_\nu\partial_\nu A_\mu(x)=-e[\bar{\psi}^{(s=1)}(x)(p_\nu-eA_\nu)(\gamma_{\mu\nu}+\delta_{\mu\nu})\psi^{(s=1)}(x) -(p_\nu+eA_\nu)\bar{\psi}^{(s=1)}(x)(\gamma_{\mu\nu}+\delta_{\mu\nu})\psi^{(s=1)}(x)]\,. \tag{2.25}$$

[16] S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1962), Chap. 13, Sec. d.

In what follows the superscript $s=1$ will be dropped since it is only the spin-1 field that will be considered. Also it will be convenient to define

$$\Gamma_{\mu\nu}=\gamma_{\mu\nu}+\delta_{\mu\nu}. \tag{2.26}$$

It has been shown by Shay and Good[13] that a vector meson described by Eq. (2.24) has an intrinsic magnetic dipole moment of $e/4M$ and an intrinsic electric quadrupole moment of $-e/2M^2$. To study the interaction of vector mesons with arbitrary dipole and quadrupole moments, Shay and Good have shown that Eqs. (2.24) and (2.25) are to be replaced by

$$\left[(p_\mu-eA_\mu)(p_\nu-eA_\nu)\Gamma_{\mu\nu}+2M^2+\frac{e\lambda}{12}\gamma_{5,\alpha\beta}F_{\alpha\beta}+\frac{eq}{6M^2}\gamma_{6,\alpha\beta,\mu\nu}\frac{\partial F_{\alpha\beta}}{\partial x_\mu}(p_\nu-eA_\nu)\right]\psi(x)=0, \tag{2.27}$$

$$\partial_\nu\partial_\nu A_\mu(x)=-e\left[\bar\psi(x)(p_\nu-eA_\nu)\Gamma_{\mu\nu}\psi(x)-(p_\nu+eA_\nu)\bar\psi(x)\Gamma_{\mu\nu}\psi(x)+\frac{eq}{6M^2}\bar\psi(x)\gamma_{6,\alpha\beta,\nu\mu}\frac{\partial F_{\alpha\beta}}{\partial x_\nu}\psi(x)\right] \tag{2.28}$$

(see Appendix A for a discussion of the 6×6 matrices $\gamma_{5,\alpha\beta}$ and $\gamma_{6,\alpha\beta,\mu\nu}$). The tensor $F_{\alpha\beta}$ is the electromagnetic field tensor. A massive spin-1 particle described by Eqs. (2.27) and (2.28) has[13] a magnetic dipole moment of $e(1+\lambda)/4M$ and an electric quadrupole moment of $(-1+\lambda+2q)/2M^2$.

For simplicity we will take $\lambda=q=0$ in the derivations; at their conclusion it will be clear how to generalize the Feynman rules to include arbitrary λ and q.

For the interacting fields $\psi(x)$ and $A_\mu(x)$, the Fock-space operator P_μ will be defined from the conserved current $J_\mu{}^{(s)}(x)$ by

$$P_\mu=\frac{1}{2}\int d\sigma_\nu(x)[J_\mu(\bar\psi(x),p_\mu\psi(x))-J_\mu(p_\mu\bar\psi(x),\psi(x))]+\frac{1}{2}\int d\sigma_\nu K_\nu(A_\alpha,p_\mu A_\alpha), \tag{2.29}$$

where $K_\nu(A_{1\alpha},A_{2\alpha})\equiv A_{1\alpha}\partial_\nu A_{2\alpha}-(\partial_\nu A_{1\alpha})A_{2\alpha}$ is a conserved current for the photon field. The Hamiltonian H is just $-iP_4$ and can be separated into a free-particle part and an interacting part. If the hypersurface σ is taken to be flat, then

$$H=\int d\mathbf{x}\,[2M^2\bar\psi(x)\psi(x)-p_i\bar\psi(x)\Gamma_{ij}p_j\psi(x)+p_4\bar\psi(x)\Gamma_{44}p_4\psi(x)]-\frac{1}{2}\int d\mathbf{x}\,K_4(A_\alpha,p_4A_\alpha)+\int d\mathbf{x}\,\{e^2A_\mu A_\nu\bar\psi(x)\Gamma_{\mu\nu}\psi(x)-eA_\mu[\bar\psi(x)\Gamma_{\mu i}p_i\psi(x)-p_i\bar\psi(x)\Gamma_{\mu i}\psi(x)]\}. \tag{2.30}$$

III. DIFFICULTIES WITH INTERACTION REPRESENTATION

The interest here will be in scattering processes involving vector mesons and photons as described by Eqs. (2.24) and (2.25). A convenient starting point for deriving elements of the S matrix is to use Dyson's equation[17]

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dx_1\cdots\int dx_n\mathcal{P}(\mathcal{H}_I'(x_1)\cdots\mathcal{H}_I'(x_n)). \tag{3.1}$$

Here $\mathcal{P}$ is Dyson's chronological operator and $\mathcal{H}_I'(x)$ is the interaction Hamiltonian density in the interaction representation.

The field equations (2.24) and (2.25), and hence the Hamiltonian coming directly from the field equations, Eq. (2.30), are in the Heisenberg representation. It is not trivial to transform the Hamiltonian from the Heisenberg representation to the interaction representation because of the time derivatives of the massive spin-1 field in the Hamiltonian. However, a program for deriving the interaction Hamiltonian in a free-field representation given the equations of motion in the Heisenberg representation has been found by Takahashi and Umezawa.[18] Some of the difficulty associated with Hamiltonians containing time derivatives of the fields is removed by introducing a special set of operators which are related to free-field operators by a unitary transformation. These special operators are not necessarily equal to the Heisenberg operators.

Both Eqs. (2.24) and (2.25) are of the form

$$D(\partial)\varphi(x)=j(x), \tag{3.2a}$$

where a free-field operator $\varphi_{\text{in}}(x)$ satisfies

$$D(\partial)\varphi_{\text{in}}(x)=0. \tag{3.2b}$$

A Green-function solution of Eq. (3.2a) is

$$\varphi(x)=\varphi_{\text{in}}(x)-\int dx'd(\partial)\Delta_R(x-x',m^2)j(x'), \tag{3.3}$$

where $\Delta_R(x-x';m^2)$ is the retarded Green function from Klein-Gordon theory[16] and the differential operator $d(\partial)$ is defined by

$$D(\partial)d(\partial)=d(\partial)D(\partial)=\partial_\mu\partial_\mu-m^2. \tag{3.4}$$

It is convenient to define an operator

$$\tilde\varphi(x,\sigma)=\varphi_{\text{in}}(x)-\int_{-\infty}^{\sigma}dx'd(\partial)\Delta(x-x';m^2)j(x'), \tag{3.5}$$

where the notation (x,σ) means that the point x is not

[17] F. J. Dyson, Phys. Rev. **75**, 486 (1949).

[18] Y. Takahashi and H. Umezawa, Progr. Theoret. Phys. (Kyoto) **9**, 14 (1953).

on the spacelike hypersurface σ. The function

$$D(\partial)\,\tilde{\varphi}(x,\sigma)=0$$

for every σ such that x is not on σ. One can then define a unitary transformation $U(\sigma)$ which connects $\tilde{\varphi}(x,\sigma)$ with the asymptotic field $\varphi_{\rm in}(x)$:

$$\tilde{\varphi}(x,\sigma)=U^{\dagger}(\sigma)\varphi_{\rm in}(x)U(\sigma)\,. \tag{3.6}$$

The central equation for finding the interaction Hamiltonian as a functional of the $\varphi_{\rm in}(x)$ according to Takahashi and Umezawa[18] is

$$[\varphi_{\rm in}(x),\mathcal{H}_I'(x',\eta')]_- = -id(\partial)\Delta(x-x')U(\sigma)j(x')U^{\dagger}(\sigma)\,. \tag{3.7}$$

