High-Energy Behavior of a Spin-Flip Amplitude*

HUNG CHENG†‡
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139||
and
Deutsches Elektronen-Synchrotron, DESY, Hamburg, Germany

AND

TAI TSUN WU‡
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138||
and
Deutsches Elektronon-Synchrotron, DESY, Hamburg, Germany
and
HH Eidelstedt, Hinschstrasse 25, Germany

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The perturbation series for the spin-flip amplitude, where two units of spin are exchanged, of the scattering of two vector mesons through fermion–fermion–vector-meson coupling is studied in the limit of high energies. Similar to the case of elastic scattering (without spin flip) in quantum electrodynamics, there are, for this spin-flip amplitude, terms in the perturbation series of the orders of magnitudes s, $s \ln s$, $s(\ln s)^2$, $s(\ln s)^3$, etc. For each n, the leading term in the coefficient of $s(\ln s)^n$ is due to Feynman diagrams with $n+2$ closed fermion loops. These leading terms are found explicitly for the exactly forward direction, and are then summed. Because these terms alternate in sign, at high energies this sum is smaller than any one of the leading terms in the series. This sum is studied in detail for the case where massless photons are exchanged, and is found to have a simple factorization property. Some of the results are extended to the more general case of exchanging massive neutral vector mesons. The high-energy behavior of this spin-flip amplitude is qualitatively different from that of the spin-nonflip amplitude studied previously.

1. INTRODUCTION

RECENTLY, we have studied the high-energy behavior of elastic scattering amplitudes in the exactly forward direction in quantum electrodynamics.[1] This was accomplished through the method of summing the leading terms from the one-tower diagrams. Although the result, taken literally, violates the s-channel unitarity,[1,2] a suitable interpretation in terms of pionization[3] furnishes a number of predictions on some of the fundamental questions in high-energy physics.[4]

An important ingredient in this development, as previously emphasized,[5] is the fact that the leading terms, which are summed, are all positive. This fact is related closely to the optical theorem, and thus its validity has been shown only for the spin-nonflip amplitudes in the forward direction. In general, for other amplitudes, the leading terms may or may not be of the same sign. It is the purpose of this paper to carry out essentially the same considerations as those in Ref. 1 for the spin-flip amplitude in photon-photon scattering. As in Ref. 1, we shall restrict ourselves to the exactly forward direction, but the "photon" may be a massive neutral vector meson.

To the lowest nontrivial order, namely, the eighth order, photon-photon scattering in the forward direction has been studied in detail.[6] To this order, it has been found that there are, at high energies, two large amplitudes of comparable magnitude. One of these two is of course the spin-nonflip amplitude, while the other one is the spin-flip amplitude involving the exchange of two units of spin. It is this latter amplitude that we study here to higher orders.

The motivation for carrying out the present study is as follows. We have been trying to learn about the high-energy behavior of hadronic scattering processes from relativistic field theory in general and quantum electrodynamics in particular. In this learning process, two very different but equally important kinds of steps are necessary: calculations and interpretations. Examples of the former are the initial ones on two-body processes in lowest nontrivial orders[7] and that on logarithmic factors,[1] while examples of the latter are those on the impact picture[8] and limit of cross sections.[4] At present we have a satisfactory understanding of two-body elastic processes without the exchange of any quantum number,[4] but the situation is much less clear when quantum numbers are exchanged. In this paper, we provide a *calculation* involving the exchange of spins. It is possible for an amplitude involving the exchange of spins to have similar high-energy behavior as the spin-nonflip amplitude[1] already studied. An example of such a case is provided by photon-photon scattering in scalar

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‡ John S. Guggenheim Memorial Fellow.
|| Permanent address.
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[5] See (1) of Ref. 4.
[6] H. Cheng and T. T. Wu, Phys. Rev. D 1, 3414 (1970).
[7] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 666 (1969); Phys. Rev. **182**, 1852 (1969); **182**, 1868 (1969); **182**, 1873 (1969); **182**, 1899 (1969).
[8] H. Cheng and T. T. Wu, Phys. Rev. Letters **23**, 670 (1969).

electrodynamics. But it is also possible for a spin-flip amplitude to have a *different* high-energy behavior. The result of the present paper shows an example of this *high-energy behavior of the second type.*

The calculation in this paper is somewhat complicated. However, these two kinds of high-energy behavior can be understood as follows with very little computation. In the analysis of the high-energy behavior of the spin-nonflip amplitude as given in Ref. 1, a kernel $K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')$ plays a central role. Let θ be the angle between the two-dimensional vectors $\mathbf{q}_\perp$ and $\mathbf{q}_\perp'$, then [see Eq. (6.1) of Ref. 1]

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=\mathbf{q}_\perp^2\mathbf{q}_\perp'^2\int_0^1 dx\int_0^1 dy \times\frac{[x(1-x)+y(1-y)]-x(1-x)y(1-y)[5+\cos 2\theta]}{x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2+m^2}. \quad (1.1)$$

Thus $K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')$ consists of two parts, a θ-independent part which is *positive,* and a $\cos 2\theta$ part which is *negative.* Since the high-energy behavior of the spin-nonflip amplitude is controlled by the spectrum[1] of the operator with kernel

$$\mathcal{K}_0(z,z')=\frac{z}{z+\lambda^2}\frac{z'}{z'+\lambda^2}\int_0^1 dx\int_0^1 dy \times\frac{x(1-x)+y(1-y)-5x(1-x)y(1-y)}{x(1-x)z+y(1-y)z'+m^2}, \quad (1.2)$$

that of the spin-flip amplitude must analogously be controlled by the spectrum of

$$\mathcal{K}_1(z,z')=-\frac{1}{2}\frac{z}{z+\lambda^2}\frac{z'}{z'+\lambda^2}\int_0^1 dx\int_0^1 dy \times\frac{x(1-x)y(1-y)}{x(1-x)z+y(1-y)z'+m^2}. \quad (1.3)$$

In (1.2) and (1.3), λ is the mass of the exchanged vector meson. Suppose we define an operator $\mathcal{K}_1$ by

$$(\mathcal{K}_1 f)(z)=\int_0^\infty dz'\mathcal{K}_1(z,z')f(z'). \quad (1.4)$$

Then the spectrum of $\mathcal{K}_1$ is (see Appendix A for details)

$$[-\pi^3/128,\,0]. \quad (1.5)$$

Consequently, in order to get the desired high-energy behavior of this spin-flip amplitude, we need to study the improper eigenfunctions of $\mathcal{K}_1$ near 0. The fact that these improper eigenfunctions are rapidly oscillating functions of z is the underlying reason why the high-energy behavior for the spin-flip amplitude is different. This rapid oscillation is also the source of mathematical difficulties.

These two types of high-energy behavior can also be seen from the following slightly different point of view. Both of the operators $\mathcal{K}_0$ and $\mathcal{K}_1$ are of non-Fredholm type, as evidenced by their continuous spectra. If the large-z behavior of these kernels are reduced, such as in the Lee-Wick theory,[9] these kernels may then be of Fredholm type. In this case the eigenvalues of these kernels are discrete. While the high-energy behavior of the spin-nonflip amplitude is then controlled by the largest discrete eigenvalue of $\mathcal{K}_0$,[10] that of the spin-flip amplitude is instead controlled by the accumulation point of the eigenvalues of $\mathcal{K}_1$. Thus the high-energy behavior of the second type is further removed from that expected from a single Regge pole.[11]

On the basis of this discussion, the high-energy behavior of the spin-flip amplitude in the exactly forward direction is of the form

$$i(\text{const})s(\ln s)^{-b}(\ln\ln s)^{-c}, \quad (1.6)$$

with possible further factors of the form $(\ln\ln\ln s)$, etc., omitted. This is to be contrasted with the result of Ref. 1 in the form

$$i(\text{const})s^{1+11\alpha^2\pi/32}(\ln s)^{-2}. \quad (1.7)$$

We shall be concerned mostly with the evaluation of the exponent b. As we shall see below, the value of b depends on mass ratios.

In Secs. 2 and 3, the problem is precisely formulated. Because of the mathematical difficulties already mentioned, we first treat in Secs. 4 and 5 the special case $\lambda=0$, i.e., the special case where photons are exchanged. In this case, we are able to compute both of the exponents b and c. The much more difficult case of $\lambda>0$ is then treated in Sec. 6, where we obtain the exponent b but not the exponent c.

2. SPIN-FLIP AMPLITUDE

Consider the scattering process

$$1+2\rightarrow 1'+2',$$

where all four particles are vector mesons. If the particles 1 and 1′, and also 2 and 2′, are identical, then this is an elastic scattering process; otherwise it is inelastic. Let the masses of these vector mesons be, respectively, M_1, M_2, M_1', and M_2'.

As stated in the Introduction, we shall restrict ourselves entirely to the exactly forward direction. For the spin-flip amplitude of interest, two units of spin are exchanged. Thus, for example, the incoming vector mesons 1 and 2 are both right-circularly polarized while the outgoing vector mesons 1′ and 2′ are left-circularly polarized. We are thus only interested in the transverse-to-transverse impact factors. In the exactly forward direction, these impact factors depend only on one vari-

[9] T. D. Lee and G. C. Wick, Nucl. Phys. **B9**, 209 (1969).
[10] H. Cheng and T. T. Wu (to be published).
[11] T. Regge, Nuovo Cimento **14**, 951 (1959).

able $\mathbf{q}_\perp$ and are explicitly given by[12]

$$\begin{aligned}\mathcal{G}^{ii'}(\mathbf{q}_\perp)=\tfrac{1}{2}e^2 f_i f_i' \int_0^1 d\alpha \int_0^1 d\beta \\ \times\{-8\alpha(1-\alpha)\beta(1-\beta)(\mathbf{q}_\perp\cdot\boldsymbol{\varepsilon}_i)(\mathbf{q}_\perp\cdot\boldsymbol{\varepsilon}_i') \\ +(\boldsymbol{\varepsilon}_i\cdot\boldsymbol{\varepsilon}_i')[\mathbf{q}_\perp^2(1-8\beta(1-\beta)(\tfrac{1}{2}-\alpha)^2) \\ -\beta(1-\beta)(M_i^2+M_i'^2)]\}\{\alpha(1-\alpha)\mathbf{q}_\perp^2+m^2 \\ -\beta(1-\beta)[M_i^2\alpha+M_i'^2(1-\alpha)]\}^{-1} \quad (2.1)\end{aligned}$$

for $i=1, 2$. Here $\boldsymbol{\varepsilon}$ denotes the polarization vector, f_i and f_i' are the coupling constants of particles i and i' to the fermion of mass m, and e is that of the exchanged vector meson of mass λ. So far as the spin-flip amplitude is concerned, only the term proportional to $(\mathbf{q}_\perp\cdot\boldsymbol{\varepsilon}_i)\times(\mathbf{q}_\perp\cdot\boldsymbol{\varepsilon}_i')$ is relevant.

