Dispersion of the Veneziano Representation in Trajectory Parameters, Regge Cuts, and Unitarity Corrections

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It is suggested that the high-energy unitarity condition and Regge cuts can be built into the Veneziano model by dispersing the representation in trajectory parameters. In particular, a model of the Amati-Fubini-Stanghellini type and the optical model are considered in order to obtain the spectral functions of the integral representations.

I. INTRODUCTION

By now it has become well known that the Veneziano representation,[1] which incorporates resonance pole structure, crossing symmetry, and Regge behavior in an elegant manner, has several shortcomings which are related. It violates unitarity, the resonances have vanishing total widths, and the model does not have Regge cuts,[2] which are known to be present on both theoretical and experimental grounds. It also violates the Cerulus-Martin bound[3] at high energies and large momentum transfers.

Several attempts have been made to remove these difficulties.[4] Some of these deal with the calculations of loop diagrams with Veneziano Born terms, and others use K-matrix or N/D methods. The former lead to hitherto unresolved divergence difficulties, whereas the latter destroy to some extent the crossing symmetry which is the main elegant feature of the Veneziano model. On the other hand, it has been proposed that dispersing the representation in the trajectory parameters may be useful in this context.[5] The spectral functions in these treatments, however, have been left arbitrary. In the present work we start with a similar dispersed representation and discuss various procedures to determine the spectral functions.

In Sec. II, we discuss a model of the Amati-Fubini-Stanghellini (AFS) type which generates the Regge cuts in a well-known way. The optical model, with eikonal description of scattering, has been also quite successful in explaining the large-momentum-transfer data.[6] Several authors have used the idea of iterating a single-Regge-pole contribution via the eikonal formula to generate a series which contains automatically some of the Regge-cut contributions.[7] We use this method to determine the spectral function in Sec. III.

For simplicity we have considered the case of $\pi^+\pi^-$ scattering. The method can be extended to the other cases. Also, initially we ignore the Pomeranchuk trajectory and consider it only in the latter part. Results are discussed in Sec. IV and concluding remarks are made.

II. AFS-TYPE MODEL

Consider $\pi^+\pi^-$ scattering for which the one-term Veneziano amplitude is given by

$$F_v(s,t)=-\beta\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}. \qquad (1)$$

Here $\alpha(s)=\alpha_0+\alpha's$ is the exchange-degenerate ρ-f trajectory. In practice, of course, it may be necessary to consider several secondary terms.[8] Now we write

$$\alpha_c(s)=c\alpha'(\alpha_0-1)+1+s/c. \qquad (2)$$

The reason for this particular choice of $\alpha_c(s)$ will be clear later. The generalization of (1) will be given by

$$F(s,t)=-\int_{c_{\text{th}}}^{\infty}dc\,\rho(c,s,t)\frac{\Gamma(1-\alpha_c(s))\Gamma(1-\alpha_c(t))}{\Gamma(1-\alpha_c(s)-\alpha_c(t))}, \qquad (3)$$

where $\rho(c,s,t)$ is some spectral function to be determined. For maintaining crossing symmetry, $\rho(c,s,t)$ should be symmetric in s and t. In our approximation the dependence of $\rho(c,s,t)$ on s and t will turn out to be quite weak.

We wish to impose the unitarity constraints on $\rho(c,s,t)$. To simplify the treatment we approximate $\rho(c,s,t)$ by a sum of a series of δ functions:

$$\rho(c,s,t)=\sum_{n=1}^{\infty}\rho_n(s,t)\delta(c-c_n). \qquad (4)$$

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[2] S. Mandelstam, Nuovo Cimento **30**, 1148 (1963); D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962); V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, Phys. Rev. **139**, B184 (1965).
[3] F. Cerulus and A. Martin, Phys. Letters **8**, 80 (1964).
[4] For a list of numerous articles in this field see, e.g., a review article by D. Sivers and J. Yellin, LRL Report No. UCRL-19418 (unpublished).
[5] A Martin, Phys. Letters **29B**, 431 (1969); K. Huang, Phys. Rev. Letters **23**, 900 (1969); N. F. Bali, D. D. Coon, and J. W. Dash, *ibid.* **23**, 903 (1969).
[6] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Britten *et al.* (Interscience, New York, 1959), Vol. I; M. M. Islam, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Britten (Gordon and Breach, New York, 1968), Vol. XB.
[7] R. C. Arnold, Phys. Rev. **140**, B1022 (1965); **153**, 1523 (1967); S. Frautschi and B. Margolis, Nuovo Cimento **56A**, 1155 (1968); C. B. Chiu and J. Finkelstein, *ibid.* **57A**, 649 (1968); M. Martinis, *ibid.* **59A**, 490 (1969).
[8] C. Lovelace, Phys. Letters **28B**, 264 (1968); K. V. Vasavada, Phys. Rev. D **1**, 88 (1970).

