Relativistic Quantum Mechanics of Dyons. Exact Solution

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A relativistic quantum mechanics of dyons (hypothetical particles endowed with both electric and magnetic charges) is formulated. The hydrodynamic formulation of quantum mechanics is used to overcome all problems created by the simultaneous presence of both electric and magnetic charges. In the case of one dyon moving in the electromagnetic field of another dyon the exact solutions are found and the energy levels are determined.

I. INTRODUCTION

In view of the recent interest in the properties of dyons (hypothetical particles endowed with both electric and magnetic charges) we thought it worthwhile to work out exact solutions describing the motion of a dyon in the electromagnetic field created by another dyon.

Owing to the nonexistence of a straightforward extension of the canonical formalism to the case of magnetic monopoles, the standard formulation of quantum mechanics based on the Schrödinger equation cannot be used.

Our approach is that of the hydrodynamic formulation of quantum mechanics. In this formulation, as was shown in our earlier work, the electromagnetic potentials do not appear and the description of dyons becomes both straightforward and unambiguous. In the present paper we follow closely the procedure described in Ref. 1, but we replace the Schrödinger equation of the nonrelativistic theory by the Klein-Gordon equation. We solve the equations of motion and find the energy levels for a system composed of a spinless dyon moving in the field of another spinless pointlike dyon. The same formula for the energy levels was recently derived by group-theoretical methods by Berrondo and McIntosh[2] and from the classical mechanics of dyons by Rzążewski,[3] who applied the Bohr-Sommerfeld quantization conditions.

II. HYDRODYNAMIC FORMULATION OF RELATIVISTIC QUANTUM MECHANICS

The motion of a spinless electrically charged particle is described in relativistic quantum mechanics by the equation

$$[(i\hbar\partial_\mu-(e/c)A_\mu)^2-(Mc)^2]\phi(x)=0. \tag{1}$$

We express the complex wave function $\phi(x)$ in terms of the real amplitude R and the real phase S,

$$\phi(x)=R(x)\exp[(i/\hbar)S(x)], \tag{2}$$

and we define the hydrodynamic variables, the density ρ, and the four-velocity v_μ as follows:

$$\rho\equiv R^2, \tag{3}$$

$$v_\mu\equiv-\frac{1}{M}\left(\partial_\mu S+\frac{e}{c}A_\mu\right). \tag{4}$$

In terms of ρ and v_μ, Eq. (1) reads

$$\partial_\mu(\rho v^\mu)=0, \tag{5}$$

$$v_\mu v^\mu-(\hbar^2/M^2)\rho^{-1/2}\Box\rho^{1/2}-c^2=0. \tag{6}$$

It is convenient to differentiate Eq. (6) and to rewrite the resulting equation with the help of Eq. (4) in the form

$$v^\mu\partial_\mu v_\lambda=\frac{\hbar^2}{2M^2}\partial_\lambda(\rho^{-1/2}\Box\rho^{1/2})+\frac{e}{Mc}f_{\lambda\mu}v^\mu. \tag{7}$$

Equations (5) and (7) are those of the relativistic hydrodynamics of the charged fluid with the additional quantum force term. These equations are equivalent to the original equation (1) provided the velocity field satisfies the following phase integrability condition:

$$\oint d\xi^\mu\left(Mv_\mu+\frac{e}{c}A_\mu\right)=2\pi\hbar m, \tag{8}$$

where the integral is evaluated along any closed contour and m is an integer. Following our method described in Ref. 1, we transform the quantization condition (8) with the help of the Stokes theorem to the form which does not involve the potentials,

$$\int_\Sigma dn^{\mu\nu}\left[M(\partial_\mu v_\nu-\partial_\nu v_\mu)+\frac{e}{c}f_{\mu\nu}\right]=2\pi\hbar m, \tag{9}$$

where Σ is any two-dimensional surface. The quantization condition (9) can be satisfied only if the integrand vanishes everywhere except on some two-dimensional surfaces where it has $\delta^{(2)}$-like singularities whose strength is quantized.

In order to generalize Eqs. (5)–(7) and (9) to the case of a dyon, we must include the second force term $(g/Mc)\tilde{f}_{\lambda\mu}v^\mu$ in Eq. (7) and the corresponding term

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[1] I. Bialynicki-Birula and Z. Bialynicka-Birula, preceding paper, Phys. Rev. D 3, 2410 (1971).

[2] M. Berrondo and H. V. McIntosh, J. Math. Phys. 11, 125 (1970) [see also A. O. Barut and G. L. Bornzin (unpublished)].

[3] K. Rzążewski, Acta Phys. Polon. (to be published).