Similar to the treatment of Ref. 1, we follow the procedure of summing the leading terms, i.e., the terms with the highest power of $\ln s$. We therefore study the tower diagrams as shown in Fig. 1, where the number of fermion loops is designated as $n+2$, with $n=0, 1, 2, \ldots$. For given n, the leading contribution at high energies to the scattering amplitude is

$$i(n!)^{-1}s(\ln s)^n(J^{11'},\mathcal{K}^n J^{22'}) \quad (2.2)$$

[see Eq. (5.19) of Ref. 1]. For the exactly forward direction, we have

$$J^{ii'}(\mathbf{q}_\perp)=(\mathbf{q}_\perp^2+\lambda^2)^{-1}\mathcal{G}^{ii'}(\mathbf{q}_\perp) \quad (2.3)$$

for $i=1, 2$, and the kernel $\mathcal{K}$ is given by, with (1.1),

$$\mathcal{K}(0,\mathbf{q}_\perp,\mathbf{q}_\perp')=4e^4(2\pi)^{-3}(\mathbf{q}_\perp^2+\lambda^2)^{-1}(\mathbf{q}_\perp'^2+\lambda^2)^{-1} \times K_0(\mathbf{q}_\perp,\mathbf{q}_\perp'). \quad (2.4)$$

The formula (2.2) gives all the amplitudes. To pick out the spin-flip amplitude of interest, define, because of (2.1) and (2.3),

$$\begin{aligned}J^{(i)}(z)=-4\frac{z}{z+\lambda^2}\int_0^1 d\alpha\int_0^1 d\beta\,\alpha(1-\alpha)\beta(1-\beta) \\ \times\{\alpha(1-\alpha)z+m^2 \\ -\beta(1-\beta)[M_i^2\alpha+M_i'^2(1-\alpha)]\}^{-1} \quad (2.5)\end{aligned}$$

for $i=1, 2$. Because of (1.3) and (1.4), the spin-flip part of (2.2) is[13]

$$i(4\pi)^{-1}e^4 f_1 f_1' f_2 f_2'(n!)^{-1}s[2e^4(2\pi)^{-4}\ln s]^n \times(J^{(1)},\mathcal{K}_1{}^n J^{(2)}). \quad (2.6)$$

Because $-\mathcal{K}_1$ is positive definite (see Appendix A) this series alternates in sign, at least when $M_1=M_2$ and $M_1'=M_2'$. When these leading terms (2.6) are summed, the total spin-flip amplitude in the exactly forward direction is

$$is(4\pi)^{-1}e^4 f_1 f_1' f_2 f_2' A\,, \quad (2.7)$$

where

$$A=(J^{(1)},\exp(2\Lambda\mathcal{K}_1)\,J^{(2)})\,, \quad (2.8)$$

with

$$\Lambda=e^4(2\pi)^{-4}\ln s=\pi^{-2}\alpha^2\ln s\,. \quad (2.9)$$

In (2.9), α is the fine-structure constant.

In this paper, we study in some detail the asymptotic behavior of A, as defined by (2.8), for large positive Λ.

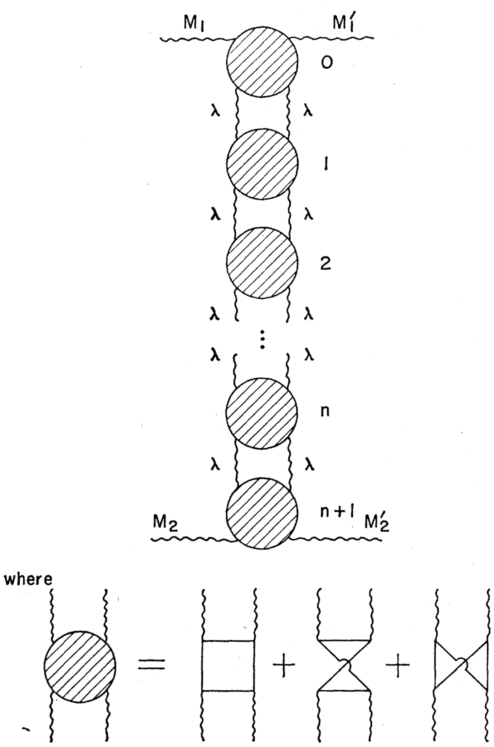

FIG. 1. Tower diagrams for the scattering of two vector mesons.

3. EIGENFUNCTION EXPANSION

In order to study the asymptotic behavior of A, we expand $J^{(1)}$ and $J^{(2)}$ in terms of the continuous eigenfunctions[14] of $\mathcal{K}_1$. It is convenient to label these eigenfunctions not by the corresponding eigenvalue but by the rate of oscillation for large values of z (see Sec. 4). We thus write in view of (1.3)

$$\mathcal{K}_1 f_t=-\tfrac{1}{2}\mu(t)f_t\,, \quad (3.1)$$

or more explicitly

$$\frac{z}{z+\lambda^2}\int_0^\infty dz' f_t(z')\frac{z'}{z'+\lambda^2} \times\int_0^1 dx\int_0^1 dy\,\frac{x(1-x)y(1-y)}{x(1-x)z+y(1-y)z'+m^2} =\mu(t)f_t(z). \quad (3.2)$$

[12] H. Cheng and T. T. Wu, Phys. Rev. D 1, 459 (1970).

[13] For $\mathcal{K}$ the operator is defined with respect to the integration $(2\pi)^{-2}\int d\mathbf{q}_\perp$, while for $\mathcal{K}_1$ the operator is defined with respect to $\int dz=\int d\mathbf{q}_\perp^2$. There is therefore a factor 4π.

[14] These "eigenfunctions" are of course not square integrable.

The labeling is such that $\mu(t)$ is a decreasing function of t for all positive t and

$$\lim_{t\to\infty}\mu(t)=0. \quad (3.3)$$

(Note that this t has nothing to do with momentum transfer, which is very small or zero in this paper.)

If we normalize the eigenfunctions by[15]

$$\int_0^\infty dz\, f_t(z)f_{t'}(z)=\delta(t-t'), \quad (3.4)$$

then

$$\int_0^\infty dt\, f_t(z)f_t(z')=\delta(z-z'). \quad (3.5)$$

This relation (3.5) follows, in a standard way, from assuming completeness and

$$\int_0^\infty dz' f_t(z')\left[\int_0^\infty dt' f_{t'}(z)f_{t'}(z')\right]$$

$$=\int_0^\infty dt' f_{t'}(z)\int_0^\infty dz' f_t(z')f_{t'}(z')$$

$$=\int_0^\infty dt' f_{t'}(z)\delta(t-t')=f_t(z).$$

With (3.5), the A of (2.8) can be recast in the form

$$A=\int_0^\infty dt\, e^{-\mu(t)\Lambda}\left[\int_0^\infty dz\, J^{(1)}(z)f_t(z)\right]$$

$$\times\left[\int_0^\infty dz' J^{(2)}(z')f_t(z')\right]. \quad (3.6)$$

In this form, the asymptotic behavior of A for large Λ is determined by the behavior of

$$\int_0^\infty dz\, J^{(i)}(z)f_t(z) \quad (i=1,2) \quad (3.7)$$

for large t.

So far as the authors are aware, there is no general method of determining asymptotically the solution of an integral equation such as (3.2). For example, the high modes of a drum are not known in general, and are believed to depend critically on the shape of the drum. Fortunately, for the present problem, a certain amount of progress can be made in spite of this mathematical difficulty. We shall begin with the quantum-electrodynamic case where all the vector mesons are massless.

4. QUANTUM ELECTRODYNAMICS

A. Reduction of Problem

We shall first study the physically interesting case of quantum electrodynamics, where there is only one mass m. More precisely, in this section we are concerned with the special case where

$$M_1=M_1'=M_2=M_2'=\lambda=0 \quad (4.1)$$

and

$$f_1=f_1'=f_2=f_2'=e. \quad (4.2)$$

In particular, it follows from (2.5) and (3.2) that

$$J(z)=J^{(1)}(z)=J^{(2)}(z)$$

$$=-\tfrac{2}{3}\int_0^1 d\alpha\, \alpha(1-\alpha)[\alpha(1-\alpha)z+m^2]^{-1} \quad (4.3)$$

and

$$\int_0^\infty dz' f_t(z')\int_0^1 dx\int_0^1 dy\frac{x(1-x)y(1-y)}{x(1-x)z+y(1-y)z'+m^2}$$

$$=\mu(t)f_t(z). \quad (4.4)$$

Moreover, in this case of quantum electrodynamics, the total spin-flip amplitude (2.7) in the exactly forward direction from the tower diagrams is

$$is(4\pi)^{-1}e^8A, \quad (4.5)$$

where

$$A=\int_0^\infty dt\, e^{-\mu(t)\Lambda}\left[\int_0^\infty dz\, J(z)f_t(z)\right]^2. \quad (4.6)$$

This formula for A can be greatly simplified by the following observation. Let $z=0$ in (4.4); then the x integration can be carried out trivially, with the result that

$$\tfrac{2}{3}\int_0^\infty dz' f_t(z')\int_0^1 dy\, y(1-y)[y(1-y)z'+m^2]^{-1}$$

$$=\mu(t)f_t(0). \quad (4.7)$$

If the dummy variables y of (4.7) and α of (4.3) are identified, it is seen that (4.7) is simply

$$\int_0^\infty dz' f_t(z')J(z')=-\mu(t)f_t(0). \quad (4.8)$$

The left-hand side of (4.8) is just the integral required in (4.6). Therefore, A is given by

$$A=\int_0^\infty dt\, e^{-\mu(t)\Lambda}[\mu(t)]^2[f_t(0)]^2. \quad (4.9)$$

At this stage, we can already get some preliminary information about the high-energy behavior of the spin-flip amplitude. Because of the analytic structure of (4.4), $\mu(t)$ is expected to be exponentially small for $t\to\infty$. If the dependence of $f_t(0)$ on t is neglected, then a comparison of (4.9) with (1.6) shows that

$$b=2. \quad (4.10)$$

In the remainder of this section, we make this more precise and also find c.

[15] Since (3.2) is real, we have chosen $f_t(z)$ to be real.

B. Discontinuity

Let

$$v=1+\tfrac{1}{2}z/m^2$$

and

$$v'=1+z'/m^2; \tag{4.11}$$

then

$$\int_1^\infty dv'\bar{f}_t(v')\int_0^1 dx\int_0^1 dy\frac{(1-x^2)(1-y^2)}{(1-x^2)v+(1-y^2)v'+x^2+y^2}=4\mu(t)\bar{f}_t(v) \tag{4.12}$$

for $v\geq 1$, where

$$\bar{f}_t(v)=\sqrt{2}mf_t(z) \tag{4.13}$$

satisfies

$$\int_1^\infty \bar{f}_t(v)\bar{f}_{t'}(v)dv=\delta(t-t') \tag{4.14}$$

because of (3.4).

For $v'\geq 1$, the kernel of (4.12)

$$\int_0^1 dx\int_0^1 dy\frac{(1-x^2)(1-y^2)}{(1-x^2)v+(1-y^2)v'+x^2+y^2} \tag{4.15}$$

is an analytic function of the complex variable v except for a branch cut from $-\infty$ to -1 along the negative real axis. Define $D(v,v')$ to be the discontinuity of this kernel across the branch cut (except for a factor 2π):

$$D(v,v')=\int_0^1 dx\int_0^1 dy(1-x^2)(1-y^2)\times\delta(-(1-x^2)v+(1-y^2)v'+x^2+y^2). \tag{4.16}$$

Note that we have changed the sign of v so that both v and v' are real and larger than 1.

In Appendix B, we explicitly carry out the integration over x and y for $D(v,v')$. The result is surprisingly simple:

$$D(v,v')=\tfrac{1}{16}[(v+1)(v'-1)]^{-3/2}\Big\{-(v-1)^{1/2}(v'-1)^{1/2}\times[3(v+v')+2]+[3(v^2+v'^2)+2vv'-8]\times\ln\frac{(v-1)^{1/2}+(v'-1)^{1/2}}{|v-v'|^{1/2}}\Big\}. \tag{4.17}$$

In particular, $D(v,v')$ has a logarithmic singularity at $v=v'$, and

$$[(v+1)/(v-1)]^{3/2}D(v,v') \tag{4.18}$$

is symmetrical in v and v'. It is also easily verified from (4.17) or (4.16) that

$$\lim_{v'\to 1}D(v,v')=\tfrac{2}{3}[(v-1)(v+1)^3]^{-1/2}. \tag{4.19}$$

C. Asymptotic Solution of Integral Equation

We want to solve (4.12) approximately for small $\mu(t)$, or by (3.3) for large t. Instead of this complicated (4.12), consider first

$$\int_1^\infty dv'\varphi(v')(v+v')^{-1}=\bar{\mu}\varphi(v), \tag{4.20}$$

which has previously been used as an example in Appendix B of Ref. 1. Notice the similarity between (4.12) and (4.20) in that the corresponding discontinuity for the kernel of (4.20) is simply $\delta(v-v')$. The solution of (4.20) has been given by Mehler[16] as[17]

$$\varphi(v)=P_{it-\frac{1}{2}}(v), \tag{4.21}$$

where P is the Legendre function of the first kind. Here the eigenvalue $\bar{\mu}$ is related to t by

$$\bar{\mu}=\pi\ \mathrm{sech}\pi t. \tag{4.22}$$

This solution is studied in some detail in Appendix C.

