High-Energy Potential Scattering. II*

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In view of the advances of knowledge, both experimental and theoretical, about hadronic scattering processes at high energies, the potential-theory model studied before is found to exhibit a number of realistic features. Because partial differential equations can be dealt with in a much more straightforward manner than operator equations, this potential-theory model is useful in understanding some features of high-energy processes. The problem of exponentiation, i.e., the possible appearance of an exponential factor, is studied in detail. It is found that the simple-exponentiation form, found recently in field theories within certain approximations, does not hold in general when the scattering particles have a finite size or internal degrees of freedom. This result in turn is applied to the field-theoretical cases to understand further the nature of the approximations involved.

1. INTRODUCTION

NEARLY fifteen years ago, one of us studied systematically a potential model for high-energy scattering.[1] Consider the scattering of a plane wave by a potential $V=V(x,y,z)$ such that the wave function $\psi=\psi(x,y,z)$ satisfies the Schrödinger equation

$$(\nabla^2+k^2-V)\psi=0\,, \tag{1.1}$$

with

$$\psi^{\mathrm{inc}}=e^{ikz}\,. \tag{1.2}$$

(We have used notation slightly different from that of I.) In Paper I, we use the following limit as the model for high-energy scattering:

$$k\to\infty,\quad V/k\quad \text{fixed}\,. \tag{1.3}$$

This limit has the following properties provided that V is neither too singular nor long-ranged: (i) The total cross section σ approaches a finite value; and (ii) if we define the Mandelstam variable[2]

$$t=-(2k\sin\tfrac{1}{2}\theta)^2\,, \tag{1.4}$$

where θ is the scattering angle, then $d\sigma/dt$ approaches a finite value for fixed t.

Although by no means clear at the time when paper I was written, there is by now impressive experimental evidence[3] that properties (i) and (ii) hold at least approximately for high-energy diffraction scattering of hadrons. Therefore, the limit (1.3) is indeed of interest as a model in connection with high-energy processes. Moreover, recent studies on the high-energy behavior[4] of field theories yield results that show remarkable similarity with the behavior of the scattering amplitude in the limit (1.3) because of the appearance of certain exponential factors. Indeed, the appearance of such exponential factors is the basis of the Glauber approximation.[5] Although there is some formal similarity, the exponential factors found by Abarbanel and Itzykson[6] in the ϕ^3 theory are actually of a rather different character[7]; contrary to their claims, the terms that they calculated are not the leading terms.

Because of the appearance of exponential factors in these different connections, it is desirable to have a more thorough understanding of the simplest case, namely, the case of the Schrödinger equation (1.1) in the limit (1.3). It is thus the purpose of the present paper to study the cases where either the incident particle or the scatterer has internal degrees of freedom. Although this generalization is mathematically trivial, it is

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[1] T. T. Wu, Phys. Rev. **108**, 466 (1957). This paper shall be referred to as I. The original title of this paper was "High-Energy Potential Scattering I." An early paper entitled "High-Energy Potential Scattering II," not related to the present paper, was never submitted for publication.
[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[3] See, for example, the rapporteur paper of G. Bellettini, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger, (CERN, Geneva, 1968).
[4] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 666 (1969); Phys. Rev. **182**, 1852 (1969); **186**, 1611 (1969). This similarity has been explicitly stressed in the last article. See also F. Englert, P. Nicoletopoulos, R. Brout, and C. Truffin, Nuovo Cimento **64A**, 561 (1969); and S.-J. Chang and S.-K. Ma, Phys. Rev. **188**, 2385 (1969).
[5] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Britten *et al.* (Interscience, New York, 1959), Vol. I.
[6] H. D. I. Abarbanel and C. Itzykson, Phys. Rev. Letters **23**, 53 (1969).
[7] H. Cheng and T. T. Wu, Phys. Rev. (to be published).

physically relevant because hadrons do have these internal degrees of freedom.

An exponential factor does appear in these more general cases. However, this exponential factor is in general complicated in various ways; for example, in the case treated in Sec. 3, this factor is of the form of an *ordered exponential.* Such ordered exponentials are not simple objects, an example being the S matrix in field theory,

$$S=\left[\exp\left(i\int d^4x\, L(x)\right)\right]_+, \tag{1.5}$$

where L is the Lagrangian density.

The question whether an exponential factor appears is therefore not the relevant one within the present context. Rather, in connection with possible application to field-theoretic calculations at high energies,[4] the question is whether the complete answer for the elastic scattering amplitude in the limit (1.3) is determined by low-order perturbation calculations. The answer to this question is *yes* for the case treated in Paper I, but is *no* in general when there are internal degrees of freedom. In Sec. 5, we discuss the implications of this result in the interpretation of the answers for the case of field theories.

The appearance of the exponential factor has been referred to as exponentiation.[4] In connection with all the existing calculations of high-energy behavior in field theory, this term exponentiation has been applied only in the case where the exponential factor takes the simplest form. Because of the appearance of an exponential factor in general, we propose, in order to avoid confusion, to use the term "simple exponentiation" to refer to this simplest case where the exponent in this exponential factor is a numerical function of $x_\perp$ (i.e., x and y) only, and hence, within the particular approximation, the complete answer for the elastic scattering amplitude can be found from the perturbation calculation of the lowest order in the potential. In this sense, simple exponentiation holds for high-energy electron-electron scattering with multiphoton exchange.[4] Whether simple exponentiation holds can be tested by carrying out the calculation of the term quadratic in the potential. Recently, a test of this variety has been attempted by Muzinich, Tiktopoulos, and Treiman.[8]

In Sec. 4, we find, on the basis of such a test, that simple exponentiation does not hold when the target, roughly speaking, has a finite size. As discussed in Sec. 5, because of this result, *simple exponentiation cannot hold in the case of* ϕ^3 *theory*, for example.

2. GENERAL REMARKS

Before considering the various cases with internal degrees of freedom, it is perhaps interesting to give an over-all view about high-energy potential scattering as defined by the limit (1.3). For the sake of definiteness, consider the case of the Schrödinger equation (1.1), although similar remarks apply to Maxwell's equations, for example. For large k, different behaviors are obtained depending on the assumption for V. In particular, we may assume that

$$V(x,y,z)/k^n \tag{2.1}$$

is fixed. Three choices of the value of n are particularly interesting: (i) $n=2$, (ii) $n=1$, and (iii) $n=0$. The limit (1.3) corresponds to case (ii).

Since cases (i) and (iii) are more familiar, we discuss here these three cases in the order (i), (iii), and (ii).

(i) When $V(x,y,z)/k^2$ is fixed, the Schrödinger equation is more conveniently written in the form

$$[\nabla^2+k^2(1-V/k^2)]\psi=0. \tag{2.2}$$

Let

$$\epsilon(x,y,z)=1-k^{-2}V(x,y,z); \tag{2.3}$$

then

$$[\nabla^2+k^2\epsilon(x,y,z)]\psi(x,y,z)=0. \tag{2.4}$$

Accordingly, $\epsilon(x,y,z)$ is a scalar dielectric constant and this case can be interpreted as the *high-frequency scattering* by a dielectric obstacle. This is therefore a case that can be dealt with by physical optics.[9] More precisely, the procedure is as follows: The rays of geometrical optics are first traced, and a phase and amplitude are assigned to each point on each ray. These phases and amplitudes are simply added together if a point in space can be reached by more than one ray.

We emphasize the following two points. (a) The rays of geometric optics, as determined by optical laws, may or may not cover the entire space. Regions not reached by these rays are the shadows. If the asymptotic behaviors in the shadow regions are desired, classical theory of diffraction needs to be used. (b) One simple extension of geometrical optics is to define additional rays when a ray reaches a singularity of $V(x,y,z)$, such as the point $x=y=z=0$ when $V(x,y,z)$ is, for example, either e^{-r}/r or e^{-r}. In general, the contributions for these additional rays decrease as some power of k for large k. If these additional rays still fail to cover the entire space, exponential decrease with k is expected in the regions not reached by any ray.[10]

(iii) When $V(x,y,z)$ is fixed, the total phase shift through the potential is small, and hence the Born approximation may be applied.[11]

The Born approximation may or may not give all the desired answers. So far as the scattering amplitude is concerned, the Born approximation is sufficient provided that $V(x,y,z)$ is neither too singular nor

[8] I. J. Muzinich, G. Tiktopoulos, and S. B. Treiman, Phys. Rev. D **3**, 1041 (1971).

