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High-Energy Scattering of a Fermion with Anomalous Magnetic Moment: Nonexponentiation*

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Although it is well known that the scattering amplitude for a Dirac particle in a potential takes a simple-exponentiation form at high energies, we show here that this is no longer the case if the Dirac particle has an anomalous magnetic moment as in the case of the proton. This serves as an example of our conclusion that simple exponentiation holds only if a particle can be treated as a point with no internal structure or internal degrees of freedom. This means that simple exponentiation is unlikely to occur for high-energy amplitudes of hadron-hadron scattering.

1. INTRODUCTION

MORE than a decade ago, a number of authors[1,2] found that, for a charged scalar meson or a Dirac particle in a static field, the scattering amplitude takes the simple-exponentiation form of Molière.[3] For this reason, in the Glauber approximation,[2] for example, this exponentiation form is assumed. Recently, interest in exponentiation forms has received a new impetus as certain evidences for its existence have been discovered in field theory. In particular, the same simple-exponentiation form was found to hold in the multiphoton exchange amplitude of electron-electron scattering[4–6] and a double-exponentiation form was found for electron-photon scattering.[4,6] These developments produced the optimism that the exponentiation form holds generally for a high-energy amplitude of hardon-hadron scattering.

We wish to point out that the simple-exponentiation form holds only in the approximation in which a particle is treated as a point with no internal structure and no internal degree of freedom. This is true for a boson satisfying the Klein-Gordon equation. This is also true for a fermion satisfying the Dirac equation with no anomalous magnetic moment. The latter is because, as it turns out, the spin of such a fermion does not flip in a scattering process. A detailed discussion of the limitation of exponentiation is given elsewhere.[7,8]

In this paper, we show that exponentiation already breaks down for the scattering of a fermion in a static field if the fermion has an anomalous magnetic moment. Simple exponentation also fails for the scattering amplitude of a charged vector meson in an external field, which will be treated elsewhere.[9]

2. FERMION IN STATIC FIELD

The Dirac equation for a fermion with an anomalous magnetic moment κ in a static field $V(\mathbf{x})$ and a magnetic field $\mathbf{H}$ is[10]

$$i\frac{\partial\psi}{\partial t}=[-i\boldsymbol{\alpha}\cdot\nabla+\beta m-\kappa\beta(\boldsymbol{\Sigma}\cdot\mathbf{H}-i\boldsymbol{\alpha}\cdot\mathbf{E})+eV]\psi,\quad(2.1)$$

where, for definiteness, we choose the representation

$$\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix},\quad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$
$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \mathbf{E}=-\nabla V.\quad(2.2)$$

We sometimes use the notation $V(\mathbf{x}_\perp,z)$ and $\mathbf{H}(\mathbf{x}_\perp,z)$ instead of $V(\mathbf{x})$ and $H(\mathbf{x})$, as the occassion demands. We assume that the potential is sufficiently smooth. The boundary condition is

$$\lim_{z\to-\infty}\psi=e^{-i(Et-pz)}u_i,$$

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[1] L. I. Schiff, Phys. Rev. **103**, 443 (1956); T. T. Wu, *ibid.* **108**, 466 (1957); D. S. Saxon and L. I. Schiff, Nuovo Cimento **6**, 614 (1957).

[2] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by E. Britten *et al.* (Interscience, New York, 1959), Vol. 1.

[3] G. Molière, Z. Naturforsch. **2**, 133 (1947).

[4] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 666 (1969); Phys. Rev. **186**, 1611 (1969).

[5] F. Englert, P. Nicoletopoulos, R. Brout, and C. Truffin, Nuovo Cimento **64A**, 561 (1969).

[6] S. J. Chang and S. Ma, Phys. Rev. **188**, 2385 (1969).

[7] H. Cheng and T. T. Wu, following paper, Phys. Rev. D **3**, 2397 (1971).

[8] H. Cheng and T. T. Wu, DESY Report Nos. 71-13, 71-16, 1971 (unpublished).

[9] H. Cheng and T. T. Wu (unpublished).

[10] See, e.g., A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience, New York 1965), p. 150, Eq. (15.12).

where u_i is the initial bispinor state and is independent of $\mathbf{x}$ and t. Also

$$p=(E^2-m^2)^{1/2}.$$

We solve (2.1) in the limit $E\to\infty$. We put

$$\psi=e^{-iE(t-z)}\phi. \tag{2.3}$$

Substituting (2.3) in (2.1), we get

$$E(1-\alpha_3)\phi=[-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta m-\kappa\beta(\boldsymbol{\Sigma}\cdot\mathbf{H}-i\boldsymbol{\alpha}\cdot\mathbf{E})+eV]\phi. \tag{2.4}$$