$(g/c)\hat{f}_{\mu\nu}$ in the quantization condition (9). Finally the equations of motion for the dyon read

$$\partial_\mu(\rho v^\mu)=0\,, \quad (10)$$

$$v^\mu\partial_\mu v_\lambda=\frac{\hbar^2}{2M^2}\partial_\lambda(\rho^{-1/2}\Box\rho^{1/2})+\frac{1}{Mc}(ef_{\lambda\mu}+g\hat{f}_{\lambda\mu})v^\mu\,, \quad (11)$$

$$\partial_\mu v_\nu-\partial_\nu v_\mu+(1/Mc)(ef_{\mu\nu}+g\hat{f}_{\mu\nu})=0\,. \quad (12)$$

The last equation holds everywhere except on some two-dimensional surfaces in the Minkowski space.

Even though the wave function of the dyon cannot be defined, the quantum theory of dyons based on Eqs. (10)–(12) possesses all the basic properties of standard quantum theory (cf. Ref. 1).

III. DYONIC ATOM

We shall now assume that the external field is created by a pointlike dyon whose electric and magnetic charges are e_1 and g_1,

$$\mathbf{E}=e_1\mathbf{r}/r^3\,,\quad \mathbf{B}=g_1\mathbf{r}/r^3\,. \quad (13)$$

Quantum-mechanical solutions with given energy and the z component of the angular momentum correspond to stationary and cylindrically symmetric motions of the probabilistic fluid.

For such motions

$$\rho=\rho(r,\vartheta)\,, \quad (14)$$

$$(v^\mu)=(v^0(r,\vartheta),\mathbf{e}_\varphi v_\varphi(r,\vartheta))\,. \quad (15)$$

Equation (12) enables us to determine v^0 and v_φ everywhere except on the singular line[4] $\vartheta=0,\,\pi$.

$$v^0(r,\vartheta)=v^0(r)=C_1-\hbar\alpha/Mr\,, \quad (16)$$

$$v_\varphi(r,\vartheta)=\frac{\hbar}{Mr}\frac{C_2+\beta\cos\vartheta}{\sin\vartheta}\,, \quad (17)$$

where

$$\alpha=(ee_1+gg_1)/\hbar c\,, \quad (18)$$

$$\beta=(ge_1-eg_1)/\hbar c\,, \quad (19)$$

and C_1 and C_2 are two integration constants. The quantization condition requires that $C_2\pm\beta$ must be integers (hence 2β is an integer),

$$C_2=m-\beta\,. \quad (20)$$

Under our assumptions, Eq. (6) for $\rho^{1/2}$ reads

$$\left\{\frac{\hbar^2}{M^2r^2}\left[\partial_r r^2\partial_r+\frac{1}{\sin\vartheta}\partial_\vartheta\sin\vartheta\,\partial_\vartheta-\left(\frac{m-\beta+\beta\cos\vartheta}{\sin\vartheta}\right)\right]^2+\left(C_1-\frac{\hbar\alpha}{Mr}\right)^2-c^2\right\}\rho^{1/2}=0\,. \quad (21)$$

[4] This line in space becomes a two-dimensional surface in space-time.

For ordinary particles, when $\beta=0$, this equation becomes identical with the appropriate form of the Klein-Gordon equation provided we identify McC_1 with the total energy,

$$C_1=E/Mc\,. \quad (22)$$

Equation (21) can be solved by the separation of variables method,

$$\rho^{1/2}(r,\vartheta)=G(r)\Theta(\vartheta)\,. \quad (23)$$

Square-integrable solutions of the angular part of this equation were obtained by Malkus[5]:

$$\left[\frac{1}{\sin\vartheta}\partial_\vartheta\sin\vartheta\,\partial_\vartheta-\left(\frac{m-\beta+\beta\cos\vartheta}{\sin\vartheta}\right)^2\right]\Theta(\vartheta)=-L^2\Theta(\vartheta)\,, \quad (24)$$

where

$$L^2=l'(l'+1)-\beta^2\,, \quad (25)$$

$$l'=l+\tfrac{1}{2}(|m|+|m-2\beta|)\,, \quad (26)$$

and l is a natural number.

The radial part of Eq. (21), which determines the energy levels, reads

$$\left[\frac{1}{x^2}\partial_x x^2\partial_x-\frac{L^2-\alpha^2}{x^2}-\frac{2\alpha E}{\hbar c\mu x}-\frac{1}{4}\right]G(x)=0\,, \quad (27)$$

where

$$x=\mu r \quad (28)$$

and

$$\mu=2\left[\left(\frac{Mc}{\hbar}\right)^2-\left(\frac{E}{\hbar c}\right)^2\right]^{1/2}. \quad (29)$$

This is the same radial equation as for the nonrelativistic hydrogen atom. Square-integrable solutions are obtained only if the coefficients $L^2-\alpha^2$ and $2\alpha E/\hbar c\mu$ are related through the formula

$$2\alpha E/\hbar c\mu=n'+\tfrac{1}{2}+(L^2-\alpha^2+\tfrac{1}{4})^{1/2}\,, \quad (30)$$

where n' is a natural number. After solving this equation with respect to E, we obtain the following formula for the energy levels:

$$E(n',l')=Mc^2\left(1+\frac{\alpha^2}{\{n'+\frac{1}{2}+[(l'+\frac{1}{2})^2-\beta^2-\alpha^2]^{1/2}\}^2}\right)^{-1/2}. \quad (31)$$

It coincides with the formulas obtained by different methods in Refs. 2 and 3.

