Bootstraps in Local Field Theory

M. L. WHIPPMAN*
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
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The problem of formulating a bootstrap in terms of vanishing renormalization constants is discussed. It is pointed out that the equations $Z=0$ need not determine masses and coupling constants, but might be identities or else possess no solution. It is argued that this always occurs in theories in which particles are treated as made up of arbitrarily many other particles, and this is illustrated in a soluble model. Finally, various consequences of this idea are mentioned.

I. SURVEY

DURING the last few years, evidence has slowly been accumulating that local field theory not only may prove to be a useful framework for describing hadron interactions, but may even exist. At the same time the bootstrap idea, originally expressed in the language of S-matrix theory, is sufficiently attractive that one would like to see it incorporated into any field theory of strong interactions. Of course, what is meant by a bootstrap is not completely defined, and to a certain extent each author defines it slightly differently, depending on the approximation scheme he is using; and it is also not clear whether field theory and S-matrix theory are merely two ways of saying the same thing or really give different descriptions of the world. Nonetheless, there is some general agreement that in the language of local field theory we may formulate a bootstrap model by requiring that all renormalization constants vanish.[1] Since the purpose of this paper is to examine this statement critically, it is worthwhile being rather more precise as to what exactly it implies.

The early work[2] in this field was largely devoted to proving what we call Jouvet's theorem in various models. To understand this theorem, let us consider an elementary particle A of mass M which couples to two other particles B and C through a three-point Yukawa coupling of strength g. The theorem states that for certain values of g and M all observable quantities in the theory are the same as those in a theory in which B and C couple directly through a four-particle coupling, and in which there exists a bound state A' of B and C of mass M. Furthermore, the limit in which this occurs is that which makes the wave-function renormalization constant of particle A equal to zero. Loosely speaking, an elementary particle with zero wave-function renormalization behaves like a bound state.[3]

Of course, this result has no content until we define more carefully what we mean by "bound state" or "elementary particle." Once again, these terms have usually been defined in terms of the model being used—an elementary particle is a pole of the N function, or gives a Kronecker δ in the complex angular momentum plane or corresponds to a field that occurs in the original Lagrangian, while a bound state is a zero of D, or lies on a Regge trajectory or does not occur in the Lagrangian. Intuitively we expect that if A is elementary, then the theory exists for a range of values of M and g, while if A is bound then M and g are determined by the other masses and coupling constants in the theory. It is precisely the equations expressing this dependence that imply that the wave function renormalization of A vanishes. Furthermore, these equations are exactly the equations that would be written down in a "traditional" N/D approach to the bootstrap.[4]

Jouvet's theorem raises the intriguing possibility that we can make all the particles in the theory bound states by requiring that all the renormalization constants vanish. Such a theory would be a true bootstrap in which no particle was more elementary than any other, and in which all ratios of masses and coupling constants would be determined. At this stage, we should mention one technical point: Since we want the renormalized masses and couplings to be finite, the unrenormalized masses and couplings will, in general, be infinite. A straightforward field-theoretic calculation usually gives the renormalization constants (which we shall collectively call Z) in terms of these unrenormalized quantities. It is tacitly understood that in carrying out the above program we must first express these unrenormalized quantities in terms of the renormalized ones, treating the Z's as finite numbers, and then solve for the Z's in terms of the renormalized quantities before requiring that the Z's vanish. Only such a procedure will give bootstrap equations for the observed physical quantities.

It is worth pointing out that some result such as Jouvet's theorem is almost necessary if the bootstrap program is to provide a means of calculating the properties of the hadrons. If bound states and elementary particles were really completely distinct objects, or if it

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[1] The first discussion of vanishing renormalization constants in this context seems to be due to B. Jouvet, Nuovo Cimento **5**, 1 (1957). The suggestion that this provides a basis for a bootstrap in field theory was made by A. Salam, *ibid.* **25**, 224 (1962).

[2] The literature in this field is enormous. The early work can be traced from the review article by K. Hayashi, M. Hirayama, T. Muta, and T. Shirafuji, Forschr. Phys. **15**, 625 (1967) or from H. Osborne, Ann. Phys. (N. Y.) **47**, 310 (1968).

[3] Some authors have found it useful to distinguish various types of elementary or bound states depending on how rapidly certain functions vanish at infinite energy. We shall not need these distinctions in this work.