Therefore $\mathcal{H}_I'(x,\eta)$ depends on a knowledge of $j(x)$, the current in the Heisenberg representation, which contains fields and their derivatives. Thus it is still necessary to know how to transform time derivatives of $\varphi(x)$ out of the Heisenberg representation. This is facilitated by taking the point x on σ in Eq. (3.5) and from it subtracting Eq. (3.3):

$$\varphi(x)=\tilde{\varphi}(x|\sigma) - \int dx'[d(\partial),\,\Theta(x-x')]_-\Delta(x-x')j(x')\,, \tag{3.8a}$$

where $(x|\sigma)$ means that the point x is definitely on σ and $\Theta(x)$ is the step function

$$\Theta(x)=1\,,\quad x_0\geq 0$$
$$=0\,,\quad x_0<0\,.$$

Similarily one can derive

$$\partial_\mu\varphi(x)=[\partial_\mu\tilde{\varphi}(x,\sigma)]_{x|\sigma} - \int dx'[\partial_\mu d(\partial),\,\Theta(x-x')]_-\Delta(x-x')j(x')\,, \tag{3.8b}$$

where $[\partial_\mu\tilde{\varphi}(x,\sigma)]_{x|\sigma}$ means that the derivative is to be taken before evaluating x on σ. The motivation for introducing the auxiliary field $\tilde{\varphi}(x,\sigma)$ is now clear, since from Eq. (3.6)

$$U(\sigma)[\partial_\mu\tilde{\varphi}(x,\sigma)]_{x|\sigma}U^{\dagger}(\sigma)=\partial_\mu\varphi_{\rm in}(x)$$

and thus some of the difficulty associated with Hamiltonians containing time derivatives of the fields has been removed. The procedure for using Eqs. (3.7) and (3.8) to find $\mathcal{H}_I'(x,\eta)$ is as follows.

(a) Use Eqs. (3.8) to find $\varphi(x)$ and its derivatives in terms of $\tilde{\varphi}(x|\sigma)$ and its derivatives.

(b) Use (a) to find the current $j(x)$ as a function of $\tilde{\varphi}(x|\sigma)$ and its derivatives.

(c) Use Eq. (3.7) and the free-field commutation relations to solve for $\mathcal{H}_I'(x,\eta)$.

Let us now use this method to find $\mathcal{H}_I'(x,\eta)$ for Eqs. (2.24) and (2.25). Since for the photon field

$$d(\partial)=1\,,$$
$$\Delta(x-x')=\Delta(x-x';\,m^2=0)\,,$$

Eq. (3.8a) yields

$$A_\mu(x)=\tilde{A}_\mu(x|\sigma)\,. \tag{3.9}$$

On the other hand, for the massive spin-1 field,

$$d(\partial)=(1/4M^2)(\gamma_{\mu\nu}p_\mu p_\nu-p^2-2M^2)\,,$$

so that

$$\psi(x)=\tilde{\psi}(x|\sigma)+(\eta_\alpha\eta_\beta/4M^2)(\gamma_{\alpha\beta}-\delta_{\alpha\beta}) \times[e^2A_\mu A_\nu\Gamma_{\mu\nu}-e\Gamma_{\mu\nu}(p_\mu A_\nu+A_\mu p_\nu)]\psi(x)\,. \tag{3.10}$$

It is clear from the stucture of Eq. (3.10) that $\psi(x)$ can be solved for $\tilde{\psi}(x|\sigma)$ and its derivatives only as an infinite series in the coupling constant. This in turn means that $\mathcal{H}_I'(x,\eta)$ will be given as an infinite series in the coupling constant, that is, an infinite number of surface terms.

It is known that for cases where the interaction Hamiltonian has a finite number of surface terms, surface terms appear in the reduction of the S matrix which exactly cancel those surface terms in $\mathcal{H}_I'(x,\eta)$.[18] The net result is that the rules for writing down the S matrix to any order in the coupling constant are identical to those given by Feynman's space-time approach to scattering processes.[3] For the case of fields coupled to the electromagnetic field by minimal coupling, this suggests that regardless of what $\mathcal{H}_I'(x,\eta)$ might be, the S matrix is always given by

$$S=\sum_{n=0}^{\infty}\frac{(ie)^n}{n!}\int dx_1\cdots\int dx_n \times\mathcal{P}^*(A_\mu(x_1)J_\mu(x_1)\cdots A_\nu(x_n)J_\nu(x_n))\,, \tag{3.11}$$

where J_μ is the conserved particle current, and the symbol $\mathcal{P}^*$ is defined by Takahashi and Umezawa[18] to mean that all surface terms appearing in the reduction of the S matrix are to be discarded. It should be emphasized that no formal proof that this is true exists, but rather that it seems to be reasonable. Indeed Weinberg[10] accepts this point of view in his theory of interacting fields. The authors therefore believe that if, after reducing the S matrix, Eq. (3.11) applies, then there ought to be a straightforward way of arriving at Feynman rules without having to bother with a staggering number of unphysical terms.

IV. REDUCTION OF S MATRIX TO FEYNMAN DIAGRAMS

A perturbative expansion for the S matrix can be derived directly from the Heisenberg representation by means of equations given by Yang and Feldman[4] without having to consider an interaction Hamiltonian.

The problem of surface terms is circumvented by working with Green-function solutions in the Heisenberg representation, so that manifest covariance is always maintained. It is found that the second-order scattering processes are consistent with the construction of the S matrix by the $\mathcal{P}^*$ method, Eq. (3.11). The proof that to all orders Feynman rules can be used to write down the S matrix relies heavily on the similarity between the equations of motion for scalar and vector electrodynamics.

The Yang-Feldman equations[4] for the vector electrodynamics described by Eqs. (2.24) and (2.25) are

$$\psi(x)=\psi_{\text{in,out}}(x)+\int dy\{eA_\mu(y)\{G_{R,A}(x-y)\Gamma_{\mu\nu}p_\nu(y)\psi(y)-[p_\nu(y)G_{R,A}(x-y)]\psi(y)\}-e^2G_{R,A}(x-y)\Gamma_{\alpha\beta}A_\alpha(y)A_\beta(y)\psi(y)\}\,,\quad(4.1\text{a})$$

$$A_\mu(x)=A_\mu{}^{\text{in,out}}(x)+\int dy\, D^{R,A}{}_{\mu\nu}(x-y)[e\bar\psi(y)(p_\alpha-eA_\alpha)\Gamma_{\nu\alpha}\psi(y)-e(p_\alpha+eA_\alpha)\bar\psi(y)\Gamma_{\nu\alpha}\psi(y)]\,,\quad(4.1\text{b})$$

where $G_{R,A}(x-y)$ is the retarded (advanced) Green function for the spin-1 field [see Eq. (2.11)] and $D^{R,A}{}_{\mu\nu}$ is a retarded (advanced) Green function for the photon field

$$D^{R,A}{}_{\mu\nu}(x)=\delta_{\mu\nu}\Delta_{R,A}(x;0)\,.\quad(4.2)$$

The operators $\psi_{\text{in,out}}(x)$ and $A_\mu{}^{\text{in,out}}(x)$ are free-field operators and represent the asymptotic behavior of the Heisenberg fields. The commutation relations satisfied by the free-field operators are

$$[\psi_{\text{in,out}}(x),\bar\psi_{\text{in,out}}(y)]_-=iG(x-y)\,,\quad(4.3\text{a})$$

$$[A_\mu{}^{\text{in,out}}(x),A_\nu{}^{\text{in,out}}(y)]_-=iD_{\mu\nu}(x-y)=i\delta_{\mu\nu}\Delta(x-y;0)\,.\quad(4.3\text{b})$$

All other commutators are zero. Furthermore the operators $\psi_{\text{in}}(x)$ and $\psi_{\text{out}}(x)$, $A_\mu{}^{\text{in}}(x)$ and $A_\mu{}^{\text{out}}(x)$ are related by a unitary transformation S:

$$\psi_{\text{out}}=S^\dagger\psi_{\text{in}}S\,,\quad(4.4\text{a})$$

$$A_\mu{}^{\text{out}}=S^\dagger A_\mu{}^{\text{in}}S\,.\quad(4.4\text{b})$$

Yang and Feldman[4] have shown that the transformation S is identical to Dyson's S matrix, Eq. (3.1), for cases in which the interaction Hamiltonian in a free-field representation has a finite number of terms. This leads them to the following conclusion. If one has two sets of field equations which have the same form, then their Green-function solutions have the same form, and hence the S matrix to any order in the coupling constant for the sets must have the same form. In other words, if it is known that certain Feynman rules apply for one set, then it follows that Feynman rules of the same form apply to the other set. This argument circumvents the necessity of dealing with surface terms because, if one solved the Yang-Feldman equations for S, where S is that transformation relating in-operators and out-operators, covariance would always be maintained and surface terms would never appear.