[9] See, for example, M. Born and E. Wolf, *Principles of Optics* (Pergamon, London, 1959).

[10] See, for example, R. W. P. King and T. T. Wu, *Scattering and Diffraction of Waves* (Harvard U. P., Cambridge, 1959).

[11] M. Born, Z. Physik **38**, 803 (1926).

analytic in the three variables x, y, and z. In the case of the singular potentials such as $(x^2+y^2+z^2)^{-2}$ the Born series needs to be summed.[12] In the opposite extreme of the analytic potential, the Born approximation may fail for large momentum transfers and application of the WKB method in the three *complex variables* x, y, and z is needed.[13] There are similarities between this extension of the Born approximation and the theory of diffraction.

(ii) The case of fixed $V(x,y,z)/k$ is intermediate between the above two. On the one hand, the present case can be considered to be an extreme situation in physical optics, where the rays are straight lines. Thus the exponential factor in the scattering amplitude is precisely the additional phase shift due to the presence of the potential. On the other hand, this exponential factor may be used to modify the integral equation from which the Born approximation is derived by iteration. This is precisely what was carried out in I.

Like the Born approximation, this high-energy approximation of I may or may not give all the desired answers. Clearly there are complications if the potential $V(x,y,z)$ is too singular. If V has weak singularities, as in the cases of e^{-r}/r or e^{-r} mentioned above, then additional rays may be introduced and the high-energy approximation gives all the desired information. In the other extreme, where $V(x,y,z)$ is an analytic function of x, y, and z, the high-energy approximation again may fail, and a WKB approximation in three complex variables is once more needed for large momentum transfers.[14]

In summary, these three cases are treated by the following methods. (i) $n=2$: physical optics supplemented, if necessary, by the theory of diffraction; (ii) $n=1$: high-energy approximation supplemented, if necessary, by the complex WKB method; and (iii) $n=0$: the Born approximation supplemented, if necessary, by the complex WKB method.

3. MATRIX CASE

A. Formulation

As a simple extension of the case treated in I, let the wave function ψ be a column matrix with N elements and the potential V be an $N\times N$ matrix. Then (1.1) still holds and (1.2) is replaced by

$$\psi^{\mathrm{inc}}=e^{ikz}u\,, \tag{3.1}$$

where u is a constant $N\times 1$ matrix. We study this case in the limit (1.3).

Because of (1.3), let

$$V(x,y,z)=gkU(x,y,z)\,, \tag{3.2}$$

[12] N. N. Khuri and A. Pais, Rev. Mod. Phys. **36**, 590 (1964); G. Tiktopoulos and S. B. Treiman, Phys. Rev. **134**, B844 (1964).

[13] T. T. Wu, Phys. Rev. **143**, 1110 (1966).

[14] This point is mentioned in Sec. 7(b) of Ref. 13.

where a coupling constant g, although not necessary, is introduced to facilitate the counting of orders. Both g and U are held fixed as $k\to\infty$. Let

$$\psi=e^{ikz}\Phi\,. \tag{3.3}$$

Then Φ satisfies the partial differential equation

$$(2ik\partial/\partial z+\nabla^2-gkU)\Phi=0\,. \tag{3.4}$$

Suppose we drop the term ∇^2 in (3.4); then

$$(\partial/\partial z+\tfrac{1}{2}igU)\Phi=0\,, \tag{3.5}$$

and hence, by (3.1),

$$\Phi(x,y,z)=\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z}dz'\,U(x,y,z')\right)\right]_+u\,. \tag{3.6}$$

Here the subscript + indicates an ordered exponential with respect to the z axis.

The source term J of I, defined by

$$J=gU\Phi\,, \tag{3.7}$$

is then given by

$$J(x,y,z)=gU(x,y,z)\times\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z}dz'\,U(x,y,z')\right)\right]_+u\,. \tag{3.8}$$

Note that, for a potential that is smooth and not long ranged, the approximation (3.8) is uniformly valid for large k although (3.6) is not. Equation (3.8) can thus be used to calculate the scattering amplitude defined by

$$f(\Delta_1,\Delta_2)=-k\int dxdydz\,v^\dagger J(x,y,z)e^{-i(\Delta_1x+\Delta_2y)}\,, \tag{3.9}$$

where $\boldsymbol{\Delta}=(\Delta_1,\Delta_2)$ is the momentum transfer and v is the outgoing state. Therefore, to leading order,

$$f(\Delta_1,\Delta_2)=2i\int_{-\infty}^{\infty}dxdy\,e^{-i(\Delta_1x+\Delta_2y)}\times v^\dagger\left\{1-\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty}dz\,U(x,y,z)\right)\right]_+\right\}u\,. \tag{3.10}$$

This is the desired answer. To this first approximation, the answer differs from that of the simple case of I, which corresponds to $N=1$, only in the appearance of the ordered exponential. We proceed to study the effects of this ordered exponential.

B. Simple Exponentiation

Suppose we make the additional, very restrictive, assumption that

$$[U(x,y,z),U(x,y,z')]=0 \tag{3.11}$$

for all x,y,z, and z'. Then the ordering becomes unnecessary, i.e.,

$$f(\Delta_1,\Delta_2)=2i\int_{-\infty}^{\infty} dxdy\, e^{-i(\Delta_1 x+\Delta_2 y)} \times v^\dagger\left[1-\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty} dz\, U(x,y,z)\right)\right]u. \quad (3.12)$$

To the lowest order in g, $f(\Delta_1,\Delta_2)$ is just given by the Born approximation. Therefore, once the Born approximation is known for all Δ_1, Δ_2, and u, we can compute from this information

$$\int_{-\infty}^{\infty} dz\, U(x,y,z) \quad (3.13)$$

for all x and y, and hence substitution into (3.12) gives the desired $f(\Delta_1,\Delta_2)$ to all orders in g. Note that in this process of finding $f(\Delta_1,\Delta_2)$ from the Born approximation, U is not determined although (3.13) is. In other words, (3.11) is a sufficient condition for simple exponentiation, as defined in Sec. 1, to hold. In particular, simple exponentiation always holds for $N=1$, since (3.11) is trivially satisfied in this case.

More generally, for the present matrix case, we say that simple exponentiation holds if and only if

$$\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty} dz\, U(x,y,z)\right)\right]_+ =\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty} dz\, U(x,y,z)\right) \quad (3.14)$$

for all x and y.

C. Perturbation Expansion

It is perhaps instructive to expand both sides of (3.14) to the second order in g:

[left-hand side of (3.14)]

$$=1-\tfrac{1}{2}ig\int_{-\infty}^{\infty} dz\, U(x,y,z)-\tfrac{1}{4}g^2\int_{-\infty}^{\infty} dz\, U(x,y,z) \times\int_{-\infty}^{z} dz' U(x,y,z')+O(g^3) \quad (3.15)$$

and

[right-hand side of (3.14)]

$$=1-\tfrac{1}{2}ig\int_{-\infty}^{\infty} dz\, U(x,y,z)-\tfrac{1}{8}g^2\int_{-\infty}^{\infty} dz\, U(x,y,z) \times\int_{-\infty}^{\infty} dz'\, U(x,y,z')+O(g^3). \quad (3.16)$$

Therefore, if (3.14) holds for all g, then in particular

$$\int_{-\infty}^{\infty} dz\int_{-\infty}^{z} dz'[U(x,y,z),U(x,y,z')]=0. \quad (3.17)$$

Similar but more complicated conditions follow from the coefficients of higher powers of g. Even (3.17) is of course not satisfied by all matrices, and hence simple exponentiation does not hold in general. We shall now give some explicit examples.