In order to make use of solution (4.21), we compute the integral

$$\int_1^\infty dv'P_{it-\frac{1}{2}}(v')\int_0^1 dx\int_0^1 dy\times\frac{(1-x^2)(1-y^2)}{(1-x^2)v+(1-y^2)v'+x^2+y^2}. \tag{4.23}$$

This is carried out in Appendix D for large t, and the result is given by (D21). In particular, from (D8) for t large and v not close to 1, we have

$$(4.23)\sim\tfrac{1}{4}\pi^2t^{-1}\ \mathrm{sech}\pi t\ P_{it-\frac{1}{2}}(v). \tag{4.24}$$

A comparison of (4.24) with (4.12) shows that, for t large,

$$\mu(t)\sim(16)^{-1}\pi^2t^{-1}\ \mathrm{sech}\pi t,$$

or more simply

$$\mu(t)\sim\tfrac{1}{8}\pi^2t^{-1}e^{-\pi t}. \tag{4.25}$$

We wish to emphasize that this relation (4.25) between t and μ is quite complicated. It is for this reason that, as stated in Sec. 3, we do not label the eigenfunctions by the eigenvalue μ. If (4.25) is inverted to express t in terms of μ, we get

$$t=t(\mu)\sim\frac{1}{\pi}\left(\ln\frac{\pi^3}{8\mu}-\ln\ln\frac{\pi^3}{8\mu}\right) \tag{4.26}$$

for small μ. In this way, logarithms of logarithms appear.

It remains to give an approximate expression for $\bar{f}_t(v)$. Since[17]

$$t\tanh\mu t\int_1^\infty P_{it-\frac{1}{2}}(v)P_{it'-\frac{1}{2}}(v)dv=\delta(t-t'), \tag{4.27}$$

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[16] F. G. Mehler, Math. Ann. **18**, 161 (1881).

[17] A much more convenient reference is *Bateman Manuscript Project, Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, pp. 174–175.

we have, for large t,

$$\bar{f}_t(v)\sim t^{1/2}P_{it-\frac{1}{2}}(v). \qquad (4.28)$$

As seen from (D21) or (D20), (4.28) is not accurate when v is close to 1. Nevertheless, for most purposes of determining the high-energy behavior of the spin-flip amplitude in quantum electrodynamics from the tower diagrams of Fig. 1 in Sec. 4 D, (4.28) is accurate enough. From (4.13) on the basis of (4.28), we get finally

$$f_t(z)\sim(\tfrac{1}{2}t)^{1/2}m^{-1}P_{it-\frac{1}{2}}(1+\tfrac{1}{2}m^{-2}z). \qquad (4.29)$$

D. High-Energy Behavior

We now have all the necessary information to get the high-energy behavior of the spin-flip amplitude in quantum electrodynamics. For A given by (4.9), we use (4.26) and[18]

$$f_t(0)\sim\tfrac{2}{3}\pi^{-1}(2t)^{1/2}m^{-1} \qquad (4.30)$$

on the basis of (D18), (D8), and (4.29). Therefore,

$$A\sim(8/9)\pi^{-2}m^{-2}\int_0^\infty tdt\, e^{-\mu(t)\Lambda}[\mu(t)]^2$$
$$\sim(8/9)\pi^{-4}m^{-2}\int\mu d\mu\, e^{-\mu\Lambda}[\ln(\tfrac{1}{8}\pi^3\mu^{-1}) -\ln\ln(\tfrac{1}{8}\pi^3\mu^{-1})-1]$$
$$\sim(8/9)\pi^{-4}m^{-2}\Lambda^{-2}[\ln(\tfrac{1}{8}\pi^3\Lambda) -\ln\ln(\tfrac{1}{8}\pi^3\Lambda)-2+\gamma], \qquad (4.31)$$

where $\gamma=0.57722$ is Euler's constant. By (2.9) and (4.5), the desired high-energy behavior for the spin-flip amplitude in the exactly forward direction is[19]

$$is(512/9)m^{-2}\pi^3(\ln s)^{-2}[\ln(\tfrac{1}{8}\pi\alpha^2\ln s) -\ln\ln(\tfrac{1}{8}\pi\alpha^2\ln s)-1+\gamma]. \qquad (4.32)$$

Finally, a comparison of (4.32) with (1.6) verifies (4.10), which has been obtained very simply in Sec. 4 A. Equation (4.32) gives the more precise information that

$$b=2 \quad\text{and}\quad c=-1. \qquad (4.33)$$

In this case of quantum electrodynamics, there is only one mass, namely, that of the fermion. For this reason, both b and c must be simple numbers. In Sec. 5 we still keep $\lambda=0$ but introduce the M's. It is then seen that b depends on mass ratios.

5. SCATTERING OF VECTOR MESONS

We next generalize the results of Sec. 4 to allow M_1, M_1', M_2, and M_2' to be positive. The restriction $\lambda=0$ is still retained.

[18] If the approximation (4.29) is used directly, the answer for the high-energy behavior of the spin-flip amplitude changes by a factor $(\frac{3}{4}\pi)^2$. Such an over-all factor is in any case not of great physical interest.

[19] Note that the leading term of this result (4.32) is independent of the coupling constant e. Compare, however, with (5.17).

A. Elastic Scattering

Instead of dealing directly with the general case of Sec. 5 B, we first consider the case where

$$M_1=M_1' \quad\text{and}\quad M_2=M_2'. \qquad (5.1)$$

In this case, (4.3) is replaced by

$$J^{(i)}(z)=-4\int_0^1 d\alpha\int_0^1 d\beta\,\alpha(1-\alpha)\beta(1-\beta) \times[\alpha(1-\alpha)z+m^2-\beta(1-\beta)M_i^2]^{-1} \qquad (5.2)$$

for $i=1$, 2. Equation (4.4) is not changed, while, instead of (4.5) and (4.6), the spin-flip amplitude is given by (2.7) with A expressed by (3.6).

It is convenient to use the mass ratios

$$R_1=\tfrac{1}{4}M_1^2/m^2 \quad\text{and}\quad R_2=\tfrac{1}{4}M_2^2/m^2. \qquad (5.3)$$

In terms of these ratios, $J^{(i)}(z)$ takes the form, after the change of variable (4.11),

$$J^{(i)}(z)=-\tfrac{1}{2}m^{-2}\int_0^1 dx\int_0^1 dy(1-x^2)(1-y^2) \times[(1-x^2)v+(1+x^2)-2(1-y^2)R_i]^{-1}. \qquad (5.4)$$

If these $J^{(i)}(z)$ are considered to be functions of the complex variable v, then they are analytic except for a branch cut from $-\infty$ to $-v_0^{(i)}$ along the real axis, where

$$v_0^{(i)}=1-2R_i. \qquad (5.5)$$

Note that $v_0^{(i)}$ may be negative. However, the stability of the vector mesons against decay into two fermions requires that

$$R_1<1 \quad\text{and}\quad R_2<1. \qquad (5.6)$$

Thus

$$v_0^{(1)}>-1 \quad\text{and}\quad v_0^{(2)}>-1. \qquad (5.7)$$

In other words, the branch cuts along the real axis of the v plane always stop below 1. Define $D^{(i)}$ to be the discontinuity across the cuts (except for a factor $-\pi m^{-2}$)

$$D^{(i)}(v)=\int_0^1 dx\int_0^1 dy(1-x^2)(1-y^2) \times\delta[-(1-x^2)v+(1+x^2)-2(1-y^2)R_i] =D(v,v_0^{(i)}) \qquad (5.8)$$

by (4.16) and (5.5). Note that, since $v_0^{(i)}<1$, (5.8) requires an extension of the domain of definition for $D(v,v')$. With this extension, the explicit result (4.17) holds only for $v\geq1$, i.e.,

$$D(v,v')=\tfrac{1}{16}[(v+1)(1-v')]^{-3/2}\{(v-1)^{1/2}(1-v')^{1/2} \times[3(v+v')+2]-[3(v^2+v'^2)+2vv'-8] \times\tan^{-1}[(1-v')/(v-1)]^{1/2}\} \qquad (5.9)$$

for $v>1>v'$. When $v<1$ and $v'<1$, the answer is much simpler:

$$D(v,v')=\tfrac{1}{16}\pi[(1+v)(1-v')]^{-3/2}\times[8-2vv'-3(v^2+v'^2)]. \quad (5.10)$$

In particular, for $v=v'$,

$$D(v,v)=\tfrac12\pi(1-v^2)^{-1/2}, \quad (5.11)$$

with $v<1$. When applied to (5.8), (5.11) gives (for $i=1$, 2)

$$D^{(i)}(v_0^{(i)})=\tfrac12\pi(1-v_0^{(i)2})^{-1/2}=\tfrac14\pi[R_i(1-R_i)]^{-1/2}. \quad (5.12)$$

We also need the asymptotic behavior of the Legendre function of the first kind. For t large and fixed v between -1 and 1, we have

$$\begin{aligned}P_{it-\frac12}(v)&=F(\tfrac12-it,\tfrac12+it;1;\tfrac12-\tfrac12v)\\&=\sum_{n=0}^{\infty}(t^2+\tfrac14)(t^2+9/4)\cdots[t^2+(n-\tfrac12)^2](\tfrac12-\tfrac12v)^n/(n!)^2\\&\sim\sum_{n=1}^{\infty}(2\pi n)^{-1}(\tfrac12-\tfrac12v)^n\exp[2t\tan^{-1}(n/t)+n\ln(1+t^2/n^2)]\\&\sim(2\pi t)^{-1}[(1+v)/(1-v)]^{1/2}\exp(t\cos^{-1}v)\sum_{n=-\infty}^{\infty}\exp\{-\tfrac12t^{-1}(1+v)^{3/2}(1-v)^{-1/2}[n-t(1-v)^{1/2}(1+v)^{-1/2}]^2\}\\&\sim(2\pi t)^{-1/2}(1-v^2)^{-1/4}\exp(t\cos^{-1}v). \quad (5.13)\end{aligned}$$

We now have all the necessary information to calculate the high-energy behavior of A. Substituting (5.12) and (5.13) in (3.6) and using (4.29) and (C5), we get

$$\begin{aligned}A&\sim(2m^2)^{-1}\int_0^\infty tdt\,e^{-\mu(t)\Lambda}\left[\int_0^\infty dz\,J^{(1)}(z)P_{it-\frac12}(1+\tfrac12m^{-2}z)\right]\left[\int_0^\infty dz'J^{(2)}(z')P_{it-\frac12}(1+\tfrac12m^{-2}z')\right]\\&=-\tfrac12m^{-2}\int_0^\infty tdt(\operatorname{sech}^2\pi t)e^{-\mu(t)\Lambda}\left\{\int_{C_1}dv\,J^{(1)}[2m^2(v-1)]P_{it-\frac12}(-v)\right\}\left\{\int_{C_1}dv'J^{(2)}[2m^2(v'-1)]P_{it-\frac12}(-v')\right\}\\&\sim\pi^2m^{-2}\int_0^\infty tdt\,e^{-2\pi t}e^{-\mu(t)\Lambda}\left[\int_{v_0^{(1)}}^\infty dv\,D^{(1)}(v)P_{it-\frac12}(v)\right]\left[\int_{v_0^{(2)}}^\infty dv'D^{(2)}(v')P_{it-\frac12}(v')\right]\\&\sim\tfrac12\pi m^{-2}D^{(1)}(v_0^{(1)})D^{(2)}(v_0^{(2)})(1-v_0^{(1)2})^{-1/4}(1-v_0^{(2)2})^{-1/4}\int_0^\infty dt\,e^{-2\pi t}e^{-\mu(t)\Lambda}\left[\int_{v_0^{(1)}}^1 dv\exp(t\cos^{-1}v)\right]\\&\quad\times\left[\int_{v_0^{(2)}}^1 dv'\exp(t\cos^{-1}v')\right]\sim\tfrac12\pi m^{-2}D^{(1)}(v_0^{(1)})D^{(2)}(v_0^{(2)})(1-v_0^{(1)2})^{1/4}(1-v_0^{(2)2})^{1/4}\\&\quad\times\int_0^\infty dt\,t^{-2}e^{-2\pi t}e^{-\mu(t)\Lambda}\exp[t(\cos^{-1}v_0^{(1)}+\cos^{-1}v_0^{(2)})]. \quad (5.14)\end{aligned}$$