Now the equations of motion for scalar electrodynamics are

$$[(p_\mu-eA_\mu)(p_\mu-eA_\mu)+m^2]\varphi(x)=0\,,\quad(4.5)$$

$$\partial_\nu\partial_\nu A_\mu(x)=-e\varphi^\dagger(p_\mu-eA_\mu)\varphi+e(p_\mu+eA_\mu)\varphi^\dagger\varphi\,.\quad(4.6)$$

Feynman rules for scalar electrodynamics have been derived by Rohrlich.[19] Then since Eqs. (4.5) and (4.6) and the equations for vector electrodynamics (2.24) and (2.25) have the same form, by Yang and Feldman's argument it would seem reasonable that rules of the same form as Rohrlich's rules for scalar electrodynamics would apply to vector electrodynamics.

However, Yang and Feldman only discussed cases where the interaction Hamiltonian contained a finite number of terms and never explicitly showed that Feynman rules could be derived from their equations. Their argument is on the same level as that of Takahashi and Umezawa[18]; namely, that for theories where the interaction Hamiltonian has a finite number of terms, surface terms never appear in the final result and Eq. (3.11) applies. Thus it seems *reasonable* that this should be the case for all theories.

It was decided that the problem could be settled if Feynman rules could be derived directly from Eqs. (4.1) and (4.4). Before the derivation is begun, however, the notation will be simplified. Let F_μ be defined by

$$\bar\psi(x)F_\mu\psi(x)=\bar\psi(x)\Gamma_{\mu\nu}p_\nu\psi(x)-[p_\nu\bar\psi(x)]\Gamma_{\mu\nu}\psi(x)\,.\quad(4.7)$$

Whenever F_μ appears it operates only on adjacent functions. The coordinate dependence of $A_\mu(x)$ will be suppressed whenever it is obvious from the context of the equations what the dependence should be. Hence, for example, Eq. (4.1) would be written

$$\psi(x)=\psi_{\text{in,out}}+\int dx_1[eA_{\mu1}G_{R,A}(x-x_1)F_{\mu1}\psi(x_1)-e^2G_{R,A}(x-x_1)A_\alpha A_\beta\Gamma_{\alpha\beta}\psi(x_1)]\,.\quad(4.8)$$

A method for finding S from the Yang-Feldman equations has been suggested by Rayski.[20] Note that an equivalent way of writing Eq. (4.8) is

$$\psi(x)=\psi^{(0)}(x)+\int dx_1[eA_{\mu1}\bar G(x-x_1)F_{\mu1}\psi(x_1)-e^2\bar G(x-x_1)\Gamma_{\alpha\beta}A_\alpha A_\beta\psi(x_1)]\quad(4.9\text{a})$$

[19] F. Rohrlich, Phys. Rev. **80**, 666 (1950).
[20] J. Rayski, Phil. Mag. **42**, 1289 (1951).

and

$$\psi_{\text{out}}-\psi_{\text{in}}=-\int dx_1[eA_{\mu1}G(x-x_1)F_{\mu1}\psi(x_1)$$
$$-e^2G(x-x_1)\Gamma_{\alpha\beta}A_\alpha A_\beta\psi(x_1)], \quad (4.9b)$$

where

$$\psi^{(0)}(x)=\tfrac{1}{2}[\psi_{\text{in}}(x)+\psi_{\text{out}}(x)]$$

and

$$\bar{G}(x)=\tfrac{1}{2}[G_R(x)+G_A(x)].$$

Parallel equations can be written for $A_\mu(x)$:

$$A_\mu(x)=A_\mu^{(0)}(x)+\int dx_1\bar{D}_{\mu\nu1}[e\bar{\psi}(x_1)F_{\nu1}\psi(x_1)$$
$$-2e^2A_\alpha\bar{\psi}(x_1)\Gamma_{\nu1\alpha}\psi(x_1)], \quad (4.10a)$$

$$A_\mu^{\text{out}}(x)-A_\mu^{\text{in}}(x)=-\int dx_1D_{\mu\nu1}[e\bar{\psi}(x_1)F_{\nu1}\psi(x_1)$$
$$-2e^2A_\alpha\bar{\psi}(x_1)\Gamma_{\nu1\alpha}\psi(x_1)]. \quad (4.10b)$$

Rayski[20] showed that Eqs. (4.4), (4.9b), and (4.10b) can be used to give

$$[\psi^{(0)}(x),S]_-=-\frac{1}{2}\int dx_1[eA_{\mu1}G(x-x_1)F_{\mu1}\psi(x_1)$$
$$-e^2G(x-x_1)\Gamma_{\alpha\beta}A_\alpha A_\beta\psi(x_1),S]_+ \quad (4.11)$$

and

$$[A_\mu^{(0)}(x),S]_-=-\frac{1}{2}\int dx_1[D_{\mu\mu1}(x-x_1)[e\bar{\psi}(x_1)F_{\mu1}\psi(x_1)$$
$$-2e^2A_\alpha\bar{\psi}(x_1)\Gamma_{\alpha\mu1}\psi(x_1)],S]_+. \quad (4.12)$$

Equations (4.11) and (4.12) can be solved by expanding S in a power series in the coupling constant and using Eqs. (4.9a) and (4.10a) to expand the Heisenberg fields in terms of the free fields. However an alteration of the form of Eqs. (4.9)–(4.12) is needed to carry this program through consistently. Although the order of the operators is not important in these equations [$A_\mu(x)$ and $\psi(x)$ commute at the same point], nevertheless in expanding these equations in terms of free fields, individual terms in the expansion of $A_\mu(x)$ will not commute with those in the expansion for $\psi(x)$. Thus, for example, it is necessary to write

$$A_{\mu1}G(x-x_1)F_{\mu1}\psi(x_1)=\tfrac{1}{2}[A_{\mu1},\, G(x-x_1)F_{\mu1}\psi(x_1)]_+.$$

Now let S and the fields A_μ and ψ be expanded in power series in the coupling constant, substituted into Eqs. (4.11) and (4.12), and the coefficients of e^n equated:

$$S=1+\sum_{n=1} e^nS_n, \quad (4.13a)$$

$$\psi(x)=\sum_{n=0} e^n\psi^{(n)}(x), \quad (4.13b)$$

and

$$A_\mu(x)=\sum_{n=0} e^nA_\mu^{(n)}(x). \quad (4.13c)$$

The coefficients of e in Eqs. (4.11) and (4.12) are

$$[\psi^{(0)}(x),S_1]_-=-\int dx_1A_\mu^{(0)}G(x-x_1)F_{\mu1}\psi^{(0)}(x_1), \quad (4.14a)$$

$$[A_\mu^{(0)}(x),S_1]_-=-\int dx_1D_{\mu\mu1}(x-x_1)\bar{\psi}^{(0)}(x_1)F_{\mu1}\psi^{(0)}(x_1). \quad (4.14b)$$

Since $\psi^{(0)}$ and $A_\mu^{(0)}(x)$ satisfy free-field commutation relations, it follows from Eqs. (4.14) that

$$S_1=i\int dx\, A_\mu^{(0)}\bar{\psi}^{(0)}(x)F_\mu\psi^{(0)}(x). \quad (4.15)$$

For $n\geq 2$, S_n can be found from

$$[\psi^{(0)}(x),S_n]_-=-\tfrac{1}{2}\sum_{j=0}\sum_{k=0}\sum_{r=0}^{n-k-j-1}\int dx_1[[A_{\mu1}^{(k)},\, G(x-x_1)F_{\mu1}\psi^{(j)}(x_1)]_+,S_r]_+$$
$$+\tfrac{1}{2}\sum_{j=0}\sum_{k=0}\sum_{m=0}\sum_{r=0}^{n-k-j-m-2}\int dx_1[[\tfrac{1}{4}[A_\alpha^{(k)},A_\beta^{(j)}]_+,\psi^{(m)}(x_1)]_+,S_r]_+ \quad (4.16a)$$

and

$$[A_\mu^{(0)}(x),S_n]_-=-\tfrac{1}{2}\sum_{j=0}\sum_{k=0}\sum_{r=0}^{n-k-j-1}\int dx_1[D_{\mu1}(x-x_1)\bar{\psi}^{(k)}(x_1)F_{\mu1}\psi^{(j)}(x_1),S_r]_+$$
$$+\sum_{j=0}\sum_{k=0}\sum_{m=0}\sum_{r=0}^{n-j-k-m-2}\int dx_1[D_{\mu\mu1}(x-x_1)\tfrac{1}{2}[[A_\mu^{(m)},\bar{\psi}^{(k)}(x_1)\Gamma_{\alpha\mu1}\psi^{(j)}(x_1)]_+,S_r]_+]. \quad (4.16b)$$

To solve Eqs. (4.16) for S_n, it is necessary to know all of the S_r, for $r<n$. However, since S_r, $A_\mu^{(k)}$, and $\psi^{(j)}$ can be written in terms of $A_\mu^{(0)}$ and $\psi^{(0)}$, it follows that S_n will be a function of free fields alone.