The integral (3) now becomes the sum

$$F(s,t)=-\sum_{n=1}^{\infty}\rho_n(s,t)\frac{\Gamma(1-\alpha_{c_n}(s))\Gamma(1-\alpha_{c_n}(t))}{\Gamma(1-\alpha_{c_n}(s)-\alpha_{c_n}(t))}. \quad (5)$$

ρ_n and c_n are arbitrary at this point. Now the unitarity condition on the amplitude $F(s,t)$ is given by

$$\mathrm{Im}F(s,t)=\frac{1}{64\pi^2(\sqrt{s})k}\int_{-4k^2}^{0}dt'\times\int_{-4K^2}^{0}dt''\frac{\theta(K)F(s,t')F^*(s,t'')}{\sqrt{K(t,t',t'',s)}}+\mathrm{Im}F_{\text{inelastic}}, \quad (6)$$

where

$$K(t,t',t'',s)=-t^2-t'^2-t''^2+2tt'+2tt''+2t't''+tt't''/k^2 \quad (7)$$

and $k=\frac{1}{2}(s-4m_\pi^2)^{1/2}$.

For large s, it can be seen that most of the contribution to the integral comes from the region $t'\approx t''\approx 0$. Thus, as a good first approximation we can neglect the last term under the square root in $K(t,t',t'',s)$ and evaluate all other factors except the powers of s occurring inside the double integral at $t'=t''=0$. Then

$$K(t,t',t'',s)=(a-t'')(t''-b), \quad (8)$$

where

$$a=-[(-t)^{1/2}-(-t')^{1/2}]^2,\quad b=-[(-t)^{1/2}+(-t')^{1/2}]^2.$$

The double integral can then be readily evaluated in the high-energy limit.

For large s, (5) becomes

$$F(s,t)=\sum_{n=1}^{\infty}\rho_n(s,t)\left(\frac{s}{s_0}\right)^{\alpha_{cn}(t)}\left(\frac{s_0}{c_n}\right)^{\alpha_{cn}(t)}\times\exp[-i\pi\alpha_{c_n}(t)]\Gamma(1-\alpha_{c_n}(t)), \quad (9)$$

where s_0 is the usual scale parameter. In the Veneziano model, it is naturally chosen to be $1/\alpha'$.

It should be noted that in deriving Eq. (9) we have assumed, as usual, that the $s\to\infty$ limit is taken along a wedge slightly off the real s axis. $\alpha(s)$ is taken as a real function. The effective trajectory will develop an imaginary part when we replace the $\delta(c-c_n)$ factor by a Breit-Wigner form. As in the ordinary dispersion relations, the spectral function $\rho_n(s,t)$ will be real for real s and t. We shall return to these points later.

A typical term on the right-hand side of (6) is of the form

$$\frac{1}{32\pi^2 s}\int_{-\infty}^{0}dt'\int_{-\infty}^{0}dt''\rho_{n1}\rho_{n2}\times\exp[-i\pi\alpha_{n_1}(0)+i\pi\alpha_{n_2}(0)]\Gamma(1-\alpha_{n_1}(0))\times\Gamma(1-\alpha_{n_2}(0))\left(\frac{s}{s_0}\right)^{\alpha_{n1}(0)+\alpha_{n2}(0)+t'/c_{n1}+t''/c_{n2}}\times\left(\frac{s_0}{c_{n1}}\right)^{\alpha_{n1}(0)}\left(\frac{s_0}{c_{n2}}\right)^{\alpha_{n2}(0)}\frac{1}{[(a-t'')(t''-b)]^{1/2}},$$