D. First Example

As a first example, take $N=2$ and consider

$$U(x,y,z)=\begin{cases}0, & \text{if } |z|>a+b\\ A(x,y)\sigma_3, & \text{if } b<|z|<a+b\\ B(x,y)\sigma_1, & \text{if } |z|<b,\end{cases} \quad (3.18)$$

where σ_i are the usual Pauli matrices, a and b are two positive numbers, and $A(x,y)$ and $B(x,y)$ are arbitrary functions that approach zero rapidly as $x^2+y^2\to\infty$. Physically, in this example, the two particles interact both with themselves and with each other through constant potentials, but the range of the two constant potentials are different.

A direct computation gives

$$\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty} dz\, U(x,y,z)\right)\right]_+ =[\cos(\tfrac{1}{2}gaA)-i\sigma_3\sin(\tfrac{1}{2}gaA)] \times[\cos(gbB)-i\sigma_1\sin(gbB)] \times[\cos(\tfrac{1}{2}gaA)-i\sigma_3\sin(\tfrac{1}{2}gaA)] =[\cos(gaA)-i\sigma_3\sin(gaA)]\cos(gbB) -i\sigma_1\sin(gbB). \quad (3.19)$$

This answer can be recast in an exponential form

$$\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty} dz\, U(x,y,z)\right)\right]_+ =\exp\{-i[\sigma_3\sin(gaA)\cos(gbB)+\sigma_1\sin(gbB)] \times[\sin^2(gaA)\cos^2(gbB)+\sin^2(gbB)]^{-1/2} \times\cos^{-1}[\cos(gaA)\cos(gbB)]\}. \quad (3.20)$$

We learn from this example that, when simple exponentiation does not hold, the ordered exponential can be written as an ordinary exponential only *at the expanse of sacrifying the linear dependence of the exponent on the strength g of the potential.*

For this example, simple exponentiation clearly holds if either $aA=0$ for all x and y or $bB=0$ for all x and y. Conversely, if simple exponentiation holds, then (3.19) should be equal to

$$\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty} dz\, U(x,y,z)\right)$$
$$=\cos[g(a^2A^2+b^2B^2)^{1/2}]-i(\sigma_3 aA+\sigma_1 bB)$$
$$\times(a^2A^2+b^2B^2)^{-1/2}\sin[g(a^2A^2+b^2B^2)^{1/2}]. \quad (3.21)$$

A comparison of (3.19) and (3.21) shows that the right-hand sides are equal to order g^2 in general, but to order g^3 only if either $aA=0$ or $bB=0$.

This example also provides a case where simple exponentiation does not hold but the condition (3.17) is satisfied.

E. Second Example

The example of Sec. 3 D can be solved rather simply because the differential equation (3.5) can be trivially solved in each of the three intervals $(-a-b, -b)$, $(-b, b)$, and $(b, a+b)$. As a second example, we consider the general case of two intervals. More precisely, we consider the case where

$$[U(x,y,z),U(x,y,z')]=0 \quad (3.22)$$

when z and z' are of the same sign. The dividing plane $z=0$ can of course be replaced by any surface $z=z_0(x,y)$ without any additional complication. When (3.22) is satisfied, we have

$$\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty} dz\, U(x,y,z)\right)\right]_+$$
$$=e^{-igA(x,y)}e^{-igB(x,y)}, \quad (3.23)$$

where

$$A(x,y)=\tfrac{1}{2}\int_0^{\infty} dz\, U(x,y,z)$$

and

$$B(x,y)=\tfrac{1}{2}\int_{-\infty}^{0} dz\, U(x,y,z). \quad (3.24)$$

Let the power-series expansion of this ordered exponential be

$$\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty} dz\, U(x,y,z)\right)\right]_+$$
$$=1-iga(x,y)-\tfrac{1}{2}g^2b(x,y)+\tfrac{1}{6}ig^3c(x,y)+\cdots; \quad (3.25)$$

then

$$a=A+B\,,$$
$$b=A^2+2AB+B^2\,, \quad (3.26)$$

and

$$c=A^3+3A^2B+3AB^2+B^3\,.$$

If simple exponentiation holds, then

$$b=a^2 \quad (3.27)$$

or

$$[A,B]=0\,. \quad (3.28)$$

Therefore, for this example, (3.17) is necessary and sufficient for simple exponentiation.

Suppose that (3.27), or equivalently (3.28), is not satisfied so that simple exponentiation does not hold. We can then ask the question whether a and b together determine c. Equation (3.26) can be written in the following form:

$$A=\tfrac{1}{2}a+\chi\,,\quad B=\tfrac{1}{2}a-\chi\,,$$
$$[\chi,a]=b-a^2\,, \quad (3.29)$$

and

$$[\chi,b]=c-\tfrac{1}{2}(ab+ba)\,.$$

Therefore, under the substitution

$$\chi\rightarrow\chi+\alpha a\,, \quad (3.30)$$

where α is a number, we have

$$a\rightarrow a\,,\quad b\rightarrow b\,,$$

but

$$c\rightarrow c+\alpha[a,b]\,. \quad (3.31)$$

Consequently a and b together cannot determine c unless

$$[a,b]=0\,. \quad (3.32)$$

We do not know the answer to any of the following questions. Under what conditions is (3.32) sufficient for a and b to determine all the higher coefficients? In general, how many coefficients in the Taylor expansion (3.25) are needed to determine all the coefficients? What is the relevance of the Lie algebra generated by A and B?

F. Higher-Order Corrections

So far we have only considered the leading approximation in the limit (1.3). We next apply the method of I to this matrix case. Almost all the calculations can be taken over without modification.

Instead of neglecting the term ∇^2 in (3.4), we convert it into the integral equation [see (2.10) of I]:

$$J(\mathbf{r})+gkU(\mathbf{r})\int\frac{e^{ik(|\mathbf{r}-\mathbf{r}'|-z+z')}}{4\pi|\mathbf{r}-\mathbf{r}'|}J(\mathbf{r}')d\mathbf{r}'=gU(\mathbf{r})u\,, \quad (3.33)$$

where $\mathbf{r}=(x,y,z)$ and the source J is defined by (3.7). The method of stationary-phase integration is then applied to this exact integral equation, and it is easily verified that every step in Sec. 3 of I applies without modification whatsoever, except the change of notation $x\leftrightarrow -z$. The final approximate equation is a trivial modification of (4.1) of I:

$$gU(x,y,z)u$$
$$=J(x,y,z)+\tfrac{1}{2}igU(x,y,z)\int_{-\infty}^{z}dz'J(x,y,z')$$
$$-k^{-1}gU(x,y,z)\left[\int_{-\infty}^{z}dz'(z-z')\bar{J}_1(x,y,z')+\tfrac{1}{4}J(x,y,z)\right]$$
$$-ik^{-2}gU(x,y,z)\left\{\int_{-\infty}^{z}dz'[\bar{J}_1(x,y,z')+2(z-z')^2\bar{J}_2(x,y,z')]-\tfrac{1}{8}(\partial/\partial z)J(x,y,z)\right\}+\cdots, \quad (3.34)$$

where $\bar{J}_1$ and $\bar{J}_2$ are defined by (3.6) and (3.4) of I. More explicitly, by (4.4) and (4.5) of I, we have

$$\bar{J}_1(x,y,z)=\tfrac{1}{4}\Delta_t J(x,y,z)$$

and (3.35)

$$\bar{J}_2(x,y,z)=\tfrac{1}{32}\Delta_t^2 J(x,y,z)\,,$$

where

$$\Delta_t=\partial^2/\partial x^2+\partial^2/\partial y^2 \quad (3.36)$$

is the transverse Laplacian.