Next consider S_2. Since it will be expressed entirely in terms of free fields, the superscript (0) will be dropped with the understanding that all fields seen henceforth will be free fields. Thus S_2 will be found from

$$[\psi(x),S_2]_- = \frac{1}{2}\int dx_1 2G(x-x_1)A_\alpha A_\beta \Gamma_{\alpha\beta}\psi(x_1) - \frac{1}{2}\int\int dx_1 dx_2 \{[\bar{D}_{\mu_1\mu_2}(x_1-x_2)\bar{\psi}(x_2)F_{\mu_2}\psi(x_2), G(x-x_1)F_{\mu_1}\psi(x_1)]_+ + [A_{\mu_1},A_{\mu_2}]_+ G(x-x_1)F_{\mu_1}\bar{G}(x_1-x_2)F_{\mu_2}\psi(x_2) + i[A_{\mu_1}G(x-x_1)F_{\mu_1}\psi(x_1), A_{\mu_2}\bar{\psi}(x_2)F_{\mu_2}\psi(x_2)]_+\}, \quad (4.17)$$

and a similar expression for $[A_\mu,S_2]_-$. The solution for S_2 is

$$S_2 = -i\int dx_1 A_\alpha A_\beta \bar{\psi}(x_1)\Gamma_{\alpha\beta}\psi(x_1) + \frac{1}{2}\int\int dx_1 dx_2 \{[i\bar{D}_{\mu_1\mu_2}(x_1-x_2) - A_{\mu_1}A_{\mu_2}] \times \bar{\psi}(x_1)F_{\mu_1}\psi(x_1)\bar{\psi}(x_2)F_{\mu_2}\psi(x_2) + i[A_{\mu_1},A_{\mu_2}]_+\bar{\psi}(x_1)F_{\mu_1}\bar{G}(x_1-x_2)F_{\mu_2}\psi(x_2)\}. \quad (4.18)$$

The final step is to expand the products of field operators in terms of normal-ordered products. This is facilitated by making use of a theorem given by Dyson.[21]

A contraction symbol, denoted by superior heavy dots, will be defined by

$$A^\bullet(x)B^\bullet(x) = \langle 0|A(x)B(x)|0\rangle, \quad (4.19)$$

i.e., the vacuum expectation of the two operators. The normal-ordered product of operators $AB\cdots YZ$ will be denoted $:AB\cdots YZ:$, i.e., a colon on either side of the product. The decomposition theorem of Dyson states that

$$AB\cdots XYZ = :AB\cdots XYZ: + :A^\bullet B^\bullet\cdots XYZ: + :A^\bullet B\cdots X^\bullet YZ: + \cdots + :(A^\bullet B^\bullet)\cdots(X^\bullet Y^\bullet)Z: + \cdots, \quad (4.20)$$

where the sum on the right-hand side of Eq. (4.20) includes all possible sets of contractions between pairs of factors. It is then convenient to define

$$\psi^\bullet(x_1)\bar{\psi}^\bullet(x_2) = iG^+(x_1-x_2), \quad (4.21a)$$

$$\bar{\psi}^\bullet(x_1)\psi^\bullet(x_2) = -iG^-(x_1-x_2), \quad (4.21b)$$

and

$$A^\bullet_{\mu_1}(x_1)A^\bullet_{\mu_2}(x_2) = iD^+_{\mu_1\mu_2}(x_1-x_2) = -iD^-_{\mu_1\mu_2}(x_2-x_1), \quad (4.21c)$$

where the superscript $+$ $(-)$ indicates the positive- (negative-)frequency part of the homogeneous Green function. Dyson's theorem will now be applied to Eq. (4.18). As usual S_2 will consist of a sum of different types of scattering processes. Emphasis will be placed on the connected diagrams.

1. *Meson-Meson Scattering*

$$S_2^{MM} = \frac{1}{2}\int\int dx_1 dx_2 [i\bar{D}_{\mu_1\mu_2}(x_1-x_2) - A^\bullet_{\mu_1}A^\bullet_{\mu_2}] \times :\bar{\psi}(x_1)F_{\mu_1}\psi(x_1)\bar{\psi}(x_2)F_{\mu_2}\psi(x_2):. \quad (4.22)$$

[21] F. J. Dyson, Phys. Rev. 82, 428 (1951).

In the standard notation,

$$D^{(1)}_{\mu_1\mu_2} \equiv i(D^+_{\mu_1\mu_2} - D^-_{\mu_1\mu_2}), \quad (4.23)$$

and it is well known that

$$D^F_{\mu_1\mu_2} = D^{(1)}_{\mu_1\mu_2} - 2i\bar{D}_{\mu_1\mu_2}, \quad (4.24)$$

where $D^F_{\mu_1\mu_2}$ is the Feynman photon Green function.[18] Thus Eq. (4.22) becomes

$$S_2^{MM} = -\frac{1}{2}\int\int \frac{D^F_{\mu_1\mu_2}(x_1-x_2)}{2} \times :\bar{\psi}(x_1)F_{\mu_1}\psi(x_1)\bar{\psi}(x_2)F_{\mu_2}\psi(x_2):. \quad (4.25)$$

This expression corresponds exactly to what would be obtained from the $\mathcal{P}^*$ method, Eq. (3.11), and to using Feynman rules of the same form as those for scalar electrodynamics.[19]

2. *Photon-Meson Scattering, Pair Production, Pair Annihilation*

$$S_2^{PM} = -i\int dx_1 A_\alpha A_\beta \bar{\psi}(x_1)\Gamma_{\alpha\beta}\psi(x_1) + \frac{1}{2}\int\int dx_1 dx_2 \times :\{+2iA_{\mu_1}A_{\mu_2}\bar{\psi}(x_1)F_{\mu_1}\bar{G}(x_1-x_2)F_{\mu_2}\psi(x_2) + A_{\mu_1}A_{\mu_2}[\bar{\psi}(x_1)F_{\mu_1}\psi^\bullet(x_1)\bar{\psi}^\bullet(x_1)F_{\mu_2}\psi(x_2) + \bar{\psi}^\bullet(x_1)F_{\mu_1}\psi(x_1)\bar{\psi}(x_2)F_{\mu_2}\psi^\bullet(x_2)]\}:. \quad (4.26)$$

Using Eqs. (4.21) and the symmetry properties of the integrand, this can be written

$$S_2^{PM} = -i\int dx_1 A_\alpha A_\beta \bar{\psi}(x_1)\Gamma_{\alpha\beta}\psi(x_1) - \int\int dx_1 dx_2 \times :A_{\mu_1}A_{\mu_2}\bar{\psi}(x_1)F_{\mu_1}\tfrac{1}{2}G_F(x_1-x_2)F_{\mu_2}\psi(x_2):, \quad (4.27)$$

where

$$G_F(x) \equiv G^{(1)}(x) - 2i\bar{G}(x) = \frac{(\gamma_{\mu\nu}-\delta_{\mu\nu})p_\mu p_\nu - 2M^2}{4M^2}[\Delta^{(1)}(x) - 2i\bar{\Delta}(x)] = \frac{(\gamma_{\mu\nu}-\delta_{\mu\nu})p_\mu p_\nu - 2M^2}{4M^2}\Delta_F(x). \quad (4.28)$$