where $\alpha_{n_2}(0)=\alpha_{cn_1}(0)$, etc. On integration this gives

$$\frac{1}{32\pi^2}\rho_{n1}\rho_{n2}\exp[i\pi(\alpha_{n_2}(0)-\alpha_{n_1}(0))]\Gamma(1-\alpha_{n_1}(0))\times\Gamma(1-\alpha_{n_2}(0))\left(\frac{s_0}{c_{n1}}\right)^{\alpha_{n1}(0)}\left(\frac{s_0}{c_{n2}}\right)^{\alpha_{n2}(0)}\frac{1}{s_0\ln(s/s_0)}\times\frac{c_{n1}c_{n2}}{c_{n1}+c_{n2}}\left(\frac{s}{s_0}\right)^{\alpha_{n1}(0)+\alpha_{n2}(0)-1+t/(c_{n1}+c_{n2})}$$

Now we notice that, if we choose $c_n=n/\alpha'$, the power behavior of the term is precisely that given by the cut due to the exchange of (n_1+n_2) ρ trajectories. This occurs because

$$\alpha_{n_1}(0)=n_1\alpha_0-n_1+1. \quad (10)$$

The term $n=1$ gives the ρ-trajectory contribution. All the higher terms are due to the various cuts. The particular choice of $\alpha_c(s)$ in (2) was made with this fact in mind. The nth term contains a product of all the terms with n_1, n_2 such that $n=n_1+n_2$. Then

$$\alpha_n(t)=n\alpha_0-n+1+\alpha' t/n. \quad (11)$$

Taking the imaginary part of (9), we have

$$\mathrm{Im}F(s,t)=\pi\sum_{n=1}^{\infty}\frac{\rho_n(s,t)}{\Gamma(\alpha_{c_n}(t))}\left(\frac{s}{s_0}\right)^{\alpha_{cn}(t)}\left(\frac{1}{n}\right)^{\alpha_{cn}(t)} \quad (12)$$

Now, within the spirit of the high-energy approximations made above, we can equate (6) and (12) near $t=0$ and obtain the coefficients ρ_n. We find, after some rearrangement,

$$\rho_n(s,t)=\frac{1}{32\pi^2\ln(s/s_0)}\sum_{n_1=1}^{n-1}(-1)^n\cos\pi(n-2n_1)\alpha_0\times\Gamma(1-\alpha_{n_1}(0))\Gamma(1-\alpha_{n-n_1}(0))\left(\frac{1}{n_1}\right)^{\alpha_{n1}(0)-1}\times\left(\frac{1}{n-n_1}\right)^{\alpha_{n-n1}(0)-1}\Gamma(\alpha_n(0))(n)^{\alpha_n(0)-1}\rho_{n1}\rho_{n-n1}. \quad (13)$$

Since we matched the coefficients near $t=0$, $\ln(s/s_0)$ can be readily replaced by $\ln[(s+t)/s_0]$ which will maintain the crossing symmetry. This is consistent since, if we had considered t-channel unitarity for large t and s near zero, we would have found the factor $1/\ln(t/s_0)$. Thus, only s or t dependence in ρ_n comes from the logarithmic factor. This factor, if strictly interpreted, would give rise to logarithmic singularities and nonpolynomial behavior of the Regge residues. However, note that the first term (ρ_1) which represents the contribution of a single Regge pole does not have this factor. The Regge cuts are presumably related to the continuum effects. Hence, nonpolynomial residues are not unexpected. Logarithmic singularities may be pushed into the unphysical sheet, if one wishes, by modifying this factor suitably such that the above expression is reproduced asymptotically. We shall return to this question in Sec. IV.

Equation (13) determines all ρ_n's in terms of ρ_1 which is the coefficient of a single Veneziano term. Defining

$$\tilde{\rho}_n=\frac{\rho_n}{(n)^{\alpha_n(0)-1}\Gamma(\alpha_n(0))}, \qquad (14)$$

we find that (13) becomes

$$\tilde{\rho}_n=X\sum_{n_1=1}^{n-1}\frac{[\cos\pi(n-2n_1)\alpha_0]\tilde{\rho}_{n_1}\tilde{\rho}_{n-n_1}}{\sin\pi n_1\alpha_0\,\sin\pi(n-n_1)\alpha_0}, \qquad (15)$$

where

$$X=1/32\ln[(s+t)/s_0].$$

This recursion relation is straightforward to solve term by term, although it is quite complicated to get an explicit solution.