[4] B. W. Lee, K. T. Mahanthappa, I. Gerstein, and M. L. Whippman, Ann. Phys. (N. Y.) **28**, 466 (1964).

was necessary to treat all particles on the same footing at all stages of the calculation, the only way to treat the bootstrap equations would be to write down their complete solution as the first step in the calculation. In such a scheme, even the number of states necessary to label the rows and columns of the S matrix would not be known until all the equations had been solved. It appears that progress can be made only if we can discuss some intermediate world in which a certain number of particles can be treated as elementary (in the sense that their properties are assumed, not calculated). The properties of the other particles can then be found in terms of the assumed properties of this initial set, and hopefully there is some limit in which the original set can be themselves treated as bound. Such a procedure has, of course, been used or implied in all bootstrap schemes to date; Jouvet's theorem provides a basis for it by suggesting that there is a well-defined limit in which an elementary particle is indistinguishable from a bound state.

Unfortunately, though Jouvet's theorem is very useful, it is not clear that it is true. Existing proofs of it are of two types—proofs in very simple models such as the Lee model[5] or the Zachariasen model,[6] and demonstrations that nothing obvious goes wrong in more realistic theories.[4] The most comprehensive proof of the latter sort is that by Kaus and Zachariasen,[7] who show that under very general conditions it is possible to write equations for all the renormalization constants in the theory, and that if these constants vanish, the resulting equations imply that all the particles lie on Regge trajectories. There is, however, one possible flaw in the argument: though it is straightforward to write the equations $Z=0$, it is not clear that these equations possess solutions corresponding to real finite values of the couplings and masses. We raise this possibility (which is, of course, mentioned by Kaus and Zachariasen) because there exists a large class of model field theories in which $Z=0$ is not an equation determining masses and couplings.

All known models with this property are static, and thus not very realistic. They do, however, satisfy many of the essential requirements of realistic theories—analyticity, unitarity, and crossing—though they do not Reggeize. As an example, we discuss the simplest such model—neutral scalar theory[8]—though similar behavior is known in the Ruijgrok–Van Hove model[9] and the model of Freeman and Rubin.[10] Neutral scalar theory considers an uncharged π, coupled to a single static nucleon with a coupling strength g. A simple calculation[8] gives for the wave-function renormalization of the nucleon

$$Z=e^{-g^2 I}, \tag{1}$$

where I is a kinematic integral over the cutoff function and does not involve g. It is clear that $Z=0$ cannot be achieved with a real finite coupling constant. In fact, $Z=0$ only in the limit of a point source, when I goes to infinity. In this case, however, Z vanishes identically, independently of the coupling. The question that now arises is whether this behavior is specific to these models, or whether it occurs in more realistic theories.

To summarize, we can distinguish three classes of theory:

(I) The equations $Z=0$ determine real, finite values for all the couplings g and masses M in the theory.
(II) The equations $Z=0$ are identities, true for all values of g and M.
(III) The equations $Z=0$ cannot be satisfied for any finite values of g and M.

Simple models which provide examples of each class are known, and it is interesting to see whether there is some general feature of all such models which allows us to say *a priori* to which class they belong. There is, in fact, a very simple way to classify models in terms of the number of particles that enter into intermediate states. This classification is best explained by examples.

In the Lee model, for instance, there are three particles—V, N, and θ, with $V\leftrightarrow N+\theta$ the only interaction. The physical V is then a superposition of bare V's and bare $N\theta$ states. As the wave-function renormalization of V tends to zero, V becomes a bound state of N and θ, which we call a two-particle bound state. Bronzan's[11] modification of the Lee model includes a fourth particle U and an interaction $U\leftrightarrow V+\theta$. A bound U would then be a composite of V and θ, or of N and 2θ—we call this a three-particle bound state. (In general, an n-particle bound state is a superposition of states of n, $n-1$, $n-2$, ..., particles.)

In neutral scalar theory, however, the physical N is a superposition of the bare N and any number of π's. If N were composite, it would be a composite of an infinite number of particles. We call this infinitely composite.[12] It turns out that of all soluble models that have been studied, all those where a given particle would be infinitely composite[13] belong to class II or III, while all those for which it would be a composite of only a finite number of states belong to class I. We hypothesize that

[5] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).
[6] M. L. Whippman and I. Gerstein, Phys. Rev. **134**, B1128 (1964).
[7] P. Kaus and F. Zachariasen, Phys. Rev. **171**, 1597 (1968).
[8] See, e.g., D. Barton, *Introduction to Advanced Field Theory* (Interscience, New York, 1963), p. 119.
[9] Th. W. Ruijgrok and L. Van Hove, Physica **22**, 880 (1956).
[10] D. Freeman and M. Rubin, Phys. Rev. D **1**, 3386 (1970). (The model is that of North.)