Since $E\to\infty$, (2.4) gives

$$(1-\alpha_3)\phi\sim0. \tag{2.5}$$

Let us multiply Eq. (2.4) by $\frac{1}{2}(1+\alpha_3)$ from the left. Then the left-hand side of (2.4) vanishes as $(1+\alpha_3)\times(1-\alpha_3)=0$. For the right-hand side of (2.4), we move $(1+\alpha_3)$ to the right until it operates on ϕ. Then, as a result of (2.5), we have

$$\begin{aligned}(1+\alpha_3)\boldsymbol{\alpha}_\perp\phi&=\boldsymbol{\alpha}_\perp(1-\alpha_3)\phi\sim0,\\(1+\alpha_3)\beta\phi&=\beta(1-\alpha_3)\phi\sim0,\\(1+\alpha_3)\beta\Sigma_3\phi&=\beta\Sigma_3(1-\alpha_3)\phi\sim0,\\(1+\alpha_3)\beta\alpha_3\phi&=\beta\alpha_3(1-\alpha_3)\phi\sim0,\end{aligned}$$

where $\boldsymbol{\alpha}_\perp=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2$ is the transverse part of $\boldsymbol{\alpha}$. Thus (2.4) becomes

$$[-i(\partial/\partial z)-\kappa\beta(\boldsymbol{\Sigma}_\perp\cdot\mathbf{H}_\perp-i\boldsymbol{\alpha}_\perp\cdot\mathbf{E}_\perp)+eV]\phi\sim0. \tag{2.6}$$

The solution of (2.6) is

$$\phi=\exp\left[-ie\int_{-\infty}^{z}V(\mathbf{x}_\perp z')dz'\right]\times\left\{\exp\left(i\kappa\int_{-\infty}^{z}[\beta\boldsymbol{\Sigma}_\perp\cdot\mathbf{H}_\perp(\mathbf{x}_\perp,z')-i\beta\boldsymbol{\alpha}_\perp\cdot\mathbf{E}_\perp(\mathbf{x}_\perp,z')]dz'\right)\right\}_+u_i, \tag{2.7}$$

where $\{\ \}_+$ denotes the ordered (with respect to z') product.

The scattering amplitude is equal to

$$\mathfrak{M}_{fi}=u_f^\dagger\int d^3x\,e^{-i\Delta\cdot\mathbf{x}}[eV(\mathbf{x})-\kappa\beta\boldsymbol{\Sigma}_\perp\cdot\mathbf{H}_\perp(\mathbf{x})+i\kappa\beta\boldsymbol{\alpha}_\perp\cdot\mathbf{E}_\perp(\mathbf{x})]\phi(\mathbf{x}), \tag{2.8}$$

where u_f is the bispinor of the final state and $\boldsymbol{\Delta}$ is the momentum transfer. Substituting (2.7) into (2.8) and carrying out the integration over z, we obtain

$$\mathfrak{M}_{fi}\sim iu_f^\dagger\int d^2x_\perp e^{-i\Delta\cdot\mathbf{x}_\perp}\left[\exp\left(-ie\int_{-\infty}^{\infty}V(\mathbf{x}_\perp,z)dz\right)\times\left\{\exp\left(\kappa\int_{-\infty}^{\infty}[i\beta\boldsymbol{\Sigma}_\perp\cdot\mathbf{H}_\perp(\mathbf{x}_\perp,z)+\beta\boldsymbol{\alpha}_\perp\cdot E_\perp(\mathbf{x}_\perp,z)]dz\right)\right\}_+-1\right]u_i. \tag{2.9}$$

In (2.9) or (2.7), the matrices are 4×4. It is possible to reduce them into 2×2 matrices. Let us first eliminate the term eV in (2.6) by the substitution

$$\phi=\exp\left[-ie\int_{-\infty}^{z}V(\mathbf{x}_\perp,z')dz'\right]F. \tag{2.10}$$

Then

$$-i(\partial/\partial z)F\sim\kappa\beta(\boldsymbol{\Sigma}_\perp\cdot\mathbf{H}_\perp-i\boldsymbol{\alpha}_\perp\cdot\mathbf{E}_\perp)F=\kappa\begin{pmatrix}\boldsymbol{\sigma}_\perp\cdot\mathbf{H}_\perp & -i\boldsymbol{\sigma}_\perp\cdot\mathbf{E}_\perp\\ i\boldsymbol{\sigma}_\perp\cdot\mathbf{E}_\perp & -\boldsymbol{\sigma}_\perp\cdot\mathbf{H}_\perp\end{pmatrix}F. \tag{2.11}$$

Next, as a result of (2.5), we set

$$F\sim(\tfrac{1}{2}E/m)^{1/2}\begin{pmatrix}\chi\\ \sigma_3\chi\end{pmatrix}. \tag{2.12}$$

Substituting (2.12) into (2.11), we get

$$-i(\partial/\partial z)\chi=A\chi, \tag{2.13}$$

where

$$\begin{aligned}A&=\kappa(\boldsymbol{\sigma}_\perp\cdot\mathbf{H}_\perp-i\boldsymbol{\sigma}_\perp\cdot\mathbf{E}_\perp\sigma_3)\\&=\kappa\sigma_1(H_1+E_2)+\kappa\sigma_2(H_2-E_1)\\&=\kappa\begin{pmatrix}0 & H_1-iH_2+i(E_1-iE_2)\\ H_1+iH_2-i(E_1+iE_2) & 0\end{pmatrix}.\end{aligned} \tag{2.14}$$

The matrix A is 2×2.