Here $G_F(x)$ is interpreted as the propagator for the massive spin-1 field. It agrees with what Takahashi and Umezawa[18] say should be the form for propagators, but differs from Weinberg's[10] propagator by the term

$$-(1/4M^2)(p^2+M^2)\Delta_F(x)=i\delta(x)/2M^2.$$

So far the derivations have been straightforward and there has been nothing too surprising. It will be seen however, that the self-energy parts of S_2 introduce new difficulties.[22]

3. *Self-Energy of the Vector Meson*

$$S_2{}^{\text{VSE}}=\frac{1}{2}\int\int dx_1dx_2$$
$$\times:\{i[A_{\mu1},A_{\mu2}]_+\bar{\psi}(x_1)F_{\mu1}\bar{G}(x_1-x_2)F_{\mu2}\psi(x_2)$$
$$+[i\bar{D}_{\mu1\mu2}(x_1-x_2)-A^{\bullet}{}_{\mu1}A^{\bullet}{}_{\mu2}]$$
$$\times[\bar{\psi}(x_1)F_{\mu1}\psi^{\bullet}(x_1)\bar{\psi}^{\bullet}(x_2)F_{\mu2}\psi(x_2)$$
$$+\bar{\psi}^{\bullet}(x_1)F_{\mu1}\psi(x_1)\bar{\psi}(x_2)F_{\mu2}\psi^{\bullet}(x_2)]\}:. \quad (4.29)$$

One would like to express Eq. (4.29) in terms of the Feynman Green functions D_F and G_F. To this end, the last two terms in Eq. (4.29) can be rewritten using the following relations for products of Green functions:

$$D^+G^++D^-G^-=\tfrac{1}{2}(DG-D^{(1)}G^{(1)}) \quad (4.30a)$$

and

$$DG=4\bar{D}\bar{G}-2(D_RG_A+D_AG_R). \quad (4.30b)$$

The final result is

$$S_2{}^{\text{VSE}}=-\frac{1}{4}\int\int dx_1dx_2D^F{}_{\mu1\mu2}(x_1-x_2)$$
$$\times:\bar{\psi}(x_1)F_{\mu1}G_F(x_1-x_2)F_{\mu2}\psi(x_2):-\frac{1}{2}\int\int dx_1dx_2$$
$$\times:[D^R{}_{\mu1\mu2}(x_1-x_2)\bar{\psi}(x_1)F_{\mu1}G_A(x_1-x_2)F_{\mu2}\psi(x_2)$$
$$+D^A{}_{\mu1\mu2}(x_1-x_2)\bar{\psi}(x_1)F_{\mu1}G_R(x_1-x_2)F_{\mu2}\psi(x_2)]:. \quad (4.31)$$

The first term in Eq. (4.31) is the usual type of expression for the self-energy. The second term, which would vanish automatically if the derivation were for scalar electrodynamics, appears to be a new type of self-energy term. This type of term is analogous to a term found by Lee and Yang[6] in their discussion of vector electrodynamics in the canonical formalism. They found that in the reduction of the S matrix, the cancellation between surface terms coming from the interaction Hamiltonian and those coming from the time-ordered products of operators was not complete.

Note that in the second term in Eq. (4.31) the integrands vanish everywhere except at the origin, where they are singular. On the other hand, the integrand of the first term in Eq. (4.31) is also singular at the origin, which simply means that the self-energy is divergent. Divergent self-energies are common to S matrix theory and can be dealt with in a standard way known as the Pauli-Villars regularization.[23] Auxiliary masses are introduced into the integrals in a special way which makes the integrals finite. The degree of divergence of an integral is recovered by letting the auxiliary masses tend to zero at the end of the calculation. We now consider the second term in Eq. (4.31) when it is regularized.

Let $|p_1\rangle$ and $|p_2\rangle$ be states of one vector meson of four-momenta p_1 and p_2, respectively:

$$\langle p_1|\int\int dx_1dx_2D^R{}_{\mu1\mu2}(x_1-x_2):\bar{\psi}(x_1)F_{\mu1}G_A(x_1-x_2)F_{\mu2}\psi(x_2):|p_2\rangle$$
$$=\frac{\delta(p_1-p_2)\bar{u}(p_1)}{(2\pi)^3 2p_0}\int\frac{d^4k\ \Gamma_{\mu\alpha}(p+k)_\alpha[k\Omega k-2M^2](p+k)_\beta\Gamma_{\beta\mu}u(p_2)}{[(p-k)^2-i\epsilon(p_0-k_0)]4M^2[k^2+M^2+i\epsilon k_0]}, \quad (4.32)$$

where $k\Omega k\equiv k_\mu(\gamma_{\mu\nu}-\delta_{\mu\nu})k_\nu$. The regularized integral of Eq. (4.32) is defined by

$$W_R=\int d^4k\ \Gamma_{\mu\alpha}(p+k)_\alpha\frac{k\Omega k-2M^2}{k^2+M^2+i\epsilon k_0}(p+k)_\beta\Gamma_{\beta\mu}$$
$$\times\sum_{i=1}^{4}c_i[(p-k)^2+m_i{}^2-i\epsilon(p_0-k_0)]^{-1}, \quad (4.33)$$

with the conditions

$$\sum c_i=0,\quad \sum c_iM_i{}^2=0,\quad \sum c_iM_i{}^4=0.$$

Making explicit use of these conditions and then writing the denominator of the integrand in Eq. (4.33) as a function of k^2 only,[24] it can be shown that the integrand is analytic in the upper k_0 plane. Then since the integral along the k_0 axis can be closed in the upper half k_0 plane, the integral vanishes by the Cauchy theorem,

$$W_R=0. \quad (4.34)$$

[22] The bubble diagrams which result from contracting two operators with the same argument are omitted as they could be taken care of initially by normal-ordering the currents in Eqs. (2.24) and (2.25).

[23] W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).

[24] S. Schweber, Ref. 16, Chap. 15, p. 520.

In a similar manner, the term with

$$D^A(x_1-x_2)G_R(x_1-x_2)$$

will vanish. The effect of the Pauli-Villars regularization method is to remove singularities at the origin. Since in the integrand in Eq. (4.31) the only nonvanishing part is for $x_1=x_2$, after removal of this singularity by regularization the integral is expected to vanish.

The point to make is that if one wished to display the order of divergence and extract any finite parts from Eq. (4.31), the usual procedure of regularizing such an expression first and then letting the auxiliary masses tend to zero after doing the integration would be followed. But as has been shown, the second term in Eq. (4.31) vanishes identically after regularization; hence it is concluded that such terms make no contribution to the self-energy.

It is also straightforward to show that the photon self-energy consists of an integral with Feynman Green functions $G_F(x)$ plus terms with integrals over products of $G_A(x)$ and $G_R(x)$. Again, if only regularized expressions are used in calculating radiative corrections, the photon self-energy is calculated only from an integral with Feynman Green functions.

It is now clear why the $\mathcal{P}^*$ method will apply to this vector electrodynamics to all orders. When S_n is found from the Yang-Feldman equations, it can be expanded in terms of normal-ordered products. Each type of scattering terms will be given by an integral whose integrand is a functional of the fields and various Green functions. These Green functions can be manipulated into a form such that the integral consists of a Feynman amplitude part plus some other terms. Since the equations of motion for scalar electrodynamics, Eqs. (4.5) and (4.6), have the same form as for vector electrodynamics, Eqs. (2.24) and (2.25), S_n will have the same form as for vector electrodynamics. But Feynman rules do apply to scalar electrodynamics[19]; hence any extra terms appearing in the derivation of S_n for scalar electrodynamics using the Yang-Feldman equations must be products of advanced and retarded Green functions since it is only these terms which vanish identically. For vector electrodynamics, integrands with products of advanced and retarded Green functions vanish everywhere except at the origin, where they are singular. As was indicated for the second-order photon and vector-meson self-energies, it is the regularized expressions which are meaningful in doing calculations. Since the process of regularization removes the singularity at the origin, the regularized integrals of products of advanced and retarded Green functions vanish. Hence the S matrix for vector electrodynamics consists entirely of Feynman-graph amplitudes.