The matrix character of the present generalization becomes important when (3.34) is solved by iteration. The leading term for J satisfies

$$gU(x,y,z)u=J(x,y,z)+\tfrac{1}{2}igU(x,y,z)\int_{-\infty}^{z}dz'J(x,y,z'), \quad (3.37)$$

and is hence given by (3.8). The next-order approximation satisfies

$$J(x,y,z)+\tfrac{1}{2}igU(x,y,z)\int_{-\infty}^{z}dz'J(x,y,z')$$
$$=gU(x,y,z)\left\{1+\tfrac{1}{4}k^{-1}gU(x,y,z)\times\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z}dz'U(x,y,z')\right)\right]_+\right.$$
$$\left.+\tfrac{1}{4}k^{-1}\int_{-\infty}^{z}dz'(z-z')\Delta_t\left(gU(x,y,z')\times\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z'}dz''U(x,y,z'')\right)\right]_+\right)\right\}u. \quad (3.38)$$

To solve (3.38), we follow the procedure of Sec. 4 of I. Let

$$f(x,y,z)=u-\tfrac{1}{2}i\int_{-\infty}^{z}dz'J(x,y,z'); \quad (3.39)$$

then

$$J(x,y,z)=2i\partial f(x,y,z)/\partial z\,, \quad (3.40)$$

and hence $f(x,y,z)$ satisfies the differential equation

$$\frac{\partial f(x,y,z)}{\partial z}+\tfrac{1}{2}igU(x,y,z)f(x,y,z)=(4k)^{-1}gU(x,y,z)\left\{-\tfrac{1}{2}igU(x,y,z)\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z}dz'U(x,y,z')\right)\right]_+\right.$$
$$\left.+\int_{-\infty}^{z}dz'\Delta_t\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z'}dz''U(x,y,z'')\right)\right]_+\right\}u, \quad (3.41)$$

together with the boundary condition

$$f(x,y,-\infty)=u\,. \quad (3.42)$$

The solution of (3.41) is

$$f(x,y,z)=\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z}dz'U(x,y,z')\right)\right]_+u+(4k)^{-1}\int_{-\infty}^{z}dz'\left[\exp\left(-\tfrac{1}{2}ig\int_{z'}^{z}dz''U(x,y,z'')\right)\right]_+gU(x,y,z')$$
$$\times\left\{-\tfrac{1}{2}igU(x,y,z')\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z'}dz''U(x,y,z'')\right)\right]_++\int_{-\infty}^{z'}dz''\Delta_t\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z''}dz'''U(x,y,z''')\right)\right]_+\right\}u$$
$$=\left[\exp-\tfrac{1}{2}ig\int_{-\infty}^{z}dz'U(x,y,z')\right]_+u+(8ik)^{-1}\int_{-\infty}^{z}dz'\left[g^2U^2(x,y,z')\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z}dz''U(x,y,z'')\right)\right]_+u$$
$$-\tfrac{1}{2}ik^{-1}\int_{-\infty}^{z}dz'\Delta_t\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z'}dz''U(x,y,z'')\right)\right]_+u$$
$$+\tfrac{1}{2}ik^{-1}\int_{-\infty}^{z}dz'\left[\exp\left(-\tfrac{1}{2}ig\int_{z'}^{z}dz''U(x,y,z'')\right)\right]_+\Delta_t\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z'}dz''U(x,y,z'')\right)\right]_+u, \quad (3.43)$$

and hence

$$J(x,y,z)=gU(x,y,z)[1+(4k)^{-1}gU(x,y,z)]\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z}dz'U(x,y,z')\right)\right]_+ u$$
$$+(8ik)^{-1}g^3U(x,y,z)\int_{-\infty}^{z}dz'\left[U^2(x,y,z')\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z}dz''U(x,y,z'')\right)\right]_+ u$$
$$+\tfrac{1}{2}ik^{-1}gU(x,y,z)\int_{-\infty}^{z}dz'\left[\exp\left(-\tfrac{1}{2}ig\int_{z'}^{z}dz''U(x,y,z'')\right)\right]_+\Delta_t\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z'}dz''U(x,y,z'')\right)\right]_+ u. \quad (3.44)$$

This is the desired result to second order. This iterative procedure can be repeated to higher orders provided that the potential $V(x,y,z)$ has a sufficient number of derivatives.

G. Scattering Amplitude

Let Δ be the momentum transfer, and v be the outgoing state. The scattering amplitude $f(\Delta_1,\Delta_2)$ has been defined by (3.9). By (3.40) and (3.42), (3.9) can be simplified to

$$f(\Delta_1,\Delta_2)=-2ik\int dxdy\, v^\dagger[f(x,y,\infty)-u]e^{-i(\Delta_1x+\Delta_2y)}. \quad (3.45)$$

It therefore follows from (3.43) that, to second order,

$$f(\Delta_1,\Delta_2)=2ik\int dxdy\, e^{-i\Delta\cdot\mathbf{x}_\perp}v^\dagger\left\{1-\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty}dz\, U(x,y,z)\right)\right]_+\right.$$
$$-(8ik)^{-1}\int_{-\infty}^{\infty}dz\left[g^2U^2(x,y,z)\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{\infty}dz'U(x,y,z')\right)\right]_+$$
$$\left.+\tfrac{1}{2}ik^{-1}\int_{-\infty}^{\infty}dz\left(1-\left[\exp\left(-\tfrac{1}{2}ig\int_{z}^{\infty}dz'U(x,y,z')\right)\right]_+\right)\Delta_t\left[\exp\left(-\tfrac{1}{2}ig\int_{-\infty}^{z}dz'U(x,y,z')\right)\right]_+\right\}u. \quad (3.46)$$

4. SCATTERING BY BOUND SYSTEMS

A. Introduction

In Sec. 3, we have treated in some detail the case where the incident particle has a discrete degree of freedom. Actually, in this matrix case, the degree of freedom can be associated with either the incident particle or the target, the mathematics being the same. Thus the ith component of the wave function can be interpreted as corresponding to either the incident particle being in state i or the target particle being in state i.

From the explicit examples of Sec. 3, we find that simple exponentiation in general does not hold if the incident particle and/or the target particle have discrete internal degrees of freedom. In the physically interesting case of the high-energy scattering of hadrons, each hadron must be thought of as made of stuff.[15] Thus each hadron has an enormous number of internal degrees of freedom. We therefore expect, from the result of Sec. 3, that in general simple exponentiation does not hold. On the other hand, from field-theoretic calculations, we have seen that exponentiation does hold at least in some cases and in certain approximations. We therefore study in this section an example of potential scattering by a bound system[5] in order to learn about circumstances where simple exponentiation holds approximately. In this way we gain a deeper understanding of the phenomenon of exponentiation found in field theory.

[15] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).

B. Formulation

Consider two particles, designated as 1 and 2, in an external potential. The Schrödinger equation is

$$(\nabla_1^2+\tfrac{1}{2}m^{-1}\nabla_2^2+E-V_1-V_2-V_{12})\psi=0\,, \quad (4.1)$$

where V_{12} is the potential between 1 and 2, while V_1 and V_2 are, respectively, the external potential as seen by particles 1 and 2. The mass of particle 1 has been taken to be $\frac{1}{2}$ without loss of generality. Let particle 1 be the incident particle of very high energy, and particle 2 be bound in the potential. If k is the momentum of the incident particle 1, and E_0 is the binding energy of particle 2, then

$$E=k^2-E_0\,. \quad (4.2)$$

Let $\psi_0(\mathbf{r}_2)$ be the wave function for particle 2; then $\psi_0(\mathbf{r}_2)$ satisfies

$$[\tfrac{1}{2}m^{-1}\nabla_2^2-E_0-V_2(\mathbf{r}_2)]\psi_0(\mathbf{r}_2)=0 \qquad (4.3)$$

and

$$\psi^{\mathrm{inc}}(\mathbf{r}_1,\mathbf{r}_2)=e^{ikz_1}\psi_0(\mathbf{r}_2)\,. \qquad (4.4)$$

We want to study (4.1) for large k when the following three quantities are kept fixed:

$$U_1(\mathbf{r}_1)=k^{-1}V_1(\mathbf{r}_1)\,,$$
$$U_{12}(\mathbf{r}_1,\mathbf{r}_2)=k^{-1}V_{12}(\mathbf{r}_1,\mathbf{r}_2)\,, \qquad (4.5)$$

and

$$V_2(\mathbf{r}_2)\,.$$

We have omitted g here.