In this derivation, the contour C_1 of integration is the one shown in Fig. 2. In this form (5.14), a comparison with (1.6) already shows that

$$\begin{aligned}b&=[2\pi-\cos^{-1}v_0^{(1)}-\cos^{-1}v_0^{(2)}]/\pi\\&=[2\pi-\cos^{-1}(1-\tfrac12M_1^2/m^2)-\cos^{-1}(1-\tfrac12M_2^2/m^2)]/\pi\\&=2[\cos^{-1}(\tfrac12M_1/m)-\cos^{-1}(\tfrac12M_2/m)]/\pi. \quad (5.15)\end{aligned}$$

Thus this exponent depends on mass ratios. In terms of this b, it follows from (4.26) and (5.14) that

$$\begin{aligned}A&\sim(\tfrac12\pi)^4m^{-2}[R_1(1-R_1)R_2(1-R_2)]^{-1/4}\\&\quad\times(\tfrac18\pi^3\Lambda)^{-b}[\ln(\tfrac18\pi^3\Lambda)]^{-2+b}\Gamma(b)\\&\quad\times\{1+(2-b)[\ln\ln(\tfrac18\pi^3\Lambda)]/[\ln(\tfrac18\pi^3\Lambda)]\}. \quad (5.16)\end{aligned}$$

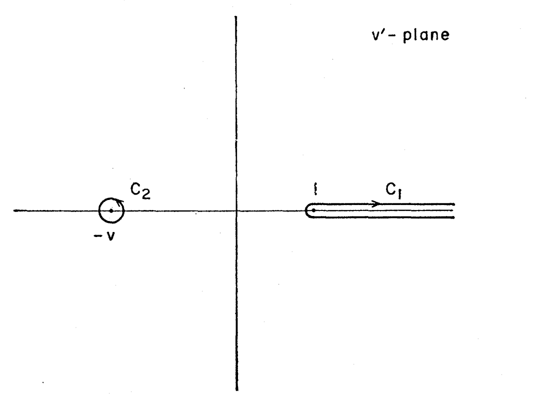

FIG. 2. Two useful contours of integration.

By (2.9) and (5.16), the desired high-energy behavior of the spin-flip amplitude in the exactly forward direction for the scattering of vector mesons is, when (5.1) is satisfied,

$$i(\tfrac{1}{4}\pi)^3 s m^{-2} e^4 f_1 f_1' f_2 f_2' \Gamma(b)[R_1(1-R_1)R_2(1-R_2)]^{-1/4} \times(\tfrac{1}{8}\pi\alpha^2 \ln s)^{-b}[\ln(\tfrac{1}{8}\pi\alpha^2 \ln s)]^{-2+b} \times\{1+(2-b)[\ln\ln(\tfrac{1}{8}\pi\alpha^2 \ln s)]/[\ln(\tfrac{1}{8}\pi\alpha^2 \ln s)]\}, \quad (5.17)$$

where b is given by (5.15). A comparison with (1.6) then gives the further information

$$c=2-b. \quad (5.18)$$

In the limit $M_1 \to 0$ and $M_2 \to 0$, we get from (5.15) that $b \to 2$. If (5.18) is used, we then get $c \to 0$. This limiting value 0 does not agree with the -1 of (4.33). This is, however, not surprising for nonuniform asymptotic expansions.

B. Inelastic Scattering

In the more general case where (5.1) is not satisfied but λ is still 0, (5.2) must be replaced by

$$J^{(i)}(z)=-4\int_0^1 d\alpha \int_0^1 d\beta\, \alpha(1-\alpha)\beta(1-\beta) \times\{\alpha(1-\alpha)z+m^2 -\beta(1-\beta)[M_i^2\alpha+M_i'^2(1-\alpha)]\}^{-1} \quad (5.19)$$

for $i=1, 2$. Equation (4.4) is still not changed, and the desired spin-flip amplitude is still given by (2.7) with (3.6).

With the change of variable (4.11), $J^{(i)}(z)$ takes the form

$$J^{(i)}(z)=-\tfrac{1}{4}m^{-2}\int_{-1}^1 dx \int_0^1 dy(1-x^2)(1-y^2) \times\{(1-x^2)v+(1+x^2)-(1-y^2) \times[R_i(1+x)+R_i'(1-x)]\}^{-1}, \quad (5.20)$$

where

$$R_i=M_i^2/(4m^2)$$

and

$$R_i'=M_i'^2/(4m^2). \quad (5.21)$$

As in Sec. 5 A, let v be a complex variable. Then $J^{(i)}(z)$ is an analytic function of v except for a branch cut from $-\infty$ to $-v_0^{(i)}$ along the real axis, where

$$v_0^{(i)}=\min_{-1\leq x\leq 1}(1-x^2)^{-1}\{(1+x^2) -[R_i(1+x)+R_i'(1-x)]\}. \quad (5.22)$$

The computation of this minimum is straightforward, but the result is somewhat complicated:

$$v_0^{(i)}=\tfrac{1}{2}[(1-R_i)^{1/2}+(1-R_i')^{1/2}]^2-1. \quad (5.23)$$

This minimum value occurs at

$$x=\frac{(1-R_i')^{\frac{1}{2}}-(1-R_i)^{\frac{1}{2}}}{(1-R_i')^{\frac{1}{2}}+(1-R_i)^{\frac{1}{2}}}. \quad (5.24)$$

As a generalization of (5.8), the discontinuity of $J^{(i)}(z)$, $i=1$ or 2, across this cut is given by

$$D^{(i)}(v)=\tfrac{1}{2}\int_{-1}^1 dx\int_0^1 dy(1-x^2)(1-y^2) \times\delta\{-(1-x^2)v+(1+x^2)-(1-y^2) \times[R_i(1+x)+R_i'(1-x)]\}. \quad (5.25)$$

Unlike (5.8), the right-hand side of (5.25) is not in any simple way related to the $D(v,v')$ of (4.16). As we learned in Sec. 5 A, the only important quantity is

$$D^{(i)}(v_0^{(i)})=\pi(1-R_i)^{1/4}(1-R_i')^{1/4} \times[(1-R_i)^{1/2}+(1+R_i')^{1/2}]^{-2} \times[1-(1-R_i)^{1/2}(1-R_i')^{1/2}]^{-1/2}. \quad (5.26)$$

Since (5.14) still holds for the present case, we get immediately, as a generalization of (5.15), that

$$\begin{aligned} b&=[2\pi-\cos^{-1}v_0^{(1)}-\cos^{-1}v_0^{(2)}]/\pi \\ &=2\pi^{-1}\{\sin^{-1}\tfrac{1}{2}[(1-R_1)^{1/2}+(1-R_1')^{1/2}] +\sin^{-1}\tfrac{1}{2}[(1-R_2)^{1/2}+(1-R_2')^{1/2}]\} \\ &=2\pi^{-1}\{\sin^{-1}\tfrac{1}{2}[(1-\tfrac{1}{4}M_1^2/m^2)^{1/2}+(1-\tfrac{1}{4}M_1'^2/m^2)^{1/2}] +\sin^{-1}\tfrac{1}{2}[(1-\tfrac{1}{4}M_2^2/m^2)^{1/2} +(1-\tfrac{1}{4}M_2'^2/m^2)^{1/2}]\}. \end{aligned} \quad (5.27)$$

Moreover, (5.18) still holds. With this b, (5.17) is only slightly modified so that the spin-flip amplitude in the exactly forward direction for the scattering of vector mesons is

$$i(\tfrac{1}{2}\pi)^3 s m^{-2} e^4 f_1 f_1' f_2 f_2' \Gamma(b)[(1-R_1)(1-R_1')(1-R_2)(1-R_2')]^{1/4}[(1-R_1)^{1/2}+(1-R_1')^{1/2}]^{-3/2} \times[1-(1-R_1)^{1/2}(1-R_1')^{1/2}]^{-1/2}\{1-\tfrac{1}{4}[(1-R_1)^{1/2}+(1-R_1')^{1/2}]^2\}^{1/4}[(1-R_2)^{1/2}+(1-R_2')^{1/2}]^{-3/2} \times[1-(1-R_2)^{1/2}(1-R_2')^{1/2}]^{-1/2}\{1-\tfrac{1}{4}[(1-R_2)^{1/2}+(1-R_2')^{1/2}]^2\}^{1/4}(\tfrac{1}{8}\pi\alpha^2 \ln s)^{-b}[\ln(\tfrac{1}{8}\pi\alpha^2 \ln s)]^{-2+b} \times\{1+(2-b)[\ln\ln(\tfrac{1}{8}\pi\alpha^2 \ln s)]/[\ln(\tfrac{1}{8}\pi\alpha^2 \ln s)]\}. \quad (5.28)$$

This is the desired result. In connection with factorization, note the appearance of $\Gamma(b)$.

6. EXCHANGE OF MASSIVE VECTOR MESONS

We now turn our attention to the case $\lambda>0$. Since this case is much more complicated than the massless one treated in Secs. 4 and 5, we shall concentrate entirely on determining the exponent b.

The additional difficulty for this case $\lambda>0$ is the following. Instead of (4.4), we must now solve the integral equation (3.2) approximately for small $\mu(t)$. Since (4.4) has been analyzed on the basis of the solution to (4.20), the corresponding simplified integral equation for $\lambda>0$

is

$$\frac{v-1}{v-v_0}\int_1^\infty dv'\,\varphi(v')\frac{v'-1}{v'-v_0}\frac{1}{v+v'}=\bar{\mu}\varphi(v)\,, \tag{6.1}$$

where

$$v_0=1-\tfrac{1}{2}\lambda^2/m^2. \tag{6.2}$$

Unfortunately, unlike (4.20), (6.1) cannot be solved exactly by the authors for a general value of v_0. We reluctantly reached this conclusion after months of unsuccessful attempts.

Because of this difficulty, we are forced to use a further simplified integral equation. One of the major differences between (6.1) and (4.20) is the presence in (6.1) of poles at v_0, with $|v_0|<1$. We therefore make up another integral equation with this feature incorporated. More precisely, we *assume* that some information about the integral equation (3.2) can be obtained by studying

$$\int_1^\infty dv'\,\varphi(v')[(v+v')^{-1}-C(v-v_0)^{-1}(v'-v_0)^{-1}]=\bar{\mu}\varphi(v)\,, \tag{6.3}$$

where $C\neq 0$ is a fixed constant. Equation (6.3) is solved in Appendix E.

Numerous questions may be raised concerning whether (6.3) is appropriate. We believe that the exponent b is sufficiently insensitive to the details of $f_t(z)$ so that this rough approximation is admissible, but we do not have any confidence of extending the results to the exponent c. We mention here only one of the simpler questions. Is it necessary to incorporate another feature of (6.1), namely, the zeros at 1? This question can be answered by considering the special case $v_0=-1$. Since

$$[(v-1)/(v+1)][(v'-1)/(v'+1)](v+v')^{-1}=(v+v')^{-1}-2(v+1)^{-1}(v'+1)^{-1}, \tag{6.4}$$

(6.1) can be solved exactly in terms of Legendre functions of the first kind for this special value of v_0. The zeros at -1 are present only if the coefficient of the second term on the right-hand side of (6.4) is precisely -2 as given there. From the explicit solution, we see that the solution does not change qualitatively when this coefficient is varied, except that $\varphi(1)=0$ only for the special value -2. Since the vanishing of f_t at $z=0$ does not directly affect the exponent b, we conclude that the zeros at 1 are unimportant for our limited purposes.