Now if, instead of taking the full amplitude on the right-hand side of the unitarity relation, we just substitute the imaginary parts, we get the relation

$$\tilde{\rho}_n=X\sum_{n_1=1}^{n-1}\tilde{\rho}_{n_1}\tilde{\rho}_{n-n_1}. \qquad (16)$$

This relation is much simpler than (15) and, in fact, it is quite similar to the one obtained by Amati, Cini, and Stanghellini.[9] These authors discuss diffraction scattering in the multiperipheral model by using the unitarity relation

$$\operatorname{Im}T_{\mathrm{el}}=(T_{\mathrm{in}}^{\dagger}T_{\mathrm{in}}+T_{\mathrm{el}}^{\dagger}T_{\mathrm{el}}), \qquad (17)$$

with the ansatz that the inelastic contribution $T_{\mathrm{in}}^{\dagger}T_{\mathrm{in}}$ is given by the imaginary part of a single Regge-pole term (Pomeranchuk term in their case). On neglecting the real parts of the Pomeranchuk term in iteration, they get the AFS cuts.[2] Applying their method to our case, we recognize that the recusion relation (16) would arise when the second-degree equation

$$Xa^2-a+\lambda\tilde{\rho}_1=0 \qquad (18)$$

is solved as a power series in λ,

$$a=\sum_{n=1}^{\infty}\lambda^n\tilde{\rho}_n. \qquad (19)$$

Solving Eq. (18) and expanding the solution in powers of λ, we can easily read off $\tilde{\rho}_n$ as coefficients of λ^n. This gives

$$\tilde{\rho}_n=\frac{\tilde{\rho}_1\Gamma(n-\frac{1}{2})}{(\sqrt{\pi})\Gamma(n+1)}(4\tilde{\rho}_1X)^{n-1}. \qquad (20)$$

In our case of the ρ trajectory, however, neglect of real parts in iteration is not justified. Hence, we should really use the more complicated recursion relation (15). Except for this fact, our model is similar in spirit to the model of Ref. 9 used within the context of the Veneziano representation.

Thus, all the ρ_n's are determined in terms of ρ_1 and the unitarity is satisfied at high energies in the sense of Ref. 9. This means that the single Veneziano term ($\operatorname{Im}\rho_1\cdots$) becomes the overlap function containing all the contributions of the inelastic states to the imaginary part of the elastic amplitude in the unitarity relation. Regge cuts, which are absent in the original Veneziano representation but are known to be present on both theoretical and experimental grounds, are automatically introduced.

It is encouraging that the entire modified Veneziano series can be built up by the application of the elastic unitarity in the high-energy limit. There are no left-over terms. Each succeeding term gives unitarity corrections to the previous terms.

It should be noted that our "elastic unitary" term itself contains many terms which would normally arise from multiparticle states. The reason is that at each vertex in the integral term, we have used a Veneziano-type Regge term. More contact with the inelastic-multiparticle unitarity can be made, when the Pomeranchuk terms are included in the present scheme.

Finally, we mention that, in principle, the approximations made here can be improved. For example, in evaluating the integrals in (6) one can make Taylor expansions of various terms around $t'=t''=0$ and consider higher-order terms.

In Sec. III we consider the optical-model approach.

III. OPTICAL-MODEL APPROACH

In the optical-model approach, the eikonal representation for the scattering amplitude is given by

$$F(s,t)=16\pi ik(\sqrt{s})\int_0^{\infty}bdb\,J_0(bq)(1-e^{2i\delta(s,b)}). \qquad (21)$$

Here b is the impact parameter and $q=(-t)^{1/2}$. $\delta(s,b)$ is the eikonal function.