C. Explicit Solution

We solve this problem by the procedure of Sec. 3 A. Similarly to (3.3), let

$$\psi(\mathbf{r}_1,\mathbf{r}_2)=e^{ikz}\Phi(\mathbf{r}_1,\mathbf{r}_2)\,; \qquad (4.6)$$

then the partial differential equation for Φ is

$$(2ik\partial/\partial z+\nabla_1^2+\tfrac{1}{2}m^{-1}\nabla_2^2-E_0-kU_1-V_2-kU_{12})\Phi =0\,. \qquad (4.7)$$

Suppose we drop all terms not proportional to k; then

$$(\partial/\partial z+\tfrac{1}{2}iU_1+\tfrac{1}{2}iU_{12})\Phi=0\,. \qquad (4.8)$$

This is a first-order ordinary differential equation, and its solution is, with the boundary condition (4.4),

$$\Phi(x_1,y_1,z_1;\,x_2,y_2,z_2) =\psi_0(x_2,y_2,z_2)\,\exp\biggl(-\tfrac{1}{2}i\int_{-\infty}^{z_1}dz_1'[U_1(x_1,y_1,z_1') +U_{12}(x_1,y_1,z_1';\,x_2,y_2,z_2)]\biggr)\,. \qquad (4.9)$$

In writing down (4.9), we have restricted ourselves to the simplest case where neither particle 1 nor particle 2 has any internal degree of freedom. Thus the exponential in (4.9) need not be ordered.

With respect to particle 1, we can define the current

$$J=k^{-1}(-\tfrac{1}{2}m^{-1}\nabla_2^2+E_0+V_1+V_2+V_{12})\Phi\,. \qquad (4.10)$$

By (4.5), the leading term of J is

$$J=(U_1+U_{12})\Phi\,. \qquad (4.11)$$

Let $\psi_1(x_2,y_2,z_2)$ be the final state for particle 2; then the scattering amplitude is defined similarly to (3.9) as

$$f(\Delta_1,\Delta_2)=-k\int dx_1dy_1dz_1dx_2dy_2dz_2J(x_1,y_1,z_1;\,x_2,y_2,z_2) \times\psi_1^*(x_2,y_2,z_2)\,\exp[-i(\Delta_1x_1+\Delta_2y_1)]\,. \qquad (4.12)$$

When (4.9) and (4.11) are used, we find that the leading approximation to the scattering amplitude is the Glauber form[5]

$$f(\Delta_1,\Delta_2)=2ik\int dx_1dy_1\int dx_2dy_2dz_2 \times e^{-i(\Delta_1x_1+\Delta_2y_1)}\psi_1^*(x_2,y_2,z_2)\psi_0(x_2,y_2,z_2) \times\biggl[1-\exp\biggl(-\tfrac{1}{2}i\int_{-\infty}^{\infty}dz_1[U_1(x_1,y_1,z_1) +U_{12}(x_1,y_1,z_1;\,x_2,y_2,z_2)]\biggr)\biggr]\,. \qquad (4.13)$$

Note that (4.13) is almost exactly the same as (3.10). Application of this procedure to other situations is obvious.

D. Simple Exponentiation

For elastic scattering, the target bound system must remain unchanged, i.e.,

$$\psi_1(x_2,y_2,z_2)=\psi_0(x_2,y_2,z_2)\,. \qquad (4.14)$$

The substitution of (4.14) into (4.13) yields the following approximate formula for the elastic scattering amplitude:

$$f(\Delta_1,\Delta_2) =2ik\int dx_1dy_1\int dx_2dy_2dz_2e^{-i(\Delta_1x_1+\Delta_2y_1)}|\psi_0(x_2,y_2,z_2)|^2 \times\biggl[1-\exp\biggl(-\tfrac{1}{2}i\int_{-\infty}^{\infty}dz_1[U_1(x_1,y_1,z_1) +U_{12}(x_1,y_1,z_1;\,x_2,y_2,z_2)]\biggr)\biggr]\,. \qquad (4.15)$$

Inspite of the simplicity of this expression, simple exponentiation as defined in the Introduction still does not hold. If we expand the right-hand side of (4.15) to the lowest order in the interaction potentials, we get

$$f_0(\Delta_1,\Delta_2) =-k\int dx_1dy_1e^{-i(\Delta_1x_1+\Delta_2y_1)}\int dx_2dy_2dz_2|\psi_0(x_2,y_2,z_2)|^2 \times\int dz_1[U_1(x_1,y_1,z_1)+U_{12}(x_1,y_1,z_1;\,x_2,y_2,z_2)]\,. \qquad (4.16)$$

Therefore, complete knowledge about $f_0(\Delta_1,\Delta_2)$ allows us to determine

$$\int dz_1 \int dx_2dy_2dz_2|\psi_0(x_2,y_2,z_2)|^2 \times[U_1(x_1,y_1,z_1)+U_{12}(x_1,y_1,z_1;\,x_2,y_2,z_2)], \quad (4.17)$$

which is in general not enough information to give $f(\Delta_1,\Delta_2)$.

In the Appendix, we discuss this point in some more detail.

E. Conditions for Simple Exponentiation

In (4.15) the term quadratic in the interaction potentials is

$$\tfrac{1}{4}ik\int dx_1dy_1e^{-i(\Delta_1x_1+\Delta_2y_1)}\int dx_2dy_2dz_2|\psi_0(x_2,y_2,z_2)|^2 \times\left(\int dz_1[U_1(x_1,y_1,z_1)+U_{12}(x_1,y_1,z_1;\,x_2,y_2,z_2)]\right)^2. \quad (4.18)$$

Therefore, a necessary condition for simple exponentiation to hold is

$$\left(\int dx_2dy_2dz_2|\psi_0(x_2,y_2,z_2)|^2\int dz_1[U_1(x_1,y_1,z_1) +U_{12}(x_1,y_1,z_1;\,x_2,y_2,z_2)]\right)^2 =\int dx_2dy_2dz_2|\psi_0(x_2,y_2,z_2)|^2\left(\int dz_1[U_1(x_1,y_1,z_1) +U_{12}(x_1,y_1,z_1;\,x_2,y_2,z_2)]\right)^2. \quad (4.19)$$

We restrict ourselves to the case where U_1 and U_{12} are real. Because of the normalization of $\psi_0(x_2,y_2,z_2)$, the left-hand side of (4.19) can never be larger than the right-hand side. Thus (4.19) holds if and only if

$$\int dz_1[U_1(x_1,y_1,z_1)+U_{12}(x_1,y_1,z_1;\,x_2,y_2,z_2)] \quad (4.20)$$

is independent of x_2, y_2, and z_2 in the support of ψ_0, i.e., in the region where $\psi_0(x_2,y_2,z_2)$ is different from zero.[16] In other words,

$$\int dz_1U_{12}(x_1,y_1,z_1;\,x_2,y_2,z_2)=F(x_1,y_1) \quad (4.21)$$

when $\psi_0(x_2,y_2,z_2)\neq 0$.

[16] The support is usually defined to be a closed set. This point is irrelevant here.

Suppose that (4.21) is satisfied; then (4.15) simplifies greatly to

$$f(\Delta_1,\Delta_2)=2ik\int dx_1dy_1 \times e^{-i(\Delta_1x_1+\Delta_2y_1)}\left\{1-\exp\left[-\tfrac{1}{2}i\left(F(x_1,y_1) +\int_{-\infty}^{\infty}dz_1U_1(x_1,y_1,z_1)\right)\right]\right\}. \quad (4.22)$$

This is the form for simple exponentiation.

In general, since $\psi_0(x_2,y_2,z_2)$ is a ground-state wave function, $\psi_0(x_2,y_2,z_2)$ is never zero. Thus simple exponentiation holds only if (4.21) is satisfied for all x_2, y_2, and z_2. Let $x_2\to\infty$, for example, and we get $F(x_1,y_1)=0$. This means, roughly speaking, that effectively there is no interaction between particle 1 and particle 2. This case is uninteresting.