We are far from claiming that only the hydrodynamic formulation of quantum mechanics is suitable for de-

[5] W. Malkus, Phys. Rev. **83**, 899 (1951).

scribing particles with both electric and magnetic charges. One can also use the standard formulation with proper modifications to obtain equivalent results. However, it seems to us that our method leads most directly to final conclusions without any need for such cumbersome artifices as potentials and wave functions with singularities along strings. Owing to its manifest gauge invariance, our method can also serve as a convenient consistency check for other approaches.

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Dual Theory for Free Fermions

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A wave equation for free fermions is proposed based on the structure of the dual theory for bosons. Its formal properties preserve the role played by the Virasoro algebra. Additional Ward-like identities, compatible with the equation, are shown to exist. Its solutions lie on linear trajectories. In particular, the parent is shown to be doubly degenerate, but these solutions lie on different sheets of the cut j plane.

INTRODUCTION

In spite of its obvious theoretical appeal, the dual model[1] has been denied full acceptance (credibility) because of its failure to include fermions. In this paper we present an extension of the model to encompass half-integer-spin states by making use of a structure evident in the dual theory of free bosons.[2] Namely, we found that the following view of duality led to no contradiction with existing results: Each "free" boson appearing in the theory is a state of a complex system. Its structure can be parametrized in terms of an internal motion which is periodic in an internal time coordinate so that each observable of the system is the average over a cycle of the internal motion of suitably generalized operators. In this way, operators appearing in the description of point particles in conventional theories must be thought of as averages over some internal motion when applied to a hadronic system. The system then becomes a point particle in the limit of the internal cycle going to zero. These precepts are illustrated by their application to the bosonic case in Sec. I. We use these guidelines to introduce a generalization of the Dirac matrices and postulate a Dirac wave equation for the free fermionic system. Its formal properties are studied in Sec. II. Section III will be devoted to a detailed study of its solutions.

I. BOSON CASE

In order to set the notation and illustrate the ideas behind our interpretation, it is desirable to first consider the (already known) free-boson theory. The free hadronic system is described in terms of an internal motion generated by the Nambu[3] Hamiltonian

$$H_B=\tfrac{1}{2}\sum_{n=0}^{\infty}\left[p^{(n)}\cdot p^{(n)}+\omega_n{}^2 q^{(n)}\cdot q^{(n)}\right], \qquad (1.1)$$

with

$$\omega_{n+1}-\omega_n=\omega\,,\quad n=0,\,1,\,2,\,\ldots \qquad (1.2)$$

and the normal-mode coordinates are four-vector operators satisfying the usual commutation relations

$$\begin{aligned}&[q_\alpha{}^{(n)},q_\beta{}^{(m)}]=[p_\alpha{}^{(n)},p_\beta{}^{(m)}]=0\,,\\&[q_\alpha{}^{(m)},p_\beta{}^{(n)}]=-ig_{\alpha\beta}\delta^{m,n}\,,\quad m,n=0,\,1,\,\ldots\end{aligned} \qquad (1.3)$$

where we use $g_{\alpha\beta}=(1,\,-1,\,-1,\,-1)$ for the Lorentz metric. The internal system carries a total momentum

$$P_\mu=\sum_{n=0}^{\infty}p_\mu{}^{(n)} \qquad (1.4)$$

corresponding to a coordinate

$$Q_\mu=\sum_{n=0}^{\infty}q_\mu{}^{(n)}\,. \qquad (1.5)$$

The variable τ which describes the evolution of the internal motions is introduced by means of the Heisenberg equations

$$[H_B,f]=i(df/d\tau)\,, \qquad (1.6)$$

where f is any operator. It is important to note that in

[1] See G. Veneziano, in Proceedings of the International School of Physics "Ettore Majorana," Erice, Italy, 1970 (unpublished).

[2] P. Ramond, National Accelerator Laboratory Report No. THY 7, 1970 (unpublished).

[3] Y. Nambu, University of Chicago Report No. EFI 69-64, 1969 (unpublished); see also L. Susskind, Yeshiva University Reports, 1969 (unpublished); S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters **29B**, 679 (1969).