By (2.5) and (5.20), for the present case of $\lambda>0$,

$$J^{(i)}(z)=-\tfrac{1}{4}m^{-2}(v-1)(v-v_0)^{-1}\int_{-1}^1 dx\int_0^1 dy(1-x^2)\times(1-y^2)\{(1-x^2)v+(1+x^2)-(1-y^2)\times[R_i(1+x)+R_i'(1-x)]\}^{-1}, \tag{6.5}$$

for $i=1$, 2, where v_0 is defined by (6.2) while R_i and R_i' are defined by (5.21). Considered as a function of the complex variable v, $J^{(i)}(z)$ has a branch cut along the real axis from $-\infty$ to $-v_0^{(i)}$, where $v_0^{(i)}$ is given by (5.23). In addition, if $v_0>v_0^{(i)}$, there is a pole at $v=v_0$. From the procedure of Sec. 5 B, we need to know the magnitudes of $\Phi(v_0)$ and $\Phi(-v_0^{(i)})$, where Φ is defined by (E10). These quantities are studied in detail in Appendix E. In particular, from the last paragraph of Appendix E, we know that $\Phi(v_0)$ is never important. Therefore, from (E23) and (E24), we get

$$\ln[N\Phi(-v_0^{(i)})]\sim -t[\pi-\cos^{-1}\min(v_0,v_0^{(i)})]. \tag{6.6}$$

Accordingly, the result (5.27) is only slightly modified to

$$b=\pi^{-1}[2\pi-\cos^{-1}\min(v_0,v_0^{(1)})-\cos^{-1}\min(v_0,v_0^{(2)})]. \tag{6.7}$$

Like (5.27), (6.7) can be written in terms of half-angles as

$$b=2\pi^{-1}\{\sin^{-1}\min[(1-\tfrac{1}{4}\lambda^2/m^2)^{1/2},\ \tfrac{1}{2}(1-\tfrac{1}{4}M_1^2/m^2)^{1/2}+\tfrac{1}{2}(1-\tfrac{1}{4}M_1'^2/m^2)^{1/2}]+\sin^{-1}\min[(1-\tfrac{1}{4}\lambda^2/m^2)^{1/2},\ \tfrac{1}{2}(1-\tfrac{1}{4}M_2^2/m^2)^{1/2}+\tfrac{1}{2}(1-\tfrac{1}{4}M_2'^2/m^2)^{1/2}]\}. \tag{6.8}$$

Since (6.8) contains (5.27) as a special case, it is the general result for the exponent b.

7. REMARKS

Since the mathematical manipulations involved in this paper are quite heuristic, it is perhaps useful to give our views on this procedure. It is clear that, compared with the treatment of the spin-nonflip amplitude in Ref. 1, the steps taken in the present paper raise many more problems and hence are much less justified mathematically. Let us list some of these problems.

(A) In Ref. 1, we sum a series where every term is of the same sign and hence no cancellation is possible. Since the nth term is proportional to $(\ln s)^n$ for large s, the sum must necessarily increase faster than $(\ln s)^n$ for all n as $s\to\infty$.[5] It is therefore reasonable that the sum should behave like s raised to a power, and this behavior is verified in the explicit calculation. This is not the case here, where terms of the series alternate in sign, leading to extensive cancellation. Indeed, since the exponent b is positive, the sum is actually smaller in every case than any one of the terms in the series. For this reason, the procedure followed here of summing the leading terms is much less justified. However, we believe that this problem is not serious, at least compared with the other difficulties to be given below.

(B) An essential step in the determination of the high-energy behavior of the spin-flip amplitude is to solve for the highly oscillatory eigenfunctions of an integral equation. So far as the authors are aware, there is no general procedure to obtain these eigenfunctions. In the massless case $\lambda=0$, this needed information is obtained in Sec. 4 C by assuming a solution in the form of a Legendre function and verifying that this assumed solution indeed satisfies the integral equation approximately over most of the range. This is true unfortunately only over most of the range $1<v<\infty$ but not over all the range, the error being large when v is close to 1. It is therefore a legitimate problem to raise whether the assumed solution is a good approximation to the desired solution. We can only answer this by simple examples where the kernel $(v+v')^{-1}$ of (4.20) is perturbed by a separable kernel. On the basis of these simple examples, the solution (4.29) is accurate enough for our purposes.

This answer by example is less satisfactory than it seems at first glance. With the detailed calculation of Appendix D, we may ask whether [see in particular the uniform asymptotic approximation (D21)]

$$f_t^{(1)}(z)=(\tfrac{1}{2}t)^{1/2}m^{-1}(v^2-1)^{-1/4}(\cosh^{-1}v)^{1/2} \times[\mathbf{E}_1(t\cosh^{-1}v)-(t\cosh^{-1}v)^{-1}\mathbf{E}_2(t\cosh^{-1}v)], \quad (7.1)$$

with $z=2m^2(v-1)$ as always, is a better approximation to $f_t(z)$ than (5.29). Clearly we should study the integral

$$\int_0^\infty dz' f_t^{(1)}(z')\int_0^1 dx\int_0^1 dy \times\frac{x(1-x)y(1-y)}{x(1-x)z+y(1-y)z'+m^2}. \quad (7.2)$$

We have been unable to find any approximate relation between (7.2) and (7.1), mostly because of some unpleasant properties of the Weber functions $\mathbf{E}_1$ and $\mathbf{E}_2$.

(C) The situation is even worse for the case $\lambda>0$ as discussed in Sec. 6. While in Sec. 4 for $\lambda=0$ we can substitute the assumed solution into the integral equation (4.12) to verify that the integral equation is approximately satisfied over most of the range and to identify the eigenvalue $\mu(t)$, this cannot be carried out for $\lambda>0$. In particular, the generalization of (4.25) to $\lambda>0$ is not known. This implies that the approximations involved in Sec. 6 are significantly less accurate than those of Sec. 4, and it is for this reason that it is not possible to calculate the exponent c for $\lambda>0$ by the procedure of Sec. 6. It is a most interesting mathematical problem to improve the most unsatisfactory procedure of Sec. 6 for $\lambda>0$.

In spite of these difficulties, we believe firmly that the results here on the exponents b and c are correct. The reason is that the exponent b is determined almost entirely by the location of singularities in the complex plane, and is hence very insensitive to the details of the approximation. Moreover, since the approximation for the massless case $\lambda=0$ is far superior to that of the massive case $\lambda>0$, it is reasonable to expect that more information can be extracted for $\lambda=0$, namely, the value of the next exponent c.

Another interesting question concerns some of the phenomena encountered in the analysis of the model integral equation (6.3) as carried out in Appendix E. As seen explicitly from (E27) and (E29), the asymptotic behavior of the denominator $\mathfrak{D}$ for large t is quite different for the two cases $v_0>0$ and $v_0<0$. By (6.2), these two cases are, respectively, $\lambda<\sqrt{2}m$ and $\lambda>\sqrt{2}m$. Does this mean that the high-energy behavior is somehow related to anomalous thresholds? Furthermore from (E27), $\mathfrak{D}$ has a zero near $v_0=2^{-1/2}$ because

$$\psi(2\pi^{-1}\cos^{-1}2^{-1/2})-\psi(1-2\pi^{-1}2^{-1/2})=0.$$

Is there anything unusual at this peculiar point $v_0=2^{-1/2}$ or $\lambda=(2-\sqrt{2})^{1/2}m$?

8. SUMMARY AND DISCUSSION

(A) All the results on b, one of exponents in the high-energy behavior (1.6) of the spin-flip amplitude for the forward scattering of two vector mesons, are contained in (6.8), since (4.10) and (5.27) are special cases. Let

$$b_1=2\pi^{-1}\sin^{-1}\{\min[(1-\tfrac{1}{4}\lambda^2/m^2)^{1/2},\ \tfrac{1}{2}(1-\tfrac{1}{4}M_1^2/m^2)^{1/2}+\tfrac{1}{2}(1-\tfrac{1}{4}M_1'^2/m^2)^{1/2}]\}$$

and

$$b_2=2\pi^{-1}\sin^{-1}\{\min[(1-\tfrac{1}{4}\lambda^2/m^2)^{1/2},\ \tfrac{1}{2}(1-\tfrac{1}{4}M_2^2/m^2)^{1/2}+\tfrac{1}{2}(1-\tfrac{1}{4}M_2'^2/m^2)^{1/2}]\}, \quad (8.1)$$

so that b_1 is independent of the nature of the particles 2 and 2′ while b_2 is independent of the natures of 1 and 1′. Then

$$b=b_1+b_2. \quad (8.2)$$

(B) So far as the next exponent c of (1.6) is concerned, we have information only for the special case $\lambda=0$, i.e., the case where photons are exchanged. The results are given in (4.23), (5.18), and (5.28). From (5.18) and (5.28) we see that the sum of the exponents $b+c$ is actually simpler, and we summarize our results on $b+c$ in Table I. It is therefore useful to define[20]

$$c_1=\begin{cases}\tfrac{1}{2}-b_1 & \text{if } M_1=M_1'=0,\\ 1-b_1 & \text{otherwise;}\end{cases}$$

and

$$c_2=\begin{cases}\tfrac{1}{2}-b_2 & \text{if } M_2=M_2'=0,\\ 1-b_2 & \text{otherwise;}\end{cases} \quad (8.3)$$

then

$$c=c_1+c_2. \quad (8.4)$$

[20] As an example of the discontinuous asymptotic behavior, consider

$$I(a)=\lim_{\Lambda\to\infty}\left(\frac{\Lambda}{\pi}\right)^{1/2}\int_a^\infty d\theta\exp(-\Lambda\sinh^2\theta).$$

Explicit evaluation gives

$$I(a)=\begin{cases}0 & \text{for } a>0,\\ \tfrac{1}{2} & \text{for } a=0,\\ 1 & \text{for } a<0.\end{cases}$$

The uniform asymptotic behavior of the integral for Λ large is of course expressed in terms of an error function.

(C) Decompositions of the forms (8.2) and (8.4) can be extended even further. From (5.17) we note that at least for the case $\lambda=0$, the high-energy behavior of the spin-flip amplitude in the exactly forward direction for the scattering of vector mesons is given by

$$is\Gamma(b_1+b_2)A_1A_2\,, \qquad (8.5)$$

where

$$A_1=(\tfrac{1}{4}\pi)^{3/2}m^{-1}e^2f_1f_1'[R_1(1-R_1)]^{-1/4} \times(\tfrac{1}{8}\pi\alpha^2\ln s)^{-b_1}[\ln(\tfrac{1}{8}\pi\alpha^2\ln s)]^{-c_1} \qquad (8.6)$$

is independent of the nature of the particles 2 and 2′, while

$$A_2=(\tfrac{1}{4}\pi)^{3/2}m^{-1}e^2f_2f_2'[R_2(1-R_2)]^{-1/4} \times(\tfrac{1}{8}\pi\alpha^2\ln s)^{-b_2}[\ln(\tfrac{1}{8}\pi\alpha^2\ln s)]^{-c_2} \qquad (8.7)$$

is independent of the nature of 1 and 1′. Thus at very high energies, the spin-flip amplitude *factors* in the simplest possible manner except for an over-all constant factor $\Gamma(b_1+b_2)$.[21] This factorization is expected to be quite general and not limited to this spin-flip amplitude. In a different connection, this factorization was discussed some time ago.[22]

(D) We have often been asked the following question: Granted that the leading singularity is a branch cut, what are the other singularities in the complex angular momentum plane? One reason for asking this question is the hope that these other singularities are of a simple nature, perhaps Regge poles. Since we are able to calculate only the high-energy behavior of scattering amplitudes, there is in general no way to answer this question. The only possibility is to choose an amplitude and a momentum transfer so that the leading singularity does not contribute at all to the high-energy behavior. This is precisely the case for the spin-flip amplitude in the exactly forward direction, and in this way we can get a glimpse of the other singularities. What we see is another branch cut, perhaps even of somewhat more complicated nature than that of the leading singularity. This does not mean, however, that the amplitudes for the exchange of other quantum numbers are necessarily also complicated, and a great deal more work is needed in this direction.

(E) On the basis of our limited knowledge in this special case of the spin-flip amplitude, we attempt to compare this high-energy behavior of the second type with that found previously[1] for the spin-nonflip amplitude (high-energy behavior of the first type). The following qualitative differences come to mind immediately.