The Born approximation is obtained by taking $1-e^{2i\delta(s,b)}\approx-2i\delta(s,b)$ and equating the amplitude to the single Regge-pole amplitude. Then

$$\delta(s,b)=\frac{1}{32\pi k\sqrt{s}}\int_0^{\infty}F_R(s,t)J_0(bq)qdq. \qquad (22)$$

Here $F_R(s,t)$ for a ρ Regge pole is given by

$$F_\rho(s,t)=\frac{i\beta(t)}{\cos\frac{1}{2}\pi\alpha(t)}\left(\frac{s}{s_0}e^{-i\pi/2}\right)^{\alpha(t)}. \qquad (23)$$

Since only values of t near zero make dominant contributions, we shall take $\beta(t)/\cos\frac{1}{2}\pi\alpha(t)$ as approxi-

[9] D. Amati, M. Cini, and A. Stanghellini, Nuovo Cimento **30**, 193 (1963).

mately constant. This assumption makes the subsequent integrations simple. On integration, we find

$$\delta_\rho(s,b)=\frac{i\beta_2 e^{\alpha_0\mu-b^2/4\alpha'\mu}}{64\pi k(\sqrt{s})\alpha'\mu\cos\frac{1}{2}\pi\alpha}, \quad (24)$$

where

$$\mu=|\ln(s/s_0)-\tfrac{1}{2}i\pi|e^{-i\phi},$$

$$\phi=\tan^{-1}\left(\frac{\pi}{2\ln(s/s_0)}\right). \quad (25)$$

As in Ref. 7, we can now expand $e^{2i\delta_\rho}$ in a power series, substitute (24), carry out the integrations, and obtain the following series in the large-s limit:

$$F(s,t)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{nn!}\frac{\beta_2}{\cos\frac{1}{2}\pi\alpha}\left(\frac{\beta_2}{16\pi|\mu|\alpha' s_0\cos\frac{1}{2}\pi\alpha}\right)^{n-1}\times\left(\frac{s}{s_0}\right)^{\alpha_n(t)}e^{i\theta}, \quad (26)$$

where

$$\theta=(n-1)\phi-\tfrac{1}{2}\pi(n\alpha_0-1+\alpha' t/n).$$

The first term of the series corresponds to the ρ contribution and the nth term corresponds to the Regge-cut contribution due to the $n\rho$ exchanges. The power behavior (in s) of the various terms in (26) is exactly the same as that of (9). This suggests that we can determine the coefficients $\rho_n(s,t)$ by equating the imaginary parts of the two series. Clearly, the two series cannot be equated at arbitrary values of t, since the functional dependence on t is different. However, the fact that the various functions are peaked near $t=0$ suggests that we can approximately equate the coefficients of $s^{\alpha_n(t)}$ for $t=0$. We follow this procedure because the idea here is not to construct just an eikonal series but to construct a dispersive Veneziano representation, while taking the optical model as a guide. Then we have

$$\rho_n=\frac{(-1)^{n-1}16\Gamma(\alpha_n(0))(n)^{\alpha_n(0)}}{nn![\ln(s/s_0)]^{n-1}}\left(\frac{\rho_1}{16\Gamma(\alpha_0)\cos\frac{1}{2}\pi\alpha_0}\right)^n\times\cos\left[\frac{n\pi\alpha_0}{2}-(n-1)\tan^{-1}\left(\frac{\pi}{2\ln(s/s_0)}\right)\right]. \quad (27)$$

Here we have expressed all the ρ_n's in terms of ρ_1. ρ_1 and β are related by

$$\rho_1=\beta\Gamma(\alpha_0)/\pi. \quad (28)$$

As discussed in Sec. II, $\ln(s/s_0)$ should be replaced by $\ln[(s+t)/s_0]$.

So far we have ignored the Pomeranchuk (P) trajectory. In recent years it has become increasingly clear that it has a very subtle nature. Indeed, it may be a fixed pole, a fixed cut, a collective effect of a number of moving cuts, or some complicated manifestation of diffraction effects which may not be simply understood in the complex l plane. Hence, iteration of the Pomeranchuk contributions may not be very meaningful. It is quite difficult to build it into a Veneziano-type representation. However, if it is treated like an ordinary Regge trajectory with nonzero slope, it is possible to build up contributions of the cuts arising from exchanges of $\rho+mP$ into the above optical-model approach. We briefly indicate this in the following, although, as mentioned above, such cuts may not have a simple manifestation.