[11] J. Bronzan, Phys. Rev. **139**, B751 (1965).
[12] There is another respect in which this theory differs from the Lee model. Here if N were composite, it would be a composite of *itself* plus π mesons—what B. Jouvet and J. Le Guillou call "Oedipian" [Nuovo Cimento **49A**, 677 (1967)]. This point is not important here—for an infinitely composite model in which this does not happen, see Sec. II.
[13] We say "would be" because in theories of class III the particle is not composite. What we mean more exactly is theories in which the physical particle state is a superposition of bare states containing arbitrarily large numbers of particles.

this is true in general—that $Z=0$ provides an equation for g^2 and M only in the case of a particle which is not infinitely composite. We mention parenthetically that we expect that only theories with some sort of cutoff will belong to class III, and hence that in all realistic local theories the renormalization constants will vanish identically.

It is, of course, exceptionally difficult to prove or disprove this hypothesis. As a step towards understanding it better, in Sec. II we discuss a simple model which allows the finite and the infinite case to be compared and in Sec. III we give some arguments that might make it more plausible in general. This is not very satisfactory, even if it is as rigorous a method of proof as has been given for most results in this field. A proof or counter example would be very valuable; our main reason for writing this paper has been to provoke further investigation of this point.

II. SOLUBLE MODEL

As we explained in Sec. I, it is interesting to look at a model in which a given particle can be a bound state of n other particles (or less) and to study the behavior of the various quantities as n tends to infinity. Our model is an extension of the Lee model closely related to the models studied by Ruijgrok and Van Hove.[9]

We consider an infinite chain of static "nucleons" N_r $(r=1,2,\ldots)$ and one boson θ, such that each nucleon can emit a boson to become the next lowest member of the chain, or can absorb one to become the next highest member,

$$N_{r+1}\leftrightarrow N_r+\theta. \tag{2}$$

Specifically, our assumed Hamiltonian is

$$H=\sum_{r=1}^{\infty} m_r\psi_r^\dagger\psi_r+\sum_{\mathbf{k}}\omega_k a_{\mathbf{k}}^\dagger a_{\mathbf{k}} +\sum_{r=1}^{\infty}\sum_{\mathbf{k}} u(\mathbf{k})g_r(\psi_{r+1}^\dagger\psi_r a_{\mathbf{k}}+\text{H.c.}), \tag{3}$$

where ψ_r is the destruction operator for N_r, $a_{\mathbf{k}}$ the destruction operator for a θ of momentum $\mathbf{k}$, $\omega_k=k^2+\mu^2$, and $u(\mathbf{k})$ is a suitable cutoff function. All the quantities appearing in this Hamiltonian are unrenormalized. Like the Lee model, the model splits up into distinct sectors labelled by the eigenvalues of the two conserved quantities—"baryon number" B defined by

$$B=\sum_{r=1}^{\infty}\psi_r^\dagger\psi_r \tag{4}$$

and "charge" Q defined by

$$Q=\sum_{r=1}^{\infty} r\psi_r^\dagger\psi_r+\sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}. \tag{5}$$

The exact vacuum $|0\rangle$ is the same as the bare vacuum, and is the only state with $Q=B=0$. To find the one-particle state, we note that the exact state with $B=1$ and $Q=n$ is a superposition of the states containing r mesons and the $(n-r)$th nucleon. Using this fact, the Schrödinger equation can be solved straightforwardly, if tediously. The algebra can be simplified considerably by a judicious choice of the bare masses m_r, and since none of our conclusions are changed by this, we choose

$$m_r=m_1+g_{r-1}^2\int d^3k\frac{|u(\mathbf{k})|^2}{\omega_k}. \tag{6}$$

The one-particle eigenstates of H can then be written in the simple form

$$|n\rangle=(\sqrt{Z_n})\left[\psi_n^\dagger|0\rangle +\sum_{r=1}^{n-1}(-)^r\frac{g_{n-1}\cdots g_{n-r}}{r!}J^r\psi_{n-r}^\dagger|0\rangle\right], \tag{7}$$

where the wave-function renormalization constant Z_n of particle n is defined by this equation, and where J is the operator

$$J=\sum_{\mathbf{k}}\frac{u(\mathbf{k})}{\omega_{\mathbf{k}}}a_{\mathbf{k}}^\dagger. \tag{8}$$

It is interesting to note that the choice (6) for the bare masses makes all the renormalized masses equal.