So far we have considered U_{12} to be a function of the six variables x_1, y_1, z_1, x_2, y_2, and z_2. Physically the most interesting case is

$$U_{12}=U_{12}(x_1-x_2,\,y_1-y_2,\,z_1-z_2)\,. \quad (4.23)$$

Thus the condition (4.21) is

$$\int dz_1U_{12}(x_1,y_1,z_1)=F(x_1+x_2,\,y_1+y_2)\,, \quad (4.24)$$

provided that $\psi_0(x_1+x_2,\,y_1+y_2,\,z_2)\neq 0$. If the support of ψ_0 is three-dimensional, (4.24) can be satisfied only if F is a constant. This again means that effectively there is no interaction between particle 1 and particle 2.

We therefore conclude that, in the presence of significant interaction between particle 1 and particle 2, simple exponentiation can hold approximately only if $\psi_0(x_2,y_2,z_2)$ is large in a small region. Since $\psi_0(x_2,y_2,z_2)$ is a ground-state wave function, physically the most important case is the situation where $\psi_0(x_2,y_2,y_2)$ is concentrated near one point. We shall study this case only, since the more general cases seem artificial and not instructive.

F. Small Bound Systems

Let $\psi_0(x_2,y_2,z_2)$ be large only in the neighborhood of the origin. Let the origin be chosen such that

$$\int dx_2dy_2dz_2x_2|\psi_0(x_2,y_2,z_2)|^2 =\int dx_2dy_2dz_2y_2|\psi_0(x_2,y_2,z_2)|^2=0\,. \quad (4.25)$$

Define the second moments by

$$M_{xx}=\int dx_2dy_2dz_2x_2^2|\psi_0(x_2,y_2,z_2)|^2,$$

$$M_{xy}=\int dx_2dy_2dz_2x_2y_2|\psi_0(x_2,y_2,z_2)|^2, \quad (4.26)$$

and

$$M_{yy}=\int dx_2dy_2dz_2y_2^2|\psi_0(x_2,y_2,z_2)|^2.$$

Suppose (4.23) holds, and we try to expand U_{12} into a Taylor series

$$\int_{-\infty}^{\infty} U_{12}(x_1-x_2,y_1-y_2,z_1-z_2)dz_1 =\int_{-\infty}^{\infty} U_{12}(x_1,y_1,z_1-z_2)dz_1-x_2W_{12x}(x_1,y_1) -y_2W_{12y}(x_1,y_1)+\tfrac{1}{2}x_2^2W_{12xx}(x_1,y_1) +x_2y_2W_{12xy}(x_1,y_1)+\tfrac{1}{2}y_2^2W_{12yy}(x_1,y_1)+\cdots. \quad (4.27)$$

By (4.26) the substitution of (4.27) into (4.15) gives approximately

$$f(\Delta_1,\Delta_2)=2ik\int dx_1dy_1 \times e^{-i(\Delta_1x_1+\Delta_2y_1)}\left\{1-\left[\exp\left(-\tfrac{1}{2}i\int_{-\infty}^{\infty}dz_1[U_1(x_1,y_1,z_1) +U_{12}(x_1,y_1,z_1)]\right)\right]\{1-\tfrac{1}{8}[M_{xx}(W_{12x}^2+2iW_{12xx}) +2M_{xy}(W_{12x}W_{12y}+2iW_{12xy}) +M_{yy}(W_{12y}^2+2iW_{12yy})]\}\right\}. \quad (4.28)$$

This is correct only to the lowest order of the size of the small bound system.

Within this approximation of keeping only the second-order moments M_{xx}, M_{xy}, and M_{yy}, (4.28) can be written as

$$f(\Delta_1,\Delta_2)=2ik\int dx_1dy_1 \times e^{-i(\Delta_1x_1+\Delta_2y_1)}\left\{1-\{1-\tfrac{1}{8}[M_{xx}W_{12x}^2(x_1,y_1) +2M_{xy}W_{12x}(x_1,y_1)W_{12y}(x_1,y_1)+M_{yy}W_{12y}^2(x_1,y_1)]\} \times\exp\left[-\tfrac{1}{2}i\left(\int_{-\infty}^{\infty}dz_1[U_1(x_1,y_1,z_1)+U_{12}(x_1,y_1,z_1)] +\tfrac{1}{2}[M_{xx}W_{12xx}(x_1,y_1)+2M_{xy}W_{12xy}(x_1,y_1) +M_{yy}W_{12yy}(x_1,y_1)]\right)\right]\right\}. \quad (4.29)$$

Therefore, the violation of simple exponentiation is measured by

$$M_{xx}W_{12x}^2(x_1,y_1)+2M_{xy}(x_1,y_1)W_{12y}(x_1,y_1) +M_{yy}W_{12}^2(x_1,y_1). \quad (4.30)$$

Since, by (4.26), the expression (4.30) is equal to

$$\int dx_2dy_2dz_2|\psi_0(x_2,y_2,z_2)|^2 \times[x_2W_{12x}(x_1,y_1)+y_2W_{12y}(x_1,y_1)]^2, \quad (4.31)$$

simple exponentiation is always violated. Very roughly, the violation is of the order of the square of the radius of the bound system.

5. APPLICATION TO FIELD THEORIES

We have learned from the explicit calculations here that, in high-energy potential scattering, simple exponentiation in general does not hold when either the incident particle or the target has internal degrees of freedom.[17] Although it may or may not be possible to extrapolate simple results in potential scattering to the more realistic cases of field theories, complications that already appear in potential theory certainly cannot be expected to disappear in the contexts of field theories and hadron physics. Thus we expect that simple exponentiation does not hold for the high-energy scattering of hadrons. In this section we discuss the implications in field theories of the failure of simple exponentiation.

In connection with exponentiation, potential theory is a particularly fertile ground in which to gain physical insight. First, the formalism of high-energy potential scattering is so simple that nothing obscure can be hidden. This is in marked contrast with the lengthy and involved field-theoretic calculations. Secondly, because of the large amount of effort required in any *reliable* field-theoretic calculation, only a small number of such calculations can be carried out. Since such calculations are further restricted to the simplest possible cases, the results, if taken literally without very careful physical interpretation, may be misleading. Finally, in many field-theoretic calculations, some assumptions are made about the region of integration from which the important contributions come. Results from high-energy potential scattering can be extremely useful in deciding which approximation may be used in field theories.

A. Quantum Electrodynamics

Among various field theories, quantum electrodynamics is the one where simple exponentiation was first found.[4] In particular, simple exponentiation holds for high-energy electron-electron scattering with multi-

[17] Note that, in the case of the high-energy scattering of a Dirac particle by a static electric field, simple exponentiation holds even though there is a spin degree of freedom.

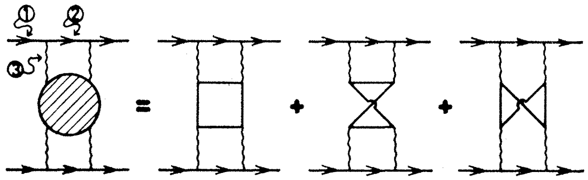

FIG. 1. Lowest-order diagrams of electron-electron scattering with one electron loop.

photon exchange, where the "photon" may be either massless or massive. Because the electron spin is not important,[17] for these diagrams the electrons have neither additional internal degrees of freedom nor size. Therefore, on the basis of our knowledge here from potential theory, this result of simple exponentiation is not surprising.