(i) The power dependence on s is a function of the coupling constant for the first type, but is independent of the coupling constant for the second type. For the examples, this s dependence is $s^{1+11\alpha^2\pi/32}$ for the spin-nonflip amplitude (1.7) but is simply s for the spin-flip amplitude (8.5).

TABLE I. Values of the sum of the exponents b and c for the scattering process $1+2\to 1'+2'$ due to the exchange of massless photons.

	$M_1=M_1'=0$	$M_1\neq 0$ or $M_1'\neq 0$
$M_2=M_2'=0$	1	$\frac{3}{2}$
$M_2\neq 0$ or $M_2'\neq 0$	$\frac{3}{2}$	2

(ii) High-energy behavior of the second type is also less sensitive to the large-momentum behavior of the underlying theory. As discussed previously,[10] if the large-momentum contributions are suppressed, the high-energy behavior of the first type changes from one appropriate for a fixed branch cut to one for a moving Regge pole. Similarly, with such a suppression, the high-energy behavior of the second type changes, in this example of the spin-flip amplitude, from one appropriate for a fixed branch cut to one for a fixed essential singularity. But the high-energy behavior for these two cases of a fixed branch cut and a fixed essential singularity are essentially the same; the value of the exponent b is in particular not changed.

(iii) The high-energy behavior of the second type depends on the highly oscillatory components of the impact factors. In the example of the spin-flip amplitude, this is the reason why the exponent b is fairly complicated and depends on mass ratios. The physical meaning of this dependence is, however, not clear to the authors.

There are many other differences, such as the appearance of the (ln ln$s$) factor in (1.6) but not in (1.7), and the discontinuous behavior of the corresponding exponent as given by (8.3). But these differences may or may not be general in distinguishing the high-energy behaviors of these two types.

Although these differences (i)–(iii) are of a major character, we believe that the difference is actually even deeper. By suppressing the large-momentum contributions, the high-energy behavior of the first type bears close resemblance to one expected from a Regge pole (to the right of $J=1$).[10] This is not at all the case for high-energy behavior of the second type. Rather, the high-energy behavior of the second type seems to be a realization of the kind of behavior from essential singularities.[23]

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[21] Since $\Gamma(b_1+b_2)=B(b_1,b_2)\Gamma(b_1)\Gamma(b_2)$, where B is the beta function, this over-all factor can alternatively be given as $B(b_1,b_2)$ while the factors $\Gamma(b_1)$ and $\Gamma(b_2)$ are included, respectively, in A_1 and A_2.

[22] The discussion on factorization was first presented by T. T. Wu, Bull. Am. Phys. Soc. 14, 49(T) (1969).

[23] A very extreme, and probably unjustified, way of stating the difference between the high-energy behaviors of these two kinds is the following. The high-energy behavior of the first kind is more dynamic than kinematic in origin, while the high-energy behavior of the second kind is more kinematic than dynamic.

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APPENDIX A

In this Appendix we study in some detail the spectrum of the operator $\mathcal{K}_1$ as defined by (1.4) and (1.3). The expression (1.3) can be written in the form

$$\mathcal{K}_1(z,z') = -\frac{1}{2}\frac{z}{z+\lambda^2}\frac{z'}{z'+\lambda^2}\int_0^\infty d\xi\, e^{-m^2\xi}F(z\xi)F(z'\xi), \quad \text{(A1)}$$

where

$$F(z\xi)=\int_0^1 dx\, x(1-x)e^{-x(1-x)z\xi} \quad \text{(A2)}$$

can be expressed in terms of the error function. Therefore,

$$(f,\mathcal{K}_1f) = \int_0^\infty dz\int_0^\infty dz' f^*(z)\mathcal{K}_1(z,z')f(z') = -\tfrac{1}{2}\int_0^\infty d\xi\, e^{-m^2\xi}\left|\int_0^\infty dz\frac{z}{z+\lambda^2}F(z\xi)f(z)\right|^2 \le 0. \quad \text{(A3)}$$

Suppose that the equality sign holds in (A3), i.e.,

$$(f,\mathcal{K}_1f)=0. \quad \text{(A4)}$$

Then

$$\int_0^\infty dz\, F(z\xi)g(z)=0 \quad \text{(A5)}$$

for all $\xi \ge 0$, where

$$g(z)=\frac{z}{z+\lambda^2}f(z). \quad \text{(A6)}$$

Therefore, as a consequence of (A5),

$$\int_0^\infty d\xi\left|\int_0^\infty dz\, F(z\xi)g(z)\right|^2=0 \quad \text{(A7)}$$

or

$$\int_0^\infty dz\, g^*(z)\int_0^\infty dz' g(z')\int_0^1 dx\int_0^1 dy\, x(1-x)y(1-y) \times[x(1-x)z+y(1-y)z']^{-1}=0. \quad \text{(A8)}$$

Define ξ, $h(\xi)$, and $H(\gamma)$ by

$$z=e^\xi, \quad \text{(A9)}$$

$$g(z)=e^{-\xi/2}h(\xi), \quad \text{(A10)}$$

and

$$H(\tau)=\int_{-\infty}^\infty e^{i\xi\tau}h(\xi)d\xi. \quad \text{(A11)}$$

Then it follows from (A8) that

$$\int_{-\infty}^\infty d\tau\,|H(\tau)|^2\frac{(1+4\tau^2)\sinh\pi\tau}{\tau(1+\tau^2)\cosh^2\pi\tau}=0. \quad \text{(A12)}$$

Therefore,

$$H(\gamma)=0$$

and hence

$$f(z)=0. \quad \text{(A13)}$$

The solution $f(z)=\delta(z)$ is excluded because it is not square integrable. We have thus shown that (A4) implies (A13). In other words, $-\mathcal{K}_1$ is positive definite.

The treatment of $\mathcal{K}_0$ as given in Sec. 6 of Ref. 1 can be taken over step by step to $\mathcal{K}_1$, and the result is that the spectrum of $\mathcal{K}_1$ is given by (1.5).

APPENDIX B

In this Appendix, we briefly describe the derivation of (4.17) from (4.16). In the xy plane, the argument of the δ function of (4.16) is zero on a hyperbola. The shape of this hyperbola is somewhat different depending on whether v or v' is larger.

Consider first the case $v>v'$ and introduce a new variable θ by

$$x=[(v-v')/(v+1)]^{1/2}\cosh\theta$$

and

$$y=[(v-v')/(v'-1)]^{1/2}\sinh\theta, \quad \text{(B1)}$$

then the range of this variable θ is from 0 to

$$\ln\{[(v-1)^{1/2}+(v'-1)^{1/2}]/(v-v')^{1/2}\}. \quad \text{(B2)}$$

With this variable θ, the integration is straightforward.

For the other case $v<v'$, we use instead of (B1)

$$x=[(v'-v)/(v+1)]^{1/2}\sinh\theta$$

and

$$y=[(v'-v)/(v'-1)]^{1/2}\cosh\theta. \quad \text{(B3)}$$

The range of this θ is from 0 to

$$\ln\{[(v-1)^{1/2}+(v'-1)^{1/2}]/(v'-v)^{1/2}\}. \quad \text{(B4)}$$

APPENDIX C

In this Appendix, we study in some detail Mehler's integral equation (4.20).

Since the Legendre function $P_\nu(v)$ of the first kind is related to the hypergeometric function F by[24]

$$P_\nu(v)=F(-\nu,\nu+1;1;\tfrac{1}{2}-\tfrac{1}{2}v). \quad \text{(C1)}$$

$P_\nu(v)$ is an analytic function of the complex variable v except for a branch cut from $-\infty$ to -1 along the negative real axis. Furthermore, if $Q_\nu(v)$ is the Legendre function of the second kind, and v is positive and larger

[24] See Eq. (3) on p. 122 of Ref. 17. Properties of the hypergeometric function are given in Chap. II of Ref. 17.

than 1, then

$$P_\nu(-v+i0)=e^{i\pi\nu}P_\nu(v)-(2/\pi)\sin\pi\nu\, Q_\nu(v) \quad \text{(C2)}$$

and

$$P_\nu(-v-i0)=e^{-i\pi\nu}P_\nu(v)-(2/\pi)\sin\pi\nu\, Q_\nu(v) \quad \text{(C3)}$$

(see p. 140 of Ref. 17). Thus in particular the discontinuity across the branch cut from $-\infty$ to -1 is

$$P_\nu(-v+i0)-P_\nu(-v-i0)=2i\sin\pi\nu\, P_\nu(v)\,. \quad \text{(C4)}$$

In view of (4.21), let

$$\nu=it-\tfrac{1}{2}\,.$$

Then

$$P_{it-\frac{1}{2}}(v)=\tfrac{1}{2}i\,\operatorname{sech}\pi t\,[P_{it-\frac{1}{2}}(-v+i0)-P_{it-\frac{1}{2}}(-v-i0)]\,. \quad \text{(C5)}$$

Relation (C5) is useful in verifying that (4.21) is indeed a solution of (4.20). Let C_1 and C_2 be the two contours of integration shown in Fig. 2; then

$$\begin{aligned}\int_1^\infty dv'(v+v')^{-1}P_{it-\frac{1}{2}}(v') &= -\tfrac{1}{2}i\,\operatorname{sech}\pi t\int_{C_1}dv'(v+v')^{-1}P_{it-\frac{1}{2}}(-v')\\ &= -\tfrac{1}{2}i\,\operatorname{sech}\pi t\int_{C_2}dv'(v+v')^{-1}P_{it-\frac{1}{2}}(-v')\\ &=\pi\,\operatorname{sech}\pi t\,P_{it-\frac{1}{2}}(v)\,. \end{aligned}\quad \text{(C6)}$$

This procedure is also used in Appendix D.

So far t is an arbitrary real number. We now concentrate on the case where t is large. In this limit

$$P_{it-\frac{1}{2}}(v)\sim(v^2-1)^{-1/4}(\cosh^{-1}v)^{1/2}J_0(t\cosh^{-1}v) \quad \text{(C7)}$$

uniformly for all $v\geq 1$. To derive (C7), we need to consider two cases. First, assume that v is not close to 1. Then [see Eq. (27) on p. 129 of Ref. 17]

$$\begin{aligned}P_{it-\frac{1}{2}}(v)=&\,\pi^{-1/2}\frac{\Gamma(-it)}{\Gamma(\frac{1}{2}-it)}[v+(v^2-1)^{1/2}]^{-it-\frac{1}{2}}F\left(\tfrac{1}{2},\tfrac{1}{2}+it;1+it;\frac{v-(v^2-1)^{1/2}}{v+(v^2-1)^{1/2}}\right)\\ &+\pi^{-1/2}\frac{\Gamma(it)}{\Gamma(\frac{1}{2}+it)}[v+(v^2-1)^{1/2}]^{it-\frac{1}{2}}F\left(\tfrac{1}{2},\tfrac{1}{2}-it;1-it;\frac{v-(v^2-1)^{1/2}}{v+(v^2-1)^{1/2}}\right)\\ &\sim\pi^{-1/2}\frac{\Gamma(-it)}{\Gamma(\frac{1}{2}-it)}[v+(v^2-1)^{1/2}]^{-it-\frac{1}{2}}\left[1-\frac{v-(v^2-1)^{1/2}}{v+(v^2-1)^{1/2}}\right]^{-1/2}\\ &+\pi^{-1/2}\frac{\Gamma(it)}{\Gamma(\frac{1}{2}+it)}[v+(v^2-1)^{1/2}]^{it-\frac{1}{2}}\left[1-\frac{v-(v^2-1)^{1/2}}{v+(v^2+1)^{1/2}}\right]^{-1/2}\\ &\sim(2\pi)^{-1/2}t^{-1/2}(v^2-1)^{-1/4}\{e^{i\pi/4}[v+(v^2-1)^{1/2}]^{-it}+e^{-i\pi/4}[v+(v^2-1)]^{it}\}\,. \end{aligned}\quad \text{(C8)}$$

Secondly, when v is close to 1,

$$P_{it-\frac{1}{2}}(v)\sim J_0(t[2(v-1)]^{1/2})\,. \quad \text{(C9)}$$

Combining these two results, we get (C7).