Let the P-exchange amplitude be given by

$$F_P(s,t)=-\frac{\beta_P(t)}{\sin\frac{1}{2}\pi\alpha_P(t)}\left(\frac{s}{s_0}e^{-i\pi/2}\right)^{\alpha_P(t)}. \quad (29)$$

Taking

$$\alpha_P(t)=1+\alpha_P' t, \quad (30)$$

we have

$$\delta_P(s,b)=-\frac{\beta_P e^{\mu-b^2/4\alpha_P'\mu}}{64\pi k(\sqrt{s})\alpha_P'\mu}. \quad (31)$$

Now we take

$$\delta(s,b)=\delta p(s,b)+\delta_\rho(s,b) \quad (32)$$

and find

$$1-e^{2i\delta(s,b)}=1-e^{2i\delta_P(s,b)}-e^{2i\delta_P(s,b)}2i\delta_\rho(s,b)-e^{2i\delta_P(s,b)}[2i\delta_\rho(s,b)]^2/2!+\cdots. \quad (33)$$

The first two terms give rise to the cuts due to the exchange of a number of P's. The third term gives the contribution from the ρ and the cuts due to exchange of the ρ, and a number of P's. Further terms can be associated with exchanges of a number of ρ's and P's. Here we will consider only the third term which gives Regge-cut corrections to the ρ-pole term. Following the procedure outlined above, we find this term to be

$$\bar{F}(s,t)=i\sum_{m=0}^{\infty}(-1)^m\left(\frac{\beta_P}{16\pi\alpha_P' s_0\mu}\right)^n\frac{\beta}{m!\cos\frac{1}{2}\pi\alpha_0}\times\frac{\alpha_P'}{m\alpha'+\alpha_P'}\left(\frac{s}{s_0}e^{-i\pi/2}\right)^{\bar{\alpha}_m(t)}, \quad (34)$$

where $\bar{\alpha}_m(t)$ is the branch point due to $\rho+mP$ exchange and is given by

$$\bar{\alpha}_m(t)=\alpha_0+\frac{\alpha_P'\alpha'}{m\alpha'+\alpha_P'}t=\alpha_0+t/\bar{c}_{m+1}. \quad (35)$$

If we write an equation similar to (5) for these corrections, we have

$$\bar{F}(s,t)=-\sum_{n=1}^{\infty}\bar{\rho}_n(s,t)\frac{\Gamma(1-\bar{\alpha}_n(s))\Gamma(1-\bar{\alpha}_n(t))}{\Gamma(1-\bar{\alpha}_n(s)-\bar{\alpha}_n(t))}. \quad (36)$$

Comparing the imaginary parts of (36) and (34) in the

$s\to\infty$, $t\to 0$ limit, we obtain

$$\bar{\rho}_n(s,t)=\frac{(-1)^{n-1}}{(n-1)!}\left(\frac{\beta_P\alpha'}{16\pi\alpha_P'|\mu|}\right)^{n-1}\left(\frac{(n-1)\alpha'+\alpha_P'}{\alpha_P'}\right)^{\alpha_0-1}\times\frac{\rho_1}{\cos\frac{1}{2}\pi\alpha_0}\cos[\tfrac{1}{2}\pi\alpha_0-(n-1)\phi]. \quad (37)$$

Thus we have obtained different spectral functions in different models. In Sec. IV we discuss various properties of these and also make concluding remarks.

IV. DISCUSSION AND CONCLUDING REMARKS

In Secs. II and III we discussed mainly the two models, (1) AFS and (2) optical, which can give some idea of the nature of the spectral function $\rho_n(s,t)$. The first one, the AFS-type rescattering model, is clearly more satisfying on theoretical grounds. The main defect is, of course, that only the elastic-unitarity part is considered explicitly. (See, however, the remarks towards the end of Sec. II.) The second model, the optical or absorption model, was chosen for (*ad hoc*) comparison because of its success in explaining large-momentum-transfer data and the ease with which the Regge cuts can be introduced. The two models give spectral functions differing in some detail. In particular, the factor $(-1)^{n-1}$ is absent in the first model. Particularly for the P exchanges, this factor makes the signs of the cut contributions different in the two cases because of the absence of other obscuring phase factors. This change of sign may indeed come out when the inelastic unitarity terms are properly taken into account. This fact has been well discussed in the literature.[10] In the case of the cuts due to the exchange of a number of ρ trajectories, the difference between the two cases is obscured by other phase factors. However, note that our cuts[11] contain a factor $1/\Gamma(\alpha_n(t))$. It can be seen that the relative sign between the ρ contribution and the cut contribution ($\rho\rho$ or $\rho P'$ in an exchange-degenerate case) is indeed negative for $0<-t<m_\rho^2$ in the first model but positive in the second model. Thus, it is quite likely that one will not have to rely on the $(-1)^{n-1}$ factor for this negative relative sign, which seems to be consistent with the presently known experimental results.

