However, if we know one more scattering amplitude where the bound system of particle 2 in the potential V_2 is left in a different state $\psi_1(x_2,y_2,z_2)$, then $\bar{T}_1$ and $\bar{T}_2$ can be determined from two different linear combinations of the form (A8). Once $\bar{T}_1$ and $\bar{T}_2$ are determined, all the scattering amplitudes can be found from (4.13).

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Magnetic Monopoles in the Hydrodynamic Formulation of Quantum Mechanics*

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The nonrelativistic quantum theory of a particle having both electric and magnetic charges moving in an arbitrary external electromagnetic field is presented. The theory is based on the hydrodynamic formulation of quantum mechanics. Dirac's quantization condition for the electric and magnetic charges is rederived as a consistency condition for the motion of the probability fluid. Neither the wave function nor the electromagnetic potential, which were the source of ambiguities in all other formulations, appears in our approach. Nevertheless, this theory has all the essential features of the standard quantum mechanics, including the superposition principle.

I. INTRODUCTION

THE main source of difficulties in formulating the quantum theory of particles carrying both electric and magnetic charges is the ambiguity in the definition of the electromagnetic potential.[1] One could hopefully avoid all these difficulties if one could develop an equivalent formulation of quantum theory in which, instead of the electromagnetic potential, only the field strengths appear. Such a formulation based on the hydrodynamic form of the Schrödinger equation will be presented here. In the absence of magnetic monopoles this form of quantum mechanics is completely equivalent to the Schrödinger theory. The generalization to include magnetic monopoles is very natural and it brings about a full symmetry between electricity and magnetism. The generalized theory will be shown to possess all the basic properties of the quantum theory, including the superposition principle; however, an equivalent description in terms of a unique wave function is no longer possible.

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[1] P. A. M. Dirac, Phys. Rev. **74**, 817 (1948); N. Cabibbo and F. Ferrari, Nuovo Cimento **23**, 1147 (1962); A. S. Goldhaber, Phys. Rev. **140**, B1407 (1965); J. Schwinger, *ibid.* **144**, 1087 (1966); **173**, 1536 (1968); B. Zumino, in *Strong and Weak Interactions—Present Problems*, edited by A. Zichichi (Academic, New York, 1966); D. Zwanziger, Phys. Rev. **176**, 1480 (1968); **176**, 1489 (1968).

II. HYDRODYNAMIC FORMULATION OF QUANTUM MECHANICS

As was observed by Madelung,[2] the Schrödinger equation can be replaced by a set of four hydrodynamic-like equations. In the presence of an external electromagnetic field those equations take on the form

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0\,, \tag{1}$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)+\frac{\hbar^2}{2m^2}\nabla(\rho^{-1/2}\Delta\rho^{1/2})\,, \tag{2}$$

where the density field $\rho(\mathbf{r},t)$ and the velocity field $\mathbf{v}(\mathbf{r},t)$ are related to the modulus and the phase of the wave function and the vector potential $\mathbf{A}$ in the following manner:

$$\psi(\mathbf{r},t)=R(\mathbf{r},t)\exp[(i/\hbar)S(\mathbf{r},t)]\,, \tag{3}$$

$$\rho(\mathbf{r},t)\equiv R^2(\mathbf{r},t)\,, \tag{4}$$

$$\mathbf{v}(\mathbf{r},t)\equiv\frac{1}{m}\left(\nabla S(\mathbf{r},t)-\frac{e}{c}\mathbf{A}(\mathbf{r},t)\right). \tag{5}$$

In the standard formulation there is one-to-one correspondence between the state of the system and a set of normalized wave functions differing by a constant phase

[2] E. Mandelung, Z. Physik **40**, 322 (1926).

factor.[3] In the hydrodynamic formulation the state will be determined by the set of four real functions ρ and $\mathbf{v}$. This description is completely equivalent to the previous one if and only if the velocity field obeys the following auxiliary condition:

$$\Gamma \equiv \oint d\mathbf{l}\cdot\left(m\mathbf{v}+\frac{e}{c}\mathbf{A}\right)=2\pi n\hbar\,,\qquad n=0,\,\pm1,\,\pm2,\,\ldots\,. \tag{6}$$