So far as the leading terms are concerned, simple exponentiation still holds when electron loops are present.[18] In the simplest case, electron loops are exchanged between the two incident elections as shown in Fig. 1. In this case, although simple exponentiation holds in the leading order, it fails in the next order, which is smaller by a factor $(\ln s)^{-1}$. This failure was previously emphasized in italics,[18] and is also the theme of Muzinich, Tiktopoulos, and Treiman[8] in a somewhat different context. From the present point of view, this failure can be easily understood as follows. Suppose we take the c.m. system for definiteness; to the leading order, the large momentum of the incident electron, labeled 1 in Fig. 1, is carried entirely by the virtual electron labeled 2. Therefore, to the leading order of approximation, the electrons have no size. To the next order, however, the contributions come from the region where the large momentum of electron 1 is shared comparably between the electron 2 and the photon 3, all as labeled in Fig. 1. Therefore, instead of a point electron, we have an electron and an electron-positron pair, all carrying large momenta of comparable magnitude. The situation is therefore very similar to the case discussed in Sec. 4. Because of the size of this system of an electron and an electron-positron pair, simple exponentiation is violated.

We may raise the following question: Since $(\ln s)^{-1} \to 0$ as $s \to \infty$, is it possible that simple exponentiation holds at energies so high that $(\ln s)^{-1}$ is neglibibly small? There are many ways to see that this is not possible. A particularly transparent way is to make use of the excellent discussion of Lee, Huang, and Yang[19] on the summing of leading terms. By that discussion, compared to the sum of the leading terms, the sum of the next-order terms is smaller by a factor of α^2, where α is the fine-structure constant. Therefore, no matter how high the energy is, strictly speaking simple exponentiation does not hold. Fortunately, because of the smallness of the fine-structure constant, the violation may not be large. In connection with the tower diagrams, the nature of this violation is actually a most interesting subject, but we shall not enter this discussion here.

This factor α^2 can be put in a different manner. To the leading order of approximation, the large momentum of the incident electron 1 is carried entirely by the electron 2, as already mentioned above. This statement applies both to the diagrams of Fig. 1 and also to the tower diagram of Fig. 2. Actually, for the tower diagram, for the important region of integration, the ratio

$$R = \frac{(\text{Momentum of the virtual photon 3})}{(\text{Momentum of the incident electron 1})} \tag{5.1}$$

is not zero but of the order α^2. This point is of great importance in connection with the ϕ^3 theory.

We emphasize that *the violation of simple exponentiation does not in any way affect our predictions about high-energy hadronic processes.*[20] The results there depend only on the large absorption at high energies and its consequent removal of particles from the incident beam; the explicit form of simple exponentiation is carefully not used. This point is not further discussed in our paper[20] simply because of the lack of space, and is the reason why we avoided discussing inelastic processes, which are more sensitive to the details of absorption. We distrust results that depend critically on simple exponentiation.

B. ϕ^3 Theory

The situation with exponentiation is actually more complicated in the case of ϕ^3 theory. First, when the coupling constant g is small,[21] the high-energy behavior

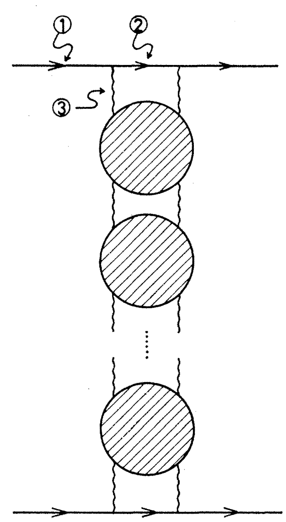

FIG. 2. The one-tower diagram for electron-electron scattering.

[18] See, in particular, the third article of Ref. 4, pp. 1617, 1618.

[19] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

[20] H. Cheng, and T. T. Wu, Phys. Rev. Letters **24** 1456 (1970).

[21] We take the mass of the scalar particle to be 1.

of the elastic scattering amplitude for fixed momentum transfer is determined by the one-particle exchange diagrams as shown in Fig. 3. Because of the simplicity of these diagrams, powers of $\ln s$ cannot appear. Consequently, for small g, the elastic differential cross section $d\sigma/dt$ is proportional to s^{-2} at high energies, and this case is thus not very interesting. In order to get other, more interesting high-energy behavior, *the coupling constant g cannot be considered to be small.*

On the other hand, let us consider, for definiteness, the ladder diagrams in the t channel as shown in Fig. 4. These diagrams were first studied by Gell-Mann and Goldberger.[22] Just as in the case of quantum electrodynamics, the large momentum of the incident particle 1 is carried entirely by particle 2 in the leading approximation, but not for the next-order terms, which are smaller by a factor $(\ln s)^{-1}$. By an argument entirely similar to that presented in Sec. 5 A, the ratio

$$R=\frac{(\text{Momentum of the virtual particle 3})}{(\text{Momentum of the incident particle 1})} \tag{5.2}$$

is found to be of the order g^2 for small coupling constants.

The simple but crucial result can be understood in a number of different ways, and we shall present some of the details in a separate paper.[7] For example, we may study asymptotically[7] the Bethe-Salpeter equation.[23] With respect to the asymptotic calculation of individual diagrams, the crucial point is the following. Let N be the number of rungs in the ladder diagram of Fig. 3; then the well-known asymptotic behavior of Gell-Mann and Goldberger,[22] and Federbush and Grisaru[24] holds for *fixed N*. By comparing with the next-order term as found by Trueman and Yao,[25] it is easily verified that the leading approximation fails when

$$N=O((\ln s)^{1/2})\,. \tag{5.3}$$

Thus the leading approximation fails before reaching

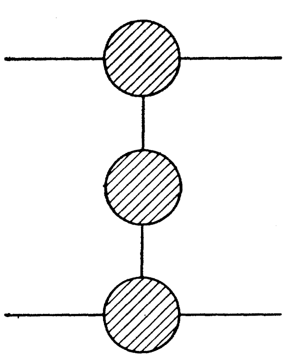

FIG. 3. One-particle exchange diagrams.

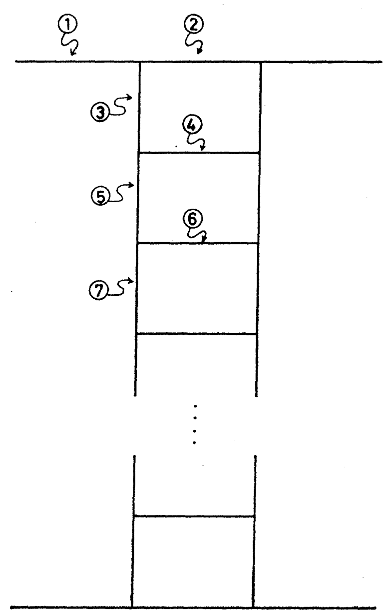

FIG. 4. Ladder diagram in the t channel.

the largest term at

$$N=O(g^2\ln s)\,. \tag{5.4}$$

This point is to be discussed in great detail.[7]

Since the coupling constant cannot be considered to be small, the ratio R of (5.2) is not small, and hence the large momentum of the incident particle 1 is divided comparably between particle 2 and particle 3. Thus the situation is similar to the one studied in Sec. 4: Each of the incident particles must be considered to have internal structure and size, and *simple exponentiation does not hold*. This result is clearly not limited to the ladder diagrams of Fig. 4. Even for the ladder diagrams, the situation is actually more complicated, because the large momentum of particle 3 is shared comparably by particles 4 and 5, while that of 5 is shared comparably by 6 and 7, etc. This repeated sharing is also of great importance.

We emphasize that the summing of leading terms is not necessarily a meaningful approximation when coupling is large. Failure to realize the limitations has misled a number of authors, including Chang and Yan.[26]

C. Remarks

We add here four simple remarks.

(a) Throughout our study of the high-energy behavior of diffraction processes, elastic or inelastic, the dominating contributions to the matrix elements always come from the region where *all the particles are not far off mass shell*, i.e., the region where all p^2 are of

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[25] T. L. Trueman and T. Yao, Phys. Rev. **132**, 2741 (1963).

[26] S.-J. Chang and T.-M. Yan, Phys. Rev. Letters **25**, 1586 (1970).

the order of m^2. This is in particular true of the tower diagrams[27] in quantum electrodynamics, and should be contrasted with the case of fermion exchange.[28] In a way that requires further clarification, the relevance of high-energy potential scattering must depend on this point.