The role of iteration of the Pomeranchuk trajectory is not clear at present. Yet, in view of its possible relevance, we have given a brief discussion of its inclusion in the optical model. We have, however, not included it in the elastic-unitarity iteration in Sec. II, although it is quite possible to do this.

Now, if we do want to continue the expressions obtained above at high energies to low and intermediate energies, it will be necessary to modify the logarithmic factors. This can be done in a number of ways. For example, $\ln[(s+t)/s_0]$ can be replaced by

$$2\ln\left[\left(\frac{s+t}{s_0}\right)^{1/2}+\lambda\right]=2\ln\left[\left(\frac{4m_\pi^2-u}{s_0}\right)^{1/2}+\lambda\right],$$

where λ is an arbitrary cutoff. It can be chosen to be at some reasonable energy ($\lambda\approx 3$ to 4, $s_0\approx 1$ GeV^2) which separates the usual low-energy resonant region from the high-energy Regge region. Actually, because of the logarithmic factor, the results will change very little by reasonable variations of λ. Also, in the low-energy region, the correction terms to the original Veneziano amplitude will not show any logarithmic variation with respect to s or t, and hence the difficulty of an unwanted infinity of "ancestor" trajectories will be removed. For very large values of s, t, or u, cuts will introduce logarithmic factors as they should. Furthermore, for $\lambda>0$, the logarithmic branch points will occur only on unphysical sheets.

Now a possible difficulty is that our spectral functions seem to blow up in some special cases when $\alpha_n(0)$ are negative integers or zero. This happens, for example, when $\alpha(0)$ is exactly $\frac{1}{2}$. This difficulty is superficial, however, and it arises because of the approximations made in equating (12) for $t=0$ to various series obtained in different cases. For a smooth $\rho_n(s,t)$ the term in question has a zero at $t=0$. Then, within the spirit of the approximations made, this difficulty can be overcome by equating the series at $t=\tilde{t}$ instead of $t=0$, where $\tilde{t}$ (a number) is the location of the maximum of

$$e^{\alpha_n(t)\ln(s/s_0)}/|\Gamma(\alpha_n(t))|.$$

For large s this will be close to zero. $\rho_n(s,t)$ will then be proportional to $\Gamma(\alpha_n(\tilde{t}))$ and there will not be any divergence.

So far we have dealt with only the discrete approximations $\rho_n(s,t)$ to a continuous function $\rho(c,s,t)$. But once we obtain $\rho_n(s,t)$, we can invoke Carlson's theorem to obtain the continuous function $\rho(c,s,t)$ as in the usual Regge theory. Then we can obtain the integral representation (3). This has a branch point at $c=c_{\rm th}$ ($\bar{c}=\bar{c}_{\rm th}$) which gives rise to branch points at[12]

$$s_j=(j-1)c_{\rm th}+c_{\rm th}^2/4m_\rho^2 \quad (38)$$

and

$$\bar{s}_j=(j-\tfrac{1}{2})\bar{c}_{\rm th}, \quad j=1,2,3,\ldots.$$

The values of $c_{\rm th}$ ($\bar{c}_{\rm th}$) can be determined by requiring that the first branch point occurs at $s_1=4m_\pi^2$ ($\bar{s}_1=4m_\pi^2$). This gives $c_{\rm th}=4m_\pi m_\rho$ and $\bar{c}_{\rm th}=8m_\pi^2$. To enforce the square-root behavior at the branch point, it will be necessary to multiply the spectral functions by a factor

[10] J. Finkelstein and M. Jacob, Nuovo Cimento **56A**, 681 (1968); L. Caneschi, Phys. Rev. Letters **23**, 254 (1969).

[11] In a recent work we have suggested that this fact could explain recurring minima and polarizations in various reactions. See K. V. Vasavada, Phys. Letters **34B**, 214 (1971).

[12] Here and in the following we have taken $\alpha_0=\frac{1}{2}$ and $\alpha'=1/2m_\rho^2$. The bars refer to the integral representations with Pomeranchuk modifications.