APPENDIX D

In this Appendix, we evaluate (4.23) for large t. We follow the procedure of verifying Mehler's solution to his integral equation as given in Appendix C. By (C5), the integral (4.23) can be rewritten in the following form:

$$\begin{aligned}(4.23)=&-\tfrac{1}{2}i\,\operatorname{sech}\pi t\int_{C_1}dv'P_{it-\frac{1}{2}}(-v')\int_0^1dx\int_0^1dy\\ &\times\frac{(1-x^2)(1-y^2)}{(1-x^2)v+(1-y^2)v'+x^2+y^2}\\ =&-\tfrac{1}{2}i\,\operatorname{sech}\pi t\int_{-C_1}dv'P_{it-\frac{1}{2}}(-v')\int_0^1dx\int_0^1dy\\ &\times\frac{(1-x^2)(1-y^2)}{(1-x^2)v+(1-y^2)v'+x^2+y^2}\,, \end{aligned}\quad \text{(D1)}$$

where $-C_1$ is the negative of the contour C_1, around the branch cut from $-\infty$ to -1. Thus

$$\begin{aligned}(4.23)=&\,\tfrac{1}{2}i\,\operatorname{sech}\pi t\int_{C_1}dv'P_{it-\frac{1}{2}}(v')\int_0^1dx\int_0^1dy\\ &\times\frac{(1-x^2)(1-y^2)}{(1-x^2)v-(\frac{1}{2}-y^2)v'+x^2+y^2}\\ =&\,\pi\,\operatorname{sech}\pi t\int_1^\infty dv'P_{it-\frac{1}{2}}(v')D(v',v)\,, \end{aligned}\quad \text{(D2)}$$

where D is the discontinuity defined by (4.16) and explicitly given by (4.17).

Because of (D2), let

$$\bar{P}(v)=\int_1^\infty dv'P_{it-\frac{1}{2}}(v')D(v',v)\,. \quad \text{(D3)}$$

We study this function $\bar{P}(v)$ in the limit of large t for two cases: (a) v not close to 1; and (b) v close to 1. It seems necessary to treat these two cases separately.

A. v Not Close to 1

We remember from (4.17) that $D(v',v)$ has a logarithmic singularity at $v=v'$. Thus, for t large and v not close to 1, there are two possible contributions to the $\bar{P}(v)$ as given by (D3): first from the vicinity of $v'=v$, and secondly from the vicinity of $v'=1$. But it follows from (4.18) and (4.19) that, as $v'\to 1$,

$$D(v',v)\sim\tfrac{2}{3}(v-1)^{-2}[\tfrac{1}{2}(v'-1)]^{3/2}. \qquad \text{(D4)}$$

Therefore, the second contribution is negligible. For v' close to v, by (4.17),

$$D(v',v)\sim\tfrac{1}{4}(v^2-1)^{-1/2}\ln|v'-v|. \qquad \text{(D5)}$$

Accordingly, by (C8),

$$\bar{P}(v)\sim-\tfrac{1}{2}(2\pi)^{-1/2}t^{-1/2}\,\mathrm{Re}\left\{e^{-i\pi/4}\int dv'[v'+(v'^2-1)]^{it}(v^2-1)^{-3/4}\ln|v-v'|\right\}. \qquad \text{(D6)}$$

Let

$$v=\cosh\xi \quad\text{and}\quad v'=\cosh\xi'. \qquad \text{(D7)}$$

Then

$$\bar{P}(v)\sim-\tfrac{1}{2}(2\pi)^{-1/2}t^{-1/2}\,\mathrm{Re}\left[e^{-i\pi/4}\int d\xi' e^{it\xi'}(v^2-1)^{-1/4}\ln|\cosh\xi-\cosh\xi'|\right]$$

$$\sim-\tfrac{1}{2}(2\pi)^{-1/2}t^{-1/2}(v^2-1)^{-1/4}\,\mathrm{Re}\left(e^{-i\pi/4}\int d\xi' e^{it\xi'}\ln|\xi-\xi'|\right)$$

$$=\tfrac{1}{2}\pi(2\pi)^{-1/2}t^{-3/2}(v^2-1)^{-1/4}\,\mathrm{Re}(e^{-i\pi/4}e^{it\xi})$$

$$=\tfrac{1}{2}\pi(2\pi)^{-1/2}t^{-3/2}(v^2-1)^{-1/4}\,\mathrm{Re}\{e^{-i\pi/4}[v+(v^2-1)^{1/2}]^{it}\}\sim\tfrac{1}{4}\pi t^{-1}P_{it-\frac{1}{2}}(v). \qquad \text{(D8)}$$

This is the desired answer for v not close to 1, and of course larger than 1.

B. v Close to 1

When v is close to 1, the contribution to $\bar{P}(v)$ as given by (D3) comes of course from the vicinity of $v'=1$. Thus the change of variable (D7) reduces to

$$v=1+\tfrac{1}{2}\xi^2 \quad\text{and}\quad v'=1+\tfrac{1}{2}\xi'^2. \qquad \text{(D9)}$$

In this case, from (4.17),

$$D(v',v)\sim\tfrac{1}{4}\xi^{-3}\{-\xi\xi'+\tfrac{1}{2}(\xi^2+\xi'^2)\ln[(\xi+\xi')/|\xi-\xi'|]\}, \qquad \text{(D10)}$$

and thus, by (C9),

$$\bar{P}(v)\sim\tfrac{1}{4}\xi^{-3}\int_0^\infty \xi' d\xi' J_0(t\xi')\left[-\xi\xi'+\tfrac{1}{2}(\xi^2+\xi'^2)\ln\frac{\xi+\xi'}{|\xi-\xi'|}\right]$$

$$=\tfrac{1}{4}t^{-1}(\xi t)^{-3}\int_0^\infty \xi' d\xi' J_0(\xi')\left[-(\xi t)\xi'+\tfrac{1}{2}(\xi^2t^2+\xi'^2)\ln\frac{\xi t+\xi'}{|\xi t-\xi'|}\right]. \qquad \text{(D11)}$$

We proceed to evaluate this integral.

For $\xi'\to\infty$, the quantity in the square brackets on the right-hand side of (D11) is approximately $4(\xi t)^3/(3\xi')$. We therefore subtract this term first:

$$\bar{P}(v)\sim\tfrac{1}{3}t^{-1}+\tfrac{1}{4}t^{-1}(\xi t)^{-3}\int_0^\infty d\xi' J_0(\xi')\{-(\xi t)\xi'^2-\tfrac{4}{3}(\xi t)^3+\tfrac{1}{2}(\xi^2t^2+\xi'^2)\xi'\ln[(\xi t+\xi')/|\xi t-\xi'|]\}. \qquad \text{(D12)}$$

In writing down (D12), we have used

$$\int_0^\infty J_0(\xi')d\xi'=1. \qquad \text{(D13)}$$

Next we use the integral representation

$$J_0(\xi')=2\pi^{-1}\int_0^{\pi/2}\cos(\xi'\cos\phi)d\phi. \qquad \text{(D14)}$$

Since $\cos\phi$ is non-negative in this range of integration, we have, with $\bar{\xi}=\xi t$,

$$\int_0^\infty d\xi'\cos(\xi'\cos\phi)\left[-\bar{\xi}\xi'^2-\tfrac{4}{3}\bar{\xi}^3+\tfrac{1}{2}(\bar{\xi}^2+\xi'^2)\xi'\ln\frac{\bar{\xi}+\xi'}{|\bar{\xi}-\xi'|}\right]$$

$$=\tfrac{1}{2}\int_{-\infty}^{\infty}d\xi' e^{i\xi'\cos\phi}\left[-\bar{\xi}\xi'^2-\tfrac{4}{3}\bar{\xi}^3+\tfrac{1}{2}(\bar{\xi}^2+\xi'^2)\xi'\ln\frac{|\bar{\xi}+\xi'|}{|\bar{\xi}-\xi'|}\right]$$

$$=\tfrac{1}{2}i\pi\int_{-\bar{\xi}}^{\bar{\xi}}d\xi' e^{i\xi'\cos\phi}\tfrac{1}{2}(\bar{\xi}^2+\xi'^2)\xi'$$

$$=-\tfrac{1}{2}\pi\int_0^{\bar{\xi}}d\xi'\sin(\xi'\cos\phi)\xi'(\bar{\xi}^2+\xi'^2).\qquad\text{(D15)}$$

Because of the appearance of $\sin(\xi'\cos\phi)$, we try to express $\bar{P}(v)$ in terms of the Struve functions $\mathbf{H}_\nu(\xi)$,[25] which are defined by

$$\mathbf{H}_\nu(\xi)=2\pi^{-1/2}(\tfrac{1}{2}\xi)^\nu[\Gamma(\nu+\tfrac{1}{2})]^{-1}\times\int_0^{\pi/2}\sin(\xi\cos\phi)(\sin\phi)^{2\nu}d\phi.\qquad\text{(D16)}$$

In particular, for $\nu=0$,

$$\mathbf{H}_0(\xi)=2\pi^{-1}\int_0^{\pi/2}\sin(\xi\cos\phi)d\phi.\qquad\text{(D17)}$$

Therefore, by (D12) and (D15),

$$\bar{P}(v)\sim\tfrac{1}{4}t^{-1}\left\{\tfrac{4}{3}-\tfrac{1}{2}\pi\bar{\xi}^{-3}\int_0^{\bar{\xi}}d\xi'\mathbf{H}_0(\xi')\xi'(\bar{\xi}^2+\xi'^2)\right\}$$
$$=\tfrac{1}{4}t^{-1}\{\tfrac{4}{3}-\pi[\mathbf{H}_1(\bar{\xi})-\bar{\xi}^{-1}\mathbf{H}_2(\bar{\xi})]\}.\qquad\text{(D18)}$$

This is the desired answer. It is somewhat more elegant to express this in terms of the Weber function $\mathbf{E}_\nu(\xi)$ defined by

$$\mathbf{E}_\nu(\xi)=2\pi^{-1}\int_0^{\pi/2}d\phi\,\sin(\nu\phi-\xi\sin\phi),\qquad\text{(D19)}$$

and the result is

$$\bar{P}(v)\sim\tfrac{1}{4}\pi t^{-1}[\mathbf{E}_1(\xi t)-(\xi t)^{-1}\mathbf{E}_2(\xi t)],\qquad\text{(D20)}$$

with $\xi=[2(v-1)]^{1/2}$. The result (D20), or equivalently (D18), holds for t large and v close to 1.

It is possible to combine the two results (D8) and (D20) in the following form which holds for t large and all v:

$$\bar{P}(v)\sim\tfrac{1}{4}\pi t^{-1}(v^2-1)^{-1/4}(\cosh^{-1}v)^{1/2}\times[\mathbf{E}_1(t\cosh^{-1}v)-(t\cosh^{-1}v)^{-1}\mathbf{E}_2(t\cosh^{-1}v)].\qquad\text{(D21)}$$

Note the similarity between (D21) and (C7).

[25] For properties of the Struve function **H** and the Weber function **E**, see pp. 35–40 of Vol. II of Ref. 17.