The scattering amplitude $\mathfrak{M}_{fi}$ can be expressed in terms of A. We have

$$\mathfrak{M}_{fi}\sim i\chi_f^\dagger\int d^2x_\perp e^{-i\Delta\cdot\mathbf{x}_\perp}\times\left[\exp\left(-ie\int_{-\infty}^{\infty}V(\mathbf{x}_\perp,z)dz\right)\times\left\{\exp\left(i\int_{-\infty}^{\infty}A(\mathbf{x}_\perp,z)dz\right)\right\}_+-1\right]\chi_iE/m, \tag{2.15}$$

where χ_i and χ_f are the *normalized*, initial- and final-state spinors.

The spinor function χ is obtained from (2.13) as

$$\chi=\left\{\exp\left[i\int_{-\infty}^{z}A(\mathbf{x}_\perp,z')dz'\right]\right\}_+\chi_i. \tag{2.16}$$

That the scattering amplitude and the wave function χ must be expressed in terms of ordered products means that simple exponentiation does not occur. As is obvious from (2.13), the anomalous magnetic moment κ couples the two spinor states of χ, and accounts for the breakdown of simple exponentiation.

In Sec. 3, we give the closed form of

$$\left\{\exp\left[i\int_{-\infty}^{z} A(\mathbf{x}_\perp,z')dz'\right]\right\}_+$$

explicitly in the special case $V(\mathbf{x})=V(|\mathbf{x}|)$ and $\mathbf{H}=0$.

3. CENTRAL STATIC POTENTIAL

In many instances it is helpful to have specific examples in which closed-form solutions are possible. We therefore give a solvable example here. Consider the case

$$H_1=H_2=0,$$

and

$$V=V(|\mathbf{x}|)$$

a central field. Then

$$E_1=-(x/|\mathbf{x}|)V'(|\mathbf{x}|), \tag{3.1}$$

$$E_2=-(y/|\mathbf{x}|)V'(|\mathbf{x}|). \tag{3.2}$$

Thus (2.14) becomes

$$A=\kappa|\mathbf{x}|^{-1}V'(|\mathbf{x}|)\begin{pmatrix} 0 & -i(x-iy) \\ i(x+iy) & 0 \end{pmatrix}. \tag{3.3}$$

Notice that aside from the factor in front, the matrix in (3.3) is independent of z. Thus $A(\mathbf{x}_\perp,z)$ and $A(\mathbf{x}_\perp,z')$ commute. This is why we can solve (2.13) in closed form.

We diagonalize the matrix in (3.3) and obtain

$$\begin{pmatrix} 0 & -i(x-iy) \\ i(x+iy) & 0 \end{pmatrix} =S\begin{pmatrix} -(x^2+y^2)^{1/2} & 0 \\ 0 & (x^2+y^2)^{1/2} \end{pmatrix}S^{-1}, \tag{3.4}$$

where

$$S=\begin{pmatrix} (x-iy)^{1/2} & (x-iy)^{1/2} \\ -i(x+iy)^{1/2} & i(x+iy)^{1/2} \end{pmatrix} \tag{3.5}$$

and

$$S^{-1}=\begin{pmatrix} i(x+iy)^{1/2} & -(x-iy)^{1/2} \\ i(x+iy)^{1/2} & (x-iy)^{1/2} \end{pmatrix}(2i)^{-1}(x^2+y^2)^{-1/2}. \tag{3.6}$$

Equations (3.3) and (3.4) imply

$$\left\{\exp\left[i\int_{-\infty}^{z} A(\mathbf{x}_\perp,z')dz'\right]\right\}_+ =\exp\left[i\int_{-\infty}^{z} A(\mathbf{x}_\perp,z')dz'\right] =S\begin{pmatrix} e^{-i\kappa W} & 0 \\ 0 & e^{i\kappa W} \end{pmatrix}S^{-1}, \tag{3.7}$$

where

$$W=(x^2+y^2)^{1/2}\int_{-\infty}^{z}(x^2+y^2+z'^2)^{-1/2} \times V'((x^2+y^2+z'^2)^{1/2})dz'. \tag{3.8}$$

Performing the matrix multiplication in (3.7), we get

$$\exp\left[i\int_{-\infty}^{z} A(\mathbf{x}_\perp,z')dz'\right]=\begin{pmatrix} \cos\kappa W & (x-iy)(x^2+y^2)^{-1/2}\sin\kappa W \\ -(x+iy)(x^2+y^2)^{-1/2}\sin\kappa W & \cos\kappa W \end{pmatrix}. \tag{3.9}$$

The wave function χ and the scattering amplitude $\mathfrak{M}_{fi}$ can be explicitly obtained from (2.15), (2.16), and (3.9).

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