Using the Stokes theorem one can transform this condition to a new form, which contains only the field strength:

$$\Gamma=\int_\Sigma d\mathbf{n}\cdot\left(m\nabla\times\mathbf{v}+\frac{e}{c}\mathbf{B}\right)=2\pi n\hbar\,, \tag{7}$$

where Σ is an arbitrary two-dimensional surface. This vorticity quantization condition means that the integrand $m\nabla\times\mathbf{v}+(e/c)\mathbf{B}$ vanishes everywhere except on certain singular lines where it has singularities of the $\delta^{(2)}$ type. Condition (7) is consistent with the equations of motion (1) and (2) because, on account of Eq. (2), the material derivative of Γ vanishes,[4] i.e.,

$$d\Gamma/dt=0\,. \tag{8}$$

Clearly the expectation value of every physical quantity can be expressed in terms of ρ and $\mathbf{v}$. We will now express also the transition probability and the superposition of two quantum states in terms of the hydrodynamic variables. The transition probability P_{12} between two states, which in the standard formulation is

$$P_{12}=|(\psi_1|\psi_2)|^2\,, \tag{9}$$

now has the form

$$P_{12}=\left|\int d^3r(\rho_1\rho_2)^{1/2}\exp\left(\frac{i}{\hbar}m\int_{\mathbf{r}_0}^{\mathbf{r}}d\mathbf{l}\cdot(\mathbf{v}_2-\mathbf{v}_1)\right)\right|^2\,, \tag{10}$$

where the line integral can be evaluated along any curve not passing through singular vorticity lines. On account of the quantization condition (7), the result does not depend on the integration contour. The change of the initial point $\mathbf{r}_0$ affects only the unobservable over-all phase of the integral.

Given two states $(\rho_1,\mathbf{v}_1)$, $(\rho_2,\mathbf{v}_2)$ in the hydrodynamic formulation, one can form a three-parameter family of states (not necessarily normalized) described by $(\rho_3,\mathbf{v}_3)$ which are defined by the following formulas:

$$\rho_3=\alpha_1^2\rho_1+\alpha_2^2\rho_2+2\alpha_1\alpha_2(\rho_1\rho_2)^{1/2}\cos\Phi\,, \tag{11}$$

$$\begin{aligned}v_3=&[\alpha_1^2\rho_1+\alpha_2^2\rho_2+2\alpha_1\alpha_2(\rho_1\rho_2)^{1/2}\cos\Phi]^{-1}\\&\times[\alpha_1^2\rho_1\mathbf{v}_1+\alpha_2^2\rho_2\mathbf{v}_2+\alpha_1\alpha_2(\mathbf{v}_1+\mathbf{v}_2)(\rho_1\rho_2)^{1/2}\cos\Phi\\&+(\hbar/m)\alpha_1\alpha_2(\rho_1^{1/2}\nabla\rho_2^{1/2}-\rho_2^{1/2}\nabla\rho_1^{1/2})\sin\Phi]\,,\end{aligned} \tag{12}$$

where

$$\Phi(\mathbf{r},t)=\frac{m}{\hbar}\int_{\mathbf{r}_0}^{\mathbf{r}}d\mathbf{l}\cdot(\mathbf{v}_2-\mathbf{v}_1)+\beta\,. \tag{13}$$

Comparing these formulas with the usual form of the superposition of two wave functions,

$$\psi_3=c_1\psi_1+c_2\psi_2\,, \tag{14}$$

we find that

$$\alpha_1=|c_1|\,,\quad \alpha_2=|c_2|\,, \tag{15}$$

and β is the phase difference between the wave functions ψ_1 and ψ_2. As in the usual formulation, the superposition of two solutions of the equations of motion satisfies again the equations of motion, although this property is now not so obvious.

Equations (1), (2), (7), and (10)–(13) form the basic set of equations in the hydrodynamic formulation of quantum mechanics. This formulation is completely equivalent to the standard one. It is best suited for the generalization to include magnetic monopoles since the electromagnetic potentials appear nowhere in the equations.

III. GENERALIZATION TO MAGNETIC MONOPOLES

If magnetic monopoles are present only as sources of the external electromagnetic field, then there is no need even to modify our basic equations. However, the consistency condition (8) will hold only when the strength of the magnetic monopole is quantized. We will study this condition later in the more general case, when also the quantized particle in addition to its charge carries a magnetic monopole. In this case we obtain the basic set of equations by simply adding to the electric charge terms the corresponding monopole terms which are obtained from the electric terms by the substitutions

$$e\to g\,,\quad \mathbf{E}\to\mathbf{B}\,,\quad \mathbf{B}\to-\mathbf{E}\,, \tag{16}$$

where g is the magnetic charge. The equations of motion and the vorticity quantization condition now read

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0\,, \tag{17}$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)+\frac{g}{m}\left(\mathbf{B}-\frac{1}{c}\mathbf{v}\times\mathbf{E}\right)+\frac{\hbar^2}{2m^2}\nabla(\rho^{-1/2}\Delta\rho^{1/2})\,, \tag{18}$$

$$\Gamma\equiv\int_\Sigma d\mathbf{n}\cdot\left(m\nabla\times\mathbf{v}+\frac{e}{c}\mathbf{B}-\frac{g}{c}\mathbf{E}\right)=2\pi n\hbar\,, \tag{19}$$

[3] There is still an arbitrariness of the phase of the wave function due to the freedom in the choice of the vector potential.