V. RENORMALIZABILITY OF ELECTRODYNAMICS OF ARBITRARY-SPIN PARTICLES

One could in principle follow Sec. IV to derive the S matrix for the electrodynamics of arbitrary-spin particles starting from Eqs. (2.22) and (2.23). It is clear from the structure of these equations that in addition to one-photon and two-photon vertices there can be 3, 4, ..., $2s$-photon vertices for the processes involving integral-spin and 3, 4, ..., $(2s+1)$-photon vertices for the processes involving half-integral spin. By the arguments of Sec. IV, Feynman rules of the same form as scalar electrodynamics must apply to the one-photon and two-photon vertices for the electrodynamics of particles with arbitrary spin. For the higher-order photon vertices, we can at this time only assume that there are no additional difficulties.

The role of the propagator is played by

$$G_F^{(s)}(x)=i[G_+^{(s)}(x)-G_-^{(s)}(x)] -i[G_A^{(s)}(x)+G_R^{(s)}(x)]. \quad (5.1)$$

Note that for large k [see Eqs. (2.11) and (2.20)]

$$G_F^{(s)}(k)\sim 1/(k^2)^{s-1}, \quad s=1, 2, \ldots \quad (5.2a)$$

$$\sim 1/(k^2)^{s+1/2}, \quad s=\tfrac{1}{2}, \tfrac{3}{2}, \ldots . \quad (5.2b)$$

A discussion of the renormalizability of Eqs. (2.22) and (2.23) can be made by using Dyson's method of power counting,[25,26] to find the "dimension" of an integral for an arbitrary graph. For the integral-spin case, an m-photon vertex will have a $(2s+1-m)$-order-derivative coupling. Let $B^{(s)}$ stand for a boson (of spin s) line, $F^{(s)}$ stand for a fermion (of spin s) line, P stand for a photon line, and C stand for the number of vertices in a diagram. The dimension D of an integral is defined as the difference between the number of powers of momenta in the numerator and the denominator. The contributions to D are as follows.

(i) There is a δ function in momentum space from each coordinate space integral; however, one δ function expresses over-all conservation of four-momentum. Thus the contribution to D is $-4(C-1)$.

(ii) There is a momentum-space integral for each internal line. The contribution to D is $4(B_i^{(s)}+P_i)$ for boson electrodynamics and $4(F_i^{(s)}+P_i)$ for fermion electrodynamics. (The subscript i denotes internal lines.)

(iii) The contribution from the photon propagator is $-2P_i$, the contribution from the massive boson propagator is $-2(s-1)B_i^{(s)}$, and the contribution from the massive fermion propagator is $-(2s+1)F_i^{(s)}$.

(iv) The derivative coupling from an m-photon vertex contributes $(2s-m)C$ for boson electrodynamics and $(2s+1-m)C$ for fermion electrodynamics. Hence

[25] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).
[26] S. Schweber, Ref. 16, Chap 16, Sec. a.

the dimension of an integrand for a diagram with m photon vertices is

$$D^{(s)}(m)=(2s-m-4)C+4+(6-2s)B_i^{(s)}+2P_i\,,\qquad s=1,2,\ldots \quad (5.3a)$$

$$=(2s-3-m)C+4+(3-2s)F_i^{(s)}+2P_i\,,\qquad s=\tfrac{1}{2},\tfrac{3}{2},\ldots. \quad (5.3b)$$

It is convenient to express Eqs. (5.3) in terms of external lines. Each vertex has two massive boson or fermion line endings; thus

$$2C=2B_i^{(s)}+B_e^{(s)}\,,\quad s=1,2,\ldots \quad (5.4a)$$

or

$$2C=2F_i^{(s)}+F_e^{(s)}\,,\quad s=\tfrac{1}{2},\tfrac{3}{2},\ldots. \quad (5.4b)$$

(The subscript e denotes external lines.) An m-photon vertex means that

$$mC=2P_i+P_e. \quad (5.4c)$$

Thus in terms of external lines

$$D^{(s)}=4+2C-(3-s)B_e^{(s)}-P_e\,,\quad s=1,2,3,\ldots \quad (5.5a)$$

$$=4+(s-\tfrac{3}{2})F_e^{(s)}-P_e\,,\qquad s=\tfrac{1}{2},\tfrac{3}{2},\ldots \quad (5.5b)$$

independent of m in each case. Consider first the integral-spin case (5.5a). The dependence of $D^{(s)}$ on C implies that there are an infinite number of primitive divergences; hence all of the integral-spin equations (2.1) are nonrenormalizable in the presence of minimal electromagnetic coupling. For the half-integral-spin case, $D^{(s)}$ is not dependent on the number of vertices in a diagram. For $s=\frac{1}{2}$, Eq. (5.5b) is $D^{(1/2)}=4-\frac{1}{2}F_e-P_e$, which implies that there are a finite number of primative divergences. Thus the spin-$\frac{1}{2}$ electrodynamics described by the equations

$$[m\gamma_\mu(p_\mu-eA_\mu)+i(p_\mu-eA_\mu)(p_\mu-eA_\mu)]\psi(x)=0\,, \quad (5.6a)$$

$$\partial_\nu\partial_\nu A_\mu(x)=iem\bar{\psi}\gamma_\mu\psi$$
$$-e[\bar{\psi}(\overrightarrow{p}_\mu-eA_\mu)\psi-\bar{\psi}(\overleftarrow{p}_\mu+eA_\mu)\psi] \quad (5.6b)$$

is renormalizable. Hammer and Moroi[27] have studied Eqs. (5.6) in some detail.

For $s\geq\frac{3}{2}$, however, there are an infinite number of primitive divergences, since given any number of external fermion lines F_e, a number of external photon lines P_e can be chosen to make $D^{(s)}\geq 0$. Thus for $s\geq\frac{3}{2}$, Eqs. (2.22b) and (2.23) are nonrenormalizable.

VI. DISCUSSION AND CONCLUSIONS

There are some distinct advantages to the theory presented here both from a formal and practical standpoint. Since there is a single wave equation with no auxiliary conditions, each component of the field is treated on an equal footing throughout derivations and calculations. The Fock-space operators create and destroy particles or antiparticles in definite spin or helicity states. The calculations are therefore as straightforward as in the spin-$\frac{1}{2}$ case. All of the machinery needed for any calculation is written down in Appendices A and B.

This should be contrasted with the canonical formalism and the formalism in which one must always take the dependent components of the field into account. Note that in the canonical formalism the Fock-space operators create and destroy particles or antiparticles in transverse or longitudinal states.

The conventional method of deriving the S matrix from Dyson's equation was abandonded to avoid the problems of surfaces terms. Although the use of Yang-Feldman equations to find the S matrix is tedious, nevertheless this has the advantage of maintaining manifest covariance throughout the derivation. Once one is certain of the Feynman rules, the $\mathcal{P}^*$ method can be used with confidence.

The extra terms in the radiative corrections found by Lee and Yang[6] in the canonical formalism appear in this theory as integrals over products of advanced and retarded Green functions. If one takes the point of view that it is only regularized expressions which are meaningful in doing calculations, then none of the extra terms contribute.

The various formalisms have different predictions about the exact behavior of vector mesons in an electromagnetic field. However, since all known vector mesons interact strongly, one cannot experimentally test for the correct theory. But if the photon field is replaced by the nucleon current $i\bar{\psi}_N\gamma_\mu\psi_N$, a calculation can be done of the scattering of a vector meson by an external nucleon field. There may be a possibility of distinguishing among the different vector-meson theories if such a calculation is incorporated into a calculation of πp scattering, where in an intermediate state one expects ρp scattering to occur. This will be the subject of future investigations.

As is the case for the other formalisms, this theory is nonrenormalizable. A way of obtaining a renormalizable theory would be to modify the advanced and retarded Green functions and the free-field commutation relations such that the massive spin-1 propagator in momentum space had a $1/k^2$ behavior in the asymptotic region. This would be analogous to the ξ-limiting formalism of Lee and Yang. One would effectively be replacing a theory with a single cutoff parameter. Then, because the equations for arbitrary spin given by Hammer *et al.*[12] all have the same general structure, after learning what sort of modifications lead to a renormalizable theory for spin 1, the same techniques can be applied to arbitrary-spin theories.