(b) Even though in general an ordered exponential is a rather complicated expression, it nevertheless shares with ordinary exponentials the property that

$$\lim_{\Lambda\to\infty}\left[\exp\left(-\Lambda\int_{-\infty}^{\infty}M(z)dz\right)\right]_+=0\,, \tag{5.5}$$

provided that $M(z)$ is positive definite. This is the basic property that is needed for our previous discussion[20] on the limiting behavior of cross sections at high energies.

(c) In the case of quantum electrodynamics, since the bare photon itself has no direct electromagnetic interaction, an electron never appears completely absorbing to a photon no matter how high the energy is. From this point of view, the case of high-energy Compton scattering is extremely interesting and is to be discussed in a separate paper.

(d) Equation (5.5) is insufficient for the study of inelastic processes. In particular we find the determination of multiplicity to be a very difficult task. Our very preliminary result, based on multitower diagrams and discussed by us elsewhere,[29] is that the multiplicity increases with energy as $(\ln s)^2$. This is consistent with the meager experimental data and contradicts the result of Chang and Yan.[26]

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APPENDIX

In this Appendix we further discuss the problem of determining the scattering amplitude $f(\Delta_1,\Delta_2)$ of (4.15) from the integral (4.17). In connection with the problem of simple exponentiation in the field-theoretical cases, we certainly cannot assume that $\psi_0(x_2,y_2,y_2)$ is known. However, we study the academic question of whether $f(\Delta_1,\Delta_2)$ can be determined from (4.17) if $\psi_0(x_2,y_2,z_2)$ is known.

In general, if $U_{12}(x_1,y_1,z_1;x_2,y_2,z_2)$ is an arbitrary function of six variables, the answer to the above question is still no. We therefore restrict ourselves here to the most interesting case (4.23). When (4.23) holds, it is convenient to carry out the z integrations. Let

$$\theta_0(x_2,y_2)=\int_{-\infty}^{\infty}dz_2|\psi_0(x_2,y_2,z_2)|^2\,, \tag{A1}$$

$$T_1(x_1,y_1)=\int_{-\infty}^{\infty}dz_1U_1(x_1,y_1,z_1)\,, \tag{A2}$$

and

$$T_{12}(x_1-x_2,\,y_1-y_2)=\int_{-\infty}^{\infty}dz_1U_{12}(x_1-x_2,\,y_1-y_2,\,z_1-z_2)\,; \tag{A3}$$

then

$$\int_{-\infty}^{\infty}dx_2dy_2\theta_0(x_2,y_2)=1\,, \tag{A4}$$

and it follows from (4.15) that

$$f(\Delta_1,\Delta_2)=2ik\int dx_1dy_1\int dx_2dy_2e^{-i(\Delta_1x_1+\Delta_2y_1)}\theta_0(x_2,y_2)\times\{1-\exp[-\tfrac{1}{2}i(T_1(x_2,y_2)+T_{12}(x_1-x_2,\,y_1-y_2))]\}\,. \tag{A5}$$

By (A4), the expression (4.17) can be rewritten in the form

$$T_1(x_1,y_1)+\int dx_2dy_2\theta_0(x_2,y_2)T_{12}(x_1-x_2,\,y_1-y_2)\,. \tag{A6}$$

Note that (A6) is in the form of a convolution integral. Let

$$\bar{T}_1(\Delta_1,\Delta_2)=\int dx_1dy_1T_1(x_1,y_1)e^{-i(\Delta_1x_1+\Delta_2y_1)}\,,$$

$$\bar{T}_{12}(\Delta_1,\Delta_2)=\int dx_1dy_1T_{12}(x_1,y_1)e^{-i(\Delta_1x_1+\Delta_2y_1)}\,, \tag{A7}$$

$$\bar{\theta}_0(\Delta_1,\Delta_2)=\int dx_2dy_2\theta_0(x_2,y_2)e^{-i(\Delta_1x_2+\Delta_2y_2)}\,;$$

then the Fourier transform of (4.17) is given by

$$\bar{T}_1(\Delta_1,\Delta_2)+\bar{\theta}_0(\Delta_1,\Delta_2)\bar{T}_{12}(\Delta_1,\Delta_2)\,. \tag{A8}$$

Therefore, if (4.17) is given, we know one linear combination (A8) of $\bar{T}_1$ and $\bar{T}_{12}$. This is still insufficient to determine $f(\Delta_1,\Delta_2)$ from (A5).

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[28] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964); H. Cheng and T. T. Wu, *ibid.* **140**, B465 (1965).

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However, if we know one more scattering amplitude where the bound system of particle 2 in the potential V_2 is left in a different state $\psi_1(x_2,y_2,z_2)$, then $\bar{T}_1$ and $\bar{T}_2$ can be determined from two different linear combinations of the form (A8). Once $\bar{T}_1$ and $\bar{T}_2$ are determined, all the scattering amplitudes can be found from (4.13).

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Magnetic Monopoles in the Hydrodynamic Formulation of Quantum Mechanics*

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The nonrelativistic quantum theory of a particle having both electric and magnetic charges moving in an arbitrary external electromagnetic field is presented. The theory is based on the hydrodynamic formulation of quantum mechanics. Dirac's quantization condition for the electric and magnetic charges is rederived as a consistency condition for the motion of the probability fluid. Neither the wave function nor the electromagnetic potential, which were the source of ambiguities in all other formulations, appears in our approach. Nevertheless, this theory has all the essential features of the standard quantum mechanics, including the superposition principle.

I. INTRODUCTION

THE main source of difficulties in formulating the quantum theory of particles carrying both electric and magnetic charges is the ambiguity in the definition of the electromagnetic potential.[1] One could hopefully avoid all these difficulties if one could develop an equivalent formulation of quantum theory in which, instead of the electromagnetic potential, only the field strengths appear. Such a formulation based on the hydrodynamic form of the Schrödinger equation will be presented here. In the absence of magnetic monopoles this form of quantum mechanics is completely equivalent to the Schrödinger theory. The generalization to include magnetic monopoles is very natural and it brings about a full symmetry between electricity and magnetism. The generalized theory will be shown to possess all the basic properties of the quantum theory, including the superposition principle; however, an equivalent description in terms of a unique wave function is no longer possible.

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[1] P. A. M. Dirac, Phys. Rev. **74**, 817 (1948); N. Cabibbo and F. Ferrari, Nuovo Cimento **23**, 1147 (1962); A. S. Goldhaber, Phys. Rev. **140**, B1407 (1965); J. Schwinger, *ibid.* **144**, 1087 (1966); **173**, 1536 (1968); B. Zumino, in *Strong and Weak Interactions—Present Problems*, edited by A. Zichichi (Academic, New York, 1966); D. Zwanziger, Phys. Rev. **176**, 1480 (1968); **176**, 1489 (1968).

II. HYDRODYNAMIC FORMULATION OF QUANTUM MECHANICS

As was observed by Madelung,[2] the Schrödinger equation can be replaced by a set of four hydrodynamic-like equations. In the presence of an external electromagnetic field those equations take on the form

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0\,, \tag{1}$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)+\frac{\hbar^2}{2m^2}\nabla(\rho^{-1/2}\Delta\rho^{1/2})\,, \tag{2}$$

where the density field $\rho(\mathbf{r},t)$ and the velocity field $\mathbf{v}(\mathbf{r},t)$ are related to the modulus and the phase of the wave function and the vector potential $\mathbf{A}$ in the following manner:

$$\psi(\mathbf{r},t)=R(\mathbf{r},t)\exp[(i/\hbar)S(\mathbf{r},t)]\,, \tag{3}$$

$$\rho(\mathbf{r},t)\equiv R^2(\mathbf{r},t)\,, \tag{4}$$

$$\mathbf{v}(\mathbf{r},t)\equiv\frac{1}{m}\left(\nabla S(\mathbf{r},t)-\frac{e}{c}\mathbf{A}(\mathbf{r},t)\right). \tag{5}$$

In the standard formulation there is one-to-one correspondence between the state of the system and a set of normalized wave functions differing by a constant phase

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