We can now calculate Z_n immediately by requiring that $|n\rangle$ be normalized. This yields[14]

$$Z_n^{-1}=1+\sum_{r=1}^{n-1}\frac{(g_{n-1}g_{n-2}\cdots g_{n-r})^2}{r!}I^r, \tag{9}$$

where

$$I=\int d^3k\frac{|u(\mathbf{k})|^2}{\omega_k^2}. \tag{10}$$

As explained in Sec. I, before discussing the limit $Z\to 0$ we must express the Z's in terms of the renormalized coupling constants. Following Lee,[15] we define the renormalized $N_r+\theta\to N_{r+1}$ coupling constant (which we denote by γ_r) by[16]

$$\gamma_r u(k)=\langle r|j_{\mathbf{k}}|r+1\rangle, \tag{11}$$

[14] From the form of the Hamiltonian, it follows at once that $Z_1=1$. In the following, equations like (9) are supposed to hold for $n=2, 3, \ldots$. We should also remind the reader that Z_1 is the wave-function renormalization of particle 1, not a vertex renormalization.

[15] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[16] Note that we define the renormalized coupling constant in terms of the vertex function with both nucleons on the mass shell but the meson off the mass shell. Some authors prefer to define this quantity with the meson on the mass shell and one of the nucleons off. (It is impossible to have all three particles on shell in this static model.) Our choice greatly simplifies the calculation of the vertex renormalization and makes no difference to our final conclusion. To avoid confusion, however, we should point out that our formulas differ in detail from those of other workers for just this reason.

where j_k is the meson current operator given by

$$j_k=[a_k,H_{int}]=\sum_s g_s\psi_s^\dagger\psi_{s+1}. \qquad (12)$$

A straightforward calculation, using the states (7), gives

$$\gamma_r=g_rZ_{r+1}^{1/2}/Z_r^{1/2}. \qquad (13)$$

Comparing this with the usual definition

$$\gamma_r=g_r\frac{Z_r^{1/2}Z_{r+1}^{1/2}}{Z_1(r,r+1)}, \qquad (14)$$

where $Z_1(r,r+1)$ is the vertex renormalization constant for this vertex, we see that there is a Ward-like identity in the model[9]

$$Z_1(r,r+1)=Z_r. \qquad (15)$$

We can now express everything in terms of the renormalized constants. Using Eqs. (9) and (13) we find

$$Z_n=1-\sum_1^{n-1}\frac{\gamma_{n-1}^2\cdots\gamma_{n-r}^2}{r!}Z_{n-r}I^r. \qquad (16)$$

These equations may be solved to give

$$Z_n=1+\sum_1^{n-1}(-)^r\frac{\gamma_{n-1}^2\gamma_{n-2}^2\cdots\gamma_{n-r}^2}{r!}I^r, \qquad (17)$$

as shown in the Appendix.

After these algebraic preliminaries, we are finally in a position to discuss the solution of the equations $Z=0$. Somewhat remarkably, these equations possess the very simple solution

$$\gamma_r^2=r/I, \qquad (18)$$

since if this is substituted into Eq. (17) the right-hand side becomes the binomial expansion of $(1-1)^{n-1}$.

If we only wish to consider a finite number of nucleon states, this result is very satisfactory from the point of view of the bootstrap hypothesis. All except the first of these states can be made composite in the $Z=0$ sense, and the resulting equations determine all the free couplings. It is straightforward to verify that the various scattering amplitudes are well defined in this limit, and are nontrivial. In these respects this model behaves exactly like those discussed in Refs. 5 and 6, although it is interesting to note that no particular problems arise in trying to make composite states of particles which are themselves composite.[17]

The picture is very different when we try to consider the infinitely composite particle, however.[18] In this case, the bootstrap equation (18) leads to an infinite coupling constant, or, in the language of our previous classification, the theory belongs to class III. There is one other possibility: If the cutoff is such that the integral I does not converge, then we see from Eq. (9) that Z_r vanishes identically (for all $r\neq 1$) and the theory belongs to class II. Our original conjecture about the classification of theories being determined by whether particles would be finitely or infinitely composite is thus confirmed in this model. In Sec. III, we discuss why we believe it is generally true.

[17] In this connection, see P. Liossatos, Phys. Rev. **172**, 1554 (1968).