ACKNOWLEDGMENTS

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[27] C. L. Hammer and D. S. Moroi, USAEC Report No. IS-2085, 1968 (unpublished).

of us (R. H. T.) would like to thank Ron Hadley for his help in checking the formulas for traces of spin-1 matrices.

APPENDIX A: COVARIANTLY DEFINED SPIN-1 MATRICES

The 6×6 matrices $\gamma_{\mu\nu}$ have been studied in detail by Sankaranarayanan and Good.[28] A complete set of covariantly defined 6×6 matrices is the unit matrix

$$\gamma_1 = 1\,, \tag{A1a}$$

and the traceless matrices

$$\gamma_2 = \gamma_5\,, \tag{A1b}$$

$$\gamma_{3,\mu\nu} = \gamma_{\mu\nu}\,, \tag{A1c}$$

$$\gamma_{4,\mu\nu} = i\gamma_5\gamma_{\mu\nu}\,, \tag{A1d}$$

$$\gamma_{5,\mu\nu} = i(\gamma_{\mu\lambda}\gamma_{\nu\lambda} - \gamma_{\nu\lambda}\gamma_{\mu\lambda})\,, \tag{A1e}$$

$$\gamma_{6,\mu\rho,\nu\sigma} = [\gamma_{\mu\nu},\gamma_{\rho\sigma}]_+ + 2\delta_{\mu\nu}\delta_{\rho\nu} - [\gamma_{\rho\nu},\gamma_{\mu\sigma}]_+ - 2\delta_{\rho\nu}\delta_{\mu\sigma}. \tag{A1f}$$

The product of two $\gamma_{\mu\nu}$ matrices can be expanded in terms of the complete set (A1), as[29]

$$\begin{aligned}\gamma_{\mu\nu}\gamma_{\alpha\beta} = {} & -\tfrac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta} + \tfrac{2}{3}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}) \\ & -\tfrac{1}{12}i(\gamma_{5,\mu\alpha}\delta_{\nu\beta} + \gamma_{5,\nu\alpha}\delta_{\mu\beta} + \gamma_{5,\mu\beta}\delta_{\nu\alpha} + \gamma_{5,\nu\beta}\delta_{\mu\alpha}) \\ & + \tfrac{1}{6}(\gamma_{6,\mu\alpha,\nu\beta} + \gamma_{6,\nu\alpha,\mu\beta})\,. \end{aligned} \tag{A2}$$

Traces of products of any number of the γ matrices [Eqs. (A1)] depend only on traces of products with $\gamma_{\mu\nu}$ and γ_5. The trace of two $\gamma_{\mu\nu}$ matrices follows from Eq. (A2) as

$$\mathrm{Tr}(\gamma_{\mu\nu}\gamma_{\alpha\beta}) = 4(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}) - 2\delta_{\mu\nu}\delta_{\alpha\beta}\,. \tag{A3}$$

The trace of four $\gamma_{\mu\nu}$ matrices is found to be

$$\begin{aligned}\mathrm{Tr}(\gamma_{\mu\nu}\gamma_{\alpha\beta}\gamma_{\rho\sigma}\gamma_{\lambda\tau}) = {} & 6\delta_{\mu\nu}\delta_{\alpha\beta}\delta_{\rho\sigma}\delta_{\lambda\tau} + S_{\mu\nu,\alpha\beta,\rho\sigma,\lambda\tau}[(\mu\nu,\alpha\rho,\beta\lambda,\sigma\tau) + (\alpha\beta,\mu\rho,\nu\lambda,\sigma\tau) + (\rho\nu,\mu\alpha,\nu\lambda,\beta\tau) \\ & + (\lambda\tau,\mu\alpha,\nu\rho,\beta\sigma) - (\mu\nu,\alpha\beta,\rho\lambda,\sigma\tau) - (\mu\nu,\lambda\tau,\alpha\rho,\beta\sigma) - (\rho\sigma,\lambda\tau,\mu\alpha,\nu\beta) \\ & - (\alpha\beta,\rho\sigma,\lambda\mu,\tau\nu) - (\mu\nu,\rho\sigma,\alpha\lambda,\beta\tau) - (\alpha\beta,\lambda\tau,\mu\rho,\nu\sigma) + 4(\mu\alpha,\nu\beta,\rho\lambda,\sigma\tau) \\ & + 4(\mu\lambda,\nu\tau,\alpha\beta,\beta\sigma) + 4(\mu\rho,\nu\sigma,\alpha\lambda,\beta\tau) - 2(\mu\beta,\alpha\tau,\rho\nu,\lambda\nu) - 2(\mu\sigma,\alpha\tau,\rho\beta,\lambda\nu)]\,, \end{aligned} \tag{A4}$$

where, for example,

$$(\mu\nu,\alpha\beta,\rho\lambda,\sigma\tau) = \delta_{\mu\nu}\delta_{\alpha\beta}\delta_{\rho\lambda}\delta_{\sigma\tau} \tag{A5a}$$

and

$$S_{\mu\nu,\alpha\beta,\rho\sigma,\lambda\tau}(\mu\nu,\alpha\beta,\rho\lambda,\sigma\tau) = 4\delta_{\mu\nu}\delta_{\alpha\beta}(\delta_{\rho\lambda}\delta_{\sigma\tau} + \delta_{\rho\tau}\delta_{\sigma\lambda})\,, \tag{A5b}$$

i.e., the operator $S_{\mu\nu,\alpha\beta,\rho\sigma,\lambda\tau}$ means to symmetrize independently with respect to the pairs $\mu\nu$, $\alpha\beta$, $\rho\sigma$, and $\lambda\tau$. As in the Dirac case,

$$\mathrm{Tr}(\text{odd number of } \gamma_{\mu\nu}\text{'s}) = 0\,. \tag{A6}$$

Traces of higher numbers of γ matrices can be reduced to traces of four $\gamma_{\mu\nu}$ matrices by the following theorem.

If G and H each consist of a product of an odd number of γ matrices, then, as shown below,

$$\mathrm{Tr}GH = \tfrac{1}{8}\,\mathrm{Tr}(G\gamma_{\mu\nu})\,\mathrm{Tr}(\gamma_{\mu\nu}H) + \tfrac{1}{8}\,\mathrm{Tr}(G\gamma_5\gamma_{\mu\nu})\,\mathrm{Tr}(\gamma_{\mu\nu}\gamma_5 H)\,. \tag{A7}$$

Thus, for example,

$$\mathrm{Tr}(6\gamma\text{'s}) = \tfrac{1}{8}\,\mathrm{Tr}(3\gamma\text{'s}\gamma_{\mu\nu})\,\mathrm{Tr}(\gamma_{\mu\nu}3\gamma\text{'s}) + \tfrac{1}{8}\,\mathrm{Tr}(3\gamma\text{'s}\gamma_5\gamma_{\mu\nu})\,\mathrm{Tr}(\gamma_{\mu\nu}\gamma_5 3\gamma\text{'s})\,. \tag{A8}$$

The first term on the right-hand side of Eq. (A8) can be calculated using Eq. (A4). The second term can be calculated using

$$\mathrm{Tr}(\gamma_5\gamma_{\mu\nu}\gamma_{\alpha\beta}\gamma_{\rho\sigma}\gamma_{\lambda\tau}) = S_{\mu\nu,\alpha\beta,\rho\sigma,\lambda\tau}\epsilon_{\mu\alpha\rho\lambda}(\delta_{\nu\beta}\delta_{\sigma\tau} + \delta_{\nu\tau}\delta_{\sigma\beta} - \delta_{\nu\sigma}\delta_{\beta\tau})\,, \tag{A9}$$

where $\epsilon_{\mu\alpha\rho\lambda}$ is the permutation symbol with four indices. The proof of the theorem proceeds as follows.