[4] $$\frac{d}{dt}\int_\Sigma d\mathbf{n}\cdot\mathbf{a}=\int_\Sigma d\mathbf{n}\cdot\left(\frac{d\mathbf{a}}{dt}-(\mathbf{a}\cdot\nabla)\mathbf{v}+(\nabla\cdot\mathbf{v})\mathbf{a}\right).$$

[See, for example, C. Truesdell and R. Toupin, *The Classical Field Theory*, in *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1960), Vol. III/1, p. 345.]

whereas the formulas for the transition probability and the superposition remain unchanged.

To study the consistency of the quantization condition (19) with the equations of motion, we will evaluate the material derivative of Γ. With the use of Eqs. (17), (18) and generalized Maxwell equations,

$$-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}+\nabla\times\mathbf{B}=\frac{1}{c}\mathbf{j}_e, \tag{20}$$

$$\nabla\cdot\mathbf{E}=\rho_e, \tag{21}$$

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}-\nabla\times\mathbf{E}=\frac{1}{c}\mathbf{j}_m, \tag{22}$$

$$\nabla\cdot\mathbf{B}=\rho_m, \tag{23}$$

we obtain

$$\frac{d\Gamma}{dt}=\int_\Sigma d\mathbf{n}\cdot\left(-\frac{e}{c}\mathbf{j}_m+\frac{g}{c}\mathbf{j}_e+\frac{e}{c}\rho_m\mathbf{v}-\frac{g}{c}\rho_e\mathbf{v}\right). \tag{24}$$

If we insist on this expression vanishing, we obtain the proportionality of the electric and magnetic charges and currents:

$$e\rho_m=g\rho_e, \tag{25}$$

$$e\mathbf{j}_m=g\mathbf{j}_e. \tag{26}$$

These conditions imply the universality of the ratio of the electric to the magnetic charges and lead to a theory which is not essentially different from the usual theory with no magnetic charges. However, the vanishing of $d\Gamma/dt$ is not a necessary condition for Eq. (19) to be consistent with the equations of motion. The value of Γ can also change in time discontinuously from one value of n to another, i.e.,

$$d\Gamma/dt=2\pi\hbar\sum_i \Delta n_i\delta(t-t_i). \tag{27}$$

This condition will be satisfied if and only if the sources of the electromagnetic field are pointlike,

$$\rho_e=\sum_i e_i\delta^{(3)}(\mathbf{r}-\mathbf{r}_i), \tag{28}$$

$$\mathbf{j}_e=\sum_i e_i\mathbf{v}_i\delta^{(3)}(\mathbf{r}-\mathbf{r}_i), \tag{29}$$

$$\rho_m=\sum_i g_i\delta^{(3)}(\mathbf{r}-\mathbf{r}_i), \tag{30}$$

$$\mathbf{j}_m=\sum_i g_i\mathbf{v}_i\delta^{(3)}(\mathbf{r}-\mathbf{r}_i), \tag{31}$$

and the electric and magnetic charges of particles satisfy the Dirac[5] conditions in the weaker form given by Schwinger and Zwanziger[1]:

$$(eg_i-ge_i)/4\pi\hbar c=\tfrac{1}{2}n_i,\quad n_i=0,\pm1,\pm2,\ldots. \tag{32}$$

IV. DISCUSSION

The quantum theory of magnetic monopoles described in this paper can be regarded as an explicit realization of Dirac's idea presented in his classic paper.[5] In that paper Dirac pointed out that the existence of a well-defined phase factor of the wave function is not necessary for a consistent and complete quantum theory. We believe that the existence of the wave function obeying the Schrödinger equation is not a precondition for a quantum theory. In the presence of magnetic monopoles the wave function cannot be introduced in an unambiguous way, but we can still have a complete theory having all essential properties of the standard quantum mechanics.

Our formulation of the quantum theory of particles having both electric and magnetic charges is not restricted to the nonrelativistic domain and to the one-particle states. Generalizations to the Klein-Gordon particle with magnetic monopole and to many nonrelativistic particles are straightforward and will be presented elsewhere. However, we have not been able so far to include the spin in the hydrodynamic description.

[5] P. A. M. Dirac, Proc. Roy. Soc. (London) **A133**, 60 (1931).