APPENDIX E

In this Appendix, we solve (6.3) exactly and then study its asymptotic properties for small $\bar{\mu}$. This can be accomplished by using the transformation pair[16,17]

$$\bar{\varphi}(t')=t'\tanh\pi t'\int_1^\infty P_{it'-\frac{1}{2}}(v)\varphi(v)dv\qquad\text{(E1)}$$

and

$$\varphi(v)=\int_0^\infty P_{it'-\frac{1}{2}}(v)\bar{\varphi}(t')dt'.\qquad\text{(E2)}$$

From (C6), we know that

$$\int_1^\infty dv'(v'-v_0)^{-1}P_{it'-\frac{1}{2}}(v')=\pi\,\mathrm{sech}\pi t'\,P_{it'-\frac{1}{2}}(-v_0)\qquad\text{(E3)}$$

because $v_0<1$. Therefore, after transformation, (6.3) is

$$(\pi\,\mathrm{sech}\pi t'-\bar{\mu})\bar{\varphi}(t')=CI\pi t'\tanh\pi t'\,\mathrm{sech}\pi t'\,P_{it-\frac{1}{2}}(-v_0),\qquad\text{(E4)}$$

where

$$I=\int_1^\infty dv\,\varphi(v)(v-v_0)^{-1}=\pi\int_0^\infty dt'\bar{\varphi}(t')\,\mathrm{sech}\pi t'\,P_{it'-\frac{1}{2}}(-v_0).\qquad\text{(E5)}$$

Given $\bar{\mu}$, define t by (4.22). Then the solution of (E4) is, with suitable normalization,

$$\bar{\varphi}(t')=\delta(t'-t)+CIt'\tanh\pi t'\cosh\pi t\times(\cosh\pi t-\cosh\pi t')^{-1}P_{it'-\frac{1}{2}}(-v_0).\qquad\text{(E6)}$$

The substitution of (E6) into (E5) gives

$$I=\pi\,\mathrm{sech}\pi t\,P_{it-\frac{1}{2}}(-v_0)+CI\pi\cosh\pi t\int_0^\infty dt't'\tanh\pi t'\,\mathrm{sech}\pi t'\times(\cosh\pi t-\cosh\pi t')^{-1}[P_{it'-\frac{1}{2}}(-v_0)]^2,\qquad\text{(E7)}$$

where the last integral should be interpreted as a principal-value integral at $t'=t$. Solving (E7) for I and substituting into (E6), we get

$$\bar{\varphi}(t')=\delta(t'-t)+C\mathfrak{D}^{-1}\pi t' \tanh\pi t' \,(\cosh\pi t-\cosh\pi t')^{-1} \times P_{it-\frac{1}{2}}(-v_0)P_{it'-\frac{1}{2}}(-v_0)\,, \quad \text{(E8)}$$

where

$$\mathfrak{D}=1-C\pi \cosh\pi t\int_0^\infty dt' t' \tanh\pi t' \operatorname{sech}\pi t' \times(\cosh\pi t-\cosh\pi t')^{-1}[P_{it'-\frac{1}{2}}(-v_0)]^2. \quad \text{(E9)}$$

Equation (E8), with (E2), gives the desired answers.

In order to verify that this is indeed a solution of the integral equation (6.3), we make use of (C5) once more. Define

$$\Phi(v)=\int_0^\infty dt' \bar{\varphi}(t')[\pi \operatorname{sech}\pi t' \, P_{it'-\frac{1}{2}}(-v)] \quad \text{(E10)}$$

for complex v. Then the discontinuity of $\Phi(v)$ across the branch cut from 1 to ∞ along the positive real axis is

$$\Phi(v+i0)-\Phi(v-i0)=2\pi i\varphi(v)\,. \quad \text{(E11)}$$

This function $\Phi(v)$ also has the following property:

$$\begin{aligned}\Phi(-v)-\bar{\mu}\varphi(v)&=\int_0^\infty dt' \bar{\varphi}(t')(\pi \operatorname{sech}\pi t'-\pi \operatorname{sech}\pi t)P_{it'-\frac{1}{2}}(v)\\ &=C\mathfrak{D}^{-1}\pi^2 \operatorname{sech}\pi t\, P_{it-\frac{1}{2}}(-v_0)\int_0^\infty dt' t' \tanh\pi t' \operatorname{sech}\pi t' \, P_{it'-\frac{1}{2}}(-v_0)P_{it'-\frac{1}{2}}(v)\\ &=C\mathfrak{D}^{-1}\pi \operatorname{sech}\pi t\, P_{it-\frac{1}{2}}(-v_0)(v-v_0)^{-1}\,, \end{aligned} \quad \text{(E12)}$$

where, in the last step, we have used the Mehler formula for integrating the product of Legendre functions over their order.[16,17] We therefore have

$$\begin{aligned}&\int_1^\infty dv' \varphi(v')[(v+v')^{-1}-C(v-v_0)^{-1}(v'-v_0)^{-1}]\\ &=(2\pi i)^{-1}\int_{C_1} dv'\Phi(v')[(v+v')^{-1}-C(v-v_0)^{-1}(v'-v_0)^{-1}]\\ &=\Phi(-v)-C(v-v_0)^{-1}\Phi(v_0)\\ &=[\bar{\mu}\varphi(v)+C\mathfrak{D}^{-1}\pi \operatorname{sech}\pi t\, P_{it-\frac{1}{2}}(-v_0)(v-v_0)^{-1}]-C(v-v_0)^{-1}\pi \operatorname{sech}\pi t\, P_{it-\frac{1}{2}}(-v_0)[1+\mathfrak{D}^{-1}(1-\mathfrak{D})]\\ &=\bar{\mu}\varphi(v)\,. \end{aligned} \quad \text{(E13)}$$

We shall now study the asymptotic behavior of $\varphi(v)$ for large v. Since

$$P_{it'-\frac{1}{2}}(v)\sim\Gamma(2it')[\Gamma(it'+\tfrac{1}{2})]^{-2}(\tfrac{1}{2}v)^{it'-\frac{1}{2}} +\Gamma(-2it')[\Gamma(-it'+\tfrac{1}{2})]^{-2}(\tfrac{1}{2}v)^{-it'-\frac{1}{2}} \quad \text{(E14)}$$

from (C1), we have

$$\varphi(v)\sim(2v)^{-1/2} \operatorname{Re}\Gamma(2it)[\Gamma(it+\tfrac{1}{2})]^{-2}(\tfrac{1}{2}v)^{it} \times\{1-C\mathfrak{D}^{-1}\pi it \operatorname{sech}\pi t\,[P_{it-\frac{1}{2}}(-v_0)]^2\}\,. \quad \text{(E15)}$$

Therefore, the orthonormal eigenfunctions are

$$N\varphi(v)\,,$$

where

$$N=(t \tanh\pi t)^{1/2}\{1+(C\mathfrak{D}^{-1}\pi t \operatorname{sech}\pi t)^2 \times[P_{it-\frac{1}{2}}(-v_0)]^4\}^{-1/2}. \quad \text{(E16)}$$

So far the development holds for all t. We shall now concentrate on the limit of large t. For the purposes of Sec. 6, we need the behavior of $\Phi(v)$ for $-v_0<v<1$. With reference to (E15), we write, on the basis of (E8) and (E10),

$$\Phi(v)=\pi \operatorname{sech}\pi t\,\{\Phi_1(v)+C\mathfrak{D}^{-1}\pi t \operatorname{sech}\pi t \times[P_{it-\frac{1}{2}}(-v_0)]^2\Phi_2(v)\}\,, \quad \text{(E17)}$$

where

$$\Phi_1(v)=P_{it-\frac{1}{2}}(-v) \quad \text{(E18)}$$

and

$$\Phi_2(v)=t^{-1}\cosh^2\pi t\int_0^\infty dt' t' \tanh\pi t' \operatorname{sech}\pi t' \times(\cosh\pi t-\cosh\pi t')^{-1} \times P_{it'-\frac{1}{2}}(-v_0)P_{it'-\frac{1}{2}}(-v)/P_{it-\frac{1}{2}}(-v_0)\,. \quad \text{(E19)}$$

It is an immediate consequence of (5.13) that, for large t and $|v_0|<1$,

$$\Phi_1(v)\sim(2\pi t)^{-1/2}(1-v^2)^{-1/4}\exp[t(\pi-\cos^{-1}v)]\,. \quad \text{(E20)}$$

The asymptotic behavior of $\Phi_2(v)$ is more complicated for the following reason. The dominating contribution to the integral of (E19) comes from the vicinity of $t'=t$

when $v > -v_0$, but from the vicinity of $t' = 0$ when $v < -v_0$. We therefore treat these two cases separately.

For $v > -v_0$,

$$\Phi_2(v) \sim \pi^{-1}\{\psi[\pi^{-1}(\cos^{-1}v + \cos^{-1}v_0)] - \psi[1 - \pi^{-1}(\cos^{-1}v + \cos^{-1}v_0)]\}\Phi_1(v), \quad \text{(E21)}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function.

For $v < -v_0$,

$$\Phi_2(v) \sim (-v - v_0)^{-1}\Phi_1(-v_0). \quad \text{(E22)}$$

Thus, for large t, $\Phi_1(v)$ and $\Phi_2(v)$ are comparable when $-v_0 < v < 1$, but $\Phi_2(v)$ is much larger than $\Phi_1(v)$ when $-1 < v < -v_0$. In particular, it follows from (E16), (E17), and (E20)–(E22) that for $-v_0 < v < 1$,

$$\ln[N\Phi(v)] \sim -(\cos^{-1}v)t, \quad \text{(E23)}$$

and that for $-1 < v < -v_0$,

$$\ln[N\Phi(v)] \sim -(\pi - \cos^{-1}v_0)t. \quad \text{(E24)}$$

Remember that all arccosines have been taken to be between 0 and π. Equation (E23) is the result that we need in Sec. 6.

Equation (E23) does not quite tell the whole story. There is an exceptional point $v = v_0$ where the Φ_1 and Φ_2 terms in (E17) cancel each other. More explicitly,

$$\Phi(v_0) = \pi\mathfrak{D}^{-1}\,\mathrm{sech}\pi t\, P_{it-\frac{1}{2}}(-v_0) = \pi\mathfrak{D}^{-1}\,\mathrm{sech}\pi t\,\Phi_1(v_0). \quad \text{(E25)}$$

Thus $\Phi(v_0)$ is smaller by a factor $\mathfrak{D}^{-1}$ owing to the above-mentioned cancellation. For this reason, we need to estimate the order of magnitude of $\mathfrak{D}$.

A comparison of (E19) and (E9) shows that

$$\mathfrak{D} = 1 - C\pi t\,\mathrm{sech}\pi t\, P_{it-\frac{1}{2}}(-v_0)\Phi_2(v_0). \quad \text{(E26)}$$

When $v_0 > 0$, (E21) applies and we get

$$\mathfrak{D} \sim -C\pi^{-1}(1-v_0^2)^{-1/2}[\psi(2\pi^{-1}\cos^{-1}v_0) - \psi(1 - 2\pi^{-1}\cos^{-1}v_0)]\exp[t(\pi - 2\cos^{-1}v_0)]. \quad \text{(E27)}$$

Therefore, in this case $v_0 > 0$,

$$\ln[N\Phi(v_0)] \sim -(\pi - \cos^{-1}v_0)t. \quad \text{(E28)}$$

For $v_0 < 0$, (E22) is valid but not accurate enough. We need instead

$$\mathfrak{D} = 1 - C\pi\int_0^\infty dt'\,t'\,\tanh\pi t'\,[P_{it'-\frac{1}{2}}(-v_0)]^2[\mathrm{sech}\pi t' + (\cosh\pi t - \cosh\pi t')^{-1}]$$

$$= 1 + \tfrac{1}{2}Cv_0^{-1} - C\pi\int_0^\infty dt'\,t'\,\tanh\pi t'[P_{it'-\frac{1}{2}}(-v_0)]^2[\cosh\pi t - \cosh\pi t']^{-1}$$

$$\sim 1 + \tfrac{1}{2}Cv_0^{-1} - C(1-v_0^2)^{-1/2}\pi^{-1}\{\psi[1 - 2\pi^{-1}\cos^{-1}(-v_0)] - \psi[2\pi^{-1}\cos^{-1}(-v_0)]\}\exp[-2t\sin^{-1}(-v_0)]. \quad \text{(E29)}$$

Therefore, the magnitude of $\mathfrak{D}$ depends on whether or not $C = -2v_0$. However, in either case, $\ln N$ of (E16) is of order $o(t)$. Therefore, for $v_0 < 0$,

$$\ln[N\Phi(v_0)] \sim \begin{cases} -t\cos^{-1}v_0 & \text{for } C \neq -2v_0 \\ -(\pi - \cos^{-1}v_0)t & \text{for } C = 2v_0. \end{cases} \quad \text{(E30)}$$

In either case, the right-hand side of (E30) is not larger than that of (E24). The important point here is that, in all cases from (E28) and (E30), $\ln[N\Phi(v_0)]$ is never much larger than $\ln[N\Phi(-v_0)]$.