If G and H each consist of a product of an odd number of γ matrices, then in a representation in which $\gamma_{\mu\nu}$ is off-diagonal, G and H must be off-diagonal. Of the 36 linearly independent 6×6 matrices, Eqs. (A3), only $\gamma_{\mu\nu}$ and $\gamma_5\gamma_{\mu\nu}$ are off-diagonal. Hence

$$G = G_1^{\mu\nu}\gamma_{\mu\nu} + G_2^{\mu\nu}\gamma_{\mu\nu}\gamma_5\,, \tag{A10a}$$

$$H = H_1^{\mu\nu}\gamma_{\mu\nu} + H_2^{\mu\nu}\gamma_{\mu\nu}\gamma_5\,, \tag{A10b}$$

where $G_1^{\mu\nu}$, $G_2^{\mu\nu}$, $H_1^{\mu\nu}$, and $H_2^{\mu\nu}$ are expansion coefficients. One notes then that $\mathrm{Tr}(\gamma_{\mu\nu}\gamma_{\alpha\beta}\gamma_5) = 0$, so that, with the help of Eq. (A8),

$$\begin{aligned}\mathrm{Tr}(G\gamma_{\mu\nu})\,\mathrm{Tr}(\gamma_{\mu\nu}H) + \mathrm{Tr}(G\gamma_5\gamma_{\mu\nu})\,\mathrm{Tr}(\gamma_{\mu\nu}\gamma_5 H) & = 8G_1^{\mu\nu}H_1^{\alpha\beta}\,\mathrm{Tr}\gamma_{\mu\nu}\gamma_{\alpha\beta} - 8G_2^{\mu\nu}H_2^{\alpha\beta}\,\mathrm{Tr}\gamma_{\mu\nu}\gamma_{\alpha\beta} \\ & = 8\,\mathrm{Tr}[(G_1^{\mu\nu}\gamma_{\mu\nu} + G_2^{\mu\nu}\gamma_{\mu\nu}\gamma_5) \times (H_1^{\alpha\beta}\gamma_{\alpha\beta} + H_2^{\alpha\beta}\gamma_{\alpha\beta}\gamma_5)] = 8\,\mathrm{Tr}(GH)\,. \end{aligned}$$

APPENDIX B: FEYNMAN RULES FOR VECTOR ELECTRODYNAMICS

The Feynman rules for this vector electrodynamics are identical in form to Rohrlich's rules for scalar electrodynamics. The rules can be derived from an effective interaction Hamiltonian

$$\mathcal{H}_{\mathrm{EF}}'(x) = -eA_\mu\bar{\Psi}(x)F_\mu\Psi(x) + e^2A_\mu A_\nu\bar{\Psi}(x)\Gamma_{\mu\nu}\Psi(x) \tag{B1}$$

is one uses

$$S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dx_1\cdots\int dx_n \times \mathcal{P}^*(\mathcal{H}_{\mathrm{EF}}'(x_1)\cdots\mathcal{H}_{\mathrm{EF}}'(x_n))\,, \tag{B2}$$

[28] A. Sankaranarayanan and R. H. Good, Jr., Nuovo Cimento **36**, 1303 (1965).
[29] R. H. Good, Jr. (private communication).

i.e., the $\mathcal{P}^*$ method of Takahashi and Umezawa. The results can be extended to vector mesons with arbitrary magnetic dipole and electric quadrupole moments by using Eqs. (2.27) and (2.28). The effective interaction Hamiltonian is

$$\mathcal{H}_{\mathrm{EF}}'(x)=-eA_\mu\bar{\Psi}(x)F_\mu\Psi(x)+e^2A_\mu A_\nu\bar{\Psi}(x)\Gamma_{\mu\nu}\Psi(x)$$
$$+\tfrac{1}{12}e\lambda\bar{\Psi}(x)\gamma_{5,\alpha\beta}\Psi(x)F_{\alpha\beta}$$
$$+\frac{eq}{12m^2}\frac{\partial F_{\mu\nu}}{\partial x_\alpha}[\bar{\Psi}(x)\gamma_{6,\mu\nu,\alpha\beta}(p_\beta-eA_\beta)\Psi(x)$$
$$-(p_\beta+eA_\beta)\bar{\Psi}(x)\gamma_{6,\mu\nu,\alpha\beta}\Psi(x)]. \quad \text{(B3)}$$

The momentum-space Feynman rules for this interaction Hamiltonian are summarized in Fig. 1. Analogously to spin-$\frac{1}{2}$ electrodynamics, for closed meson loops the trace of the 6×6 matrix expression should be taken. Comparable rules for vector electrodynamics in the canonical formalism have been given by Aronson.[7]

The plane-wave expansions for the vector-meson field are

$$\Psi(x)=\sum_{s=1}^{3}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}2p_0}[c_s(\mathbf{p})u_s(\mathbf{p})e^{ipx}$$
$$+d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}], \quad \text{(B4a)}$$

$$\bar{\Psi}(x)=\sum_{s=1}^{3}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}2p_0}[c_s^\dagger(p)\bar{u}_s(\mathbf{p})e^{-ipx}$$
$$+d_s(p)\bar{v}_s(\mathbf{p})e^{ipx}], \quad \text{(B4b)}$$

$p_0=(p^2+M^2)^{1/2}$, where $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ are six-component column matrices for positive- and negative-energy vector mesons with polarization s. The $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ have orthogonality and completeness relations of the form

$$-(i/2p_0)j_4(\bar{u}_s(\mathbf{p})e^{-ipx},u_{s'}(\mathbf{p})e^{ipx})=\delta_{ss'},$$

or

$$-(i/p_0)\bar{u}_s(\mathbf{p})\Gamma_{4\nu}p_\nu u_{s'}(\mathbf{p})=\delta_{ss'}, \quad \text{(B5a)}$$

$$\sum_s u_s(\mathbf{p})u_s(\mathbf{p})=\sum_s v_s(\mathbf{p})v_s(\mathbf{p})$$
$$=(4M^2)^{-1}(M^2-\gamma_{\alpha\beta}p_\alpha p_\beta). \quad \text{(B5b)}$$

An explicit form for the u's and v's is

$$u_s(\mathbf{p})=v_s(\mathbf{p})=(2M^2\mathbf{p}^2)^{-1}(M^2-\gamma_{\alpha\beta}p_\alpha p_\beta)$$
$$\times\{\mathbf{p}^2-\gamma_{4i}\gamma_{4j}p_ip_j[1-(M/E)]\}u_s(0), \quad \text{(B6)}$$

where

$$u_s(0)=\frac{1}{2}\begin{pmatrix}\varphi_s\\ \varphi_s\end{pmatrix},$$

with φ_s the solution of

$$\frac{\mathbf{s}\cdot\mathbf{p}}{|\mathbf{p}|}\varphi_s=s\varphi_s.$$

The matrices $\mathbf{s}$ are the 3×3 spin-1 matrices:

$$[c_s(\mathbf{p}),c_t^\dagger(\mathbf{q})]_-=[d_s(\mathbf{p}),d_t^\dagger(\mathbf{q})]_-=2p_0\delta_{st}\delta(\mathbf{p}-\mathbf{q}), \quad \text{(B7a)}$$

$$[c_s(\mathbf{p}),c_t(\mathbf{q})]_-=[d_s(\mathbf{p}),d_t(\mathbf{q})]_-=0. \quad \text{(B7b)}$$

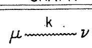
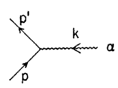
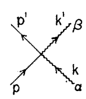
ELEMENT	GRAPH	VALUE
INTERNAL PHOTON LINE		$D^F_{\mu\nu}(k)=-i\delta_{\mu\nu}/(k^2-i\epsilon)$
INTERNAL MESON LINE		$G_F(p)=\dfrac{-i[2M^2-p_\mu p_\nu(\gamma_{\mu\nu}-\delta_{\mu\nu})]}{4M^2(p^2+M^2-i\epsilon)}$
ONE-PHOTON VERTEX		$-e\Gamma_{\alpha\beta}(p+p')_\beta-\dfrac{ie\lambda}{6}\gamma_{5,\alpha\beta}k_\beta+\dfrac{eq}{6M^2}\gamma_{6,\alpha\beta,\mu\nu}k_\beta k_\mu(p+p')_\nu$
TWO-PHOTON VERTEX		$e^2\Gamma_{\alpha\beta}+\dfrac{e^2q}{6M^2}(k_\mu k_\nu+k'_\mu k'_\nu)\gamma_{6,\mu\alpha,\nu\beta}$

FIG. 1. Feynman rules for vector electrodynamics.