$[(c-c_{\rm th})/c]^{1/2}$ ($[(\bar{c}-\bar{c}_{\rm th})/\bar{c}]^{1/2}$). This is a permissible modification, since c ($\bar{c}$)$\gg c_{\rm th}$ for the discrete values from which the representation was obtained. In particular, we have $c_n=2m_\rho^2 n$ and $\bar{c}_n=c_n$ when $\alpha_P'=\alpha'$.

Now, by duality the above branch points at s_j ($\bar{s}_j$) give rise to Regge cuts at $J=\alpha_{c_{\rm th}}(t)$, $\alpha_{c_{\rm th}}(t)-1$, ... In fact, for the zero-mass pions we can identify the first branch point with the Pomeranchuk trajectory. The vacuum quantum numbers will not come out automatically and some canceling terms will be required to prevent its exchange in channels not having these quantum numbers.

For unitarization at low energy, it is desirable to have second sheet resonance poles corresponding to particles on the ρ trajectory. This can be done by replacing the $\delta(c-c_1)$ by the Breit-Wigner form $\pi^{-1}\gamma/[(c-c_1)^2+\gamma^2]$ which becomes a δ function in the limit $\gamma\to 0$. Thus, we can multiply the spectral functions obtained above by this factor as reasonable extensions of the zero width expressions. This factor will give rise to unphysical-sheet poles at $c=c_1\pm i\gamma$ and correspondingly at $s_j^R=2m_\rho^2(j-\frac{1}{2})\pm i\gamma(j-\frac{1}{2})$, $j=1, 2, 3, \ldots$. Thus, as s^R increases, the corresponding widths of the resonances will increase as $\sqrt{s^R}$. It is interesting to note that similar results for the widths have been derived by some authors when they attempt to unitarize the Veneziano model by adding imaginary parts to α.[13] One difficulty, of course, is that all the degenerate poles have the same total widths. However, there is still some improvement over the original Veneziano model which gave zero total widths. Finally, because of the $\pm i\gamma$ factor, the Regge behavior will be obtained as $s\to\infty$ along the real axis and not just along a slightly complex direction, as required in the original Veneziano formula.

An interesting property of the original Veneziano amplitude for the π-π scattering is the satisfaction of the Adler's self-consistency condition.[8] This requires the amplitude to vanish at $s=t=u=m_\pi^2$. If $\alpha(m_\pi^2)=\frac{1}{2}$, the one-term Veneziano amplitude automatically satisfies the condition because of the blowing up of the Γ function in the denominator. Our modification also satisfies this condition approximately, since

$$\alpha_n(0)=1+n(\alpha_0-1)=1-\tfrac{1}{2}n$$

[13] See, e.g., R. Z. Roskies, Phys. Rev. Letters 21, 1851 (1968); 22, 265(E) (1969).

and

$$\bar{\alpha}_n(0)=\alpha_0=\tfrac{1}{2},$$

and the point $t=m_\pi^2$ is quite close to $t=0$.

One more point that needs to be mentioned here is the satisfaction of the Cerulus-Martin bound. As in Ref. 7, we can readily find by using the saddle-point method that the series or integral form of $F(s,t)$ has the bound

$$F(s,t)\xrightarrow[s\to\infty;\,\cos\theta_s\text{ fixed}]{} e^{-f(\cos\theta_s)\sqrt{|t|}},$$

instead of $e^{-f(\cos\theta_s)|t|}$ as in the case of the original Veneziano formula. This is true for both the models discussed in Secs. II and III. The Cerulus-Martin bound has been established on general theoretical grounds and is supported to a certain extent by large-angle proton-proton scattering data. In this respect also, the modified representation proposed here seems to be preferable to the original Veneziano representation.

In conclusion, we note that the unitarization procedures discussed here seem to have many interesting and encouraging properties. In particular, the Regge cuts and the high-energy unitarity condition are automatically built in without destroying the crossing symmetry. As mentioned before, K-matrix and N/D unitarizations do destroy some crossing symmetry. Because of the weak dependence of the spectral functions on s and t, the double spectral functions would not be quite correct. However, hopefully, the spectral functions obtained here may be some reasonable approximations to a more complete theory. Remembering that the Regge cuts arise from diagrams containing multiparticle intermediate states or from analysis of multiparticle unitarity conditions, it seems particularly appealing that they play a role in unitarization of the Veneziano model.

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