[18] That is, the limit $r\to\infty$ of Eq. (18).

III. DISCUSSION

The idea that Z may vanish identically in realistic local theories is a very old one. It seems to have been first suggested by Landau *et al.*[19] based on an analysis of the high-energy behavior of quantum electrodynamics, and was discussed again by Gell-Mann and Low[20] in a similar context. The reasoning given by these authors is persuasive, if not completely airtight,[21] though recent work by Wilson[22] does tend to support their arguments.

One of the few completely rigorous results in this field is Haag's theorem[23] which is true under very general conditions. This theorem implies that if we define wave-function renormalization constants the way we did above, as the overlap between the exact and the bare one-particle states, then they are indeed identically zero in the no-cutoff limit. At first sight, this would seem to settle the whole question, but in fact renormalization constants may also be defined in terms of integrals over the spectral functions of the exact propagator.[24] More precisely, we might define wave-function renormalization either by

$$|N\rangle_{\text{exact}}=(\sqrt{Z})\,|N\rangle_{\text{bare}}+\cdots, \qquad (19)$$

where $|N\rangle_{\text{exact}}$ and $|N\rangle_{\text{bare}}$ are one-particle states, or by

$$Z'=\left[\int\rho(s')ds'\right]^{-1}, \qquad (20)$$

where ρ is the spectral function for the one-particle Green's function. If Z defined by (19) is nonzero, and the integral in (20) converges, we can prove that these definitions are equivalent—that is, that $Z'=Z$. If Z is zero, however, or if the integral diverges, the equivalence has not been shown. Haag's theorem implies that Z vanishes in a realistic theory, but it is still possible for Z' to be well defined and to vanish only for specific values of masses and couplings. In models such as neutral scalar theory for which Haag's theorem is true in the

[19] L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 497 (1954); **95**, 773 (1954); **95**, 1177 (1954); **96**, 261 (1954). See also L. D. Landau, in *Niels Bohr and the Development of Physics*, edited by W. Pauli (Pergamon, London, 1955), p. 52.

[20] M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).

[21] M. Astaud and B. Jouvet, Nuovo Cimento **63A**, 5 (1969).

[22] K. Wilson, Phys. Rev. D **2**, 1438 (1970).

[23] R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **29**, No. 12 (1955).

[24] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

infinite-cutoff limit, this does not happen and Z and Z' are equal both for finite and for infinite cutoff. Whether this happens in general is not known; one is tempted to say, however, that Haag's theorem suggests that it would not be too surprising if our conjecture were true.

Finally, this may be an appropriate place to mention a famous problem[7] with bootstrap systems. There is one Z for each particle, and one for each primitive coupling in the theory, so that the number of equations $Z=0$ is the same as the number of masses and couplings. If none of the equations is an identity, then we have enough equations to determine all the masses and couplings. On dimensional grounds, however, we only expect to be able to predict *ratios* of masses, and hence we have one equation too many. On the other hand, a true bootstrap theory should treat all the particles on the same footing, and it is difficult to see why one of these equations should be an identity while the rest are not.[25] An elegant escape from this problem would be for *all* the equations to be identities, as we postulated.

None of these arguments is very convincing by itself, and the question is still very much open. If we throw caution to the winds, however, and assume our hypothesis is true, it is interesting to ask what the consequences are. In other words, let us assume for the moment that all the Z's do vanish identically in a realistic theory, and ask what this implies.

As we see it, there are three possibilities. The first, and least interesting, is that there exists some other bootstrap condition in local field theories which is equivalent to $Z=0$ in the simple models where Jouvet's theorem is true, but which is different from it in general. This is certainly possible, though if there is such a condition no one knows what it is.

A second and much more intriguing possibility is that the equations $Z=0$ are really equivalent to the bootstrap equations an S-matrix theorist would write down, and that they are identities. In other words, schemes like the N/D equations or the strip approximation impose conditions on the theory only because they are finite particle equations, and as more and more particles are included in intermediate states these "equations" become identities. From this point of view, the bootstrap hypothesis would be true, but also trivial, and would not constrain the theory in any way.

Finally there is the possibility that the bootstrap is nontrivial in S-matrix theory, but trivial in field theory—in other words, that these two theories are really fundamentally different ways of describing hadron physics and are not ultimately equivalent. In many ways, this would be the most interesting possibility of all.

Of course, all of this is highly conjectural and our purpose in writing this paper was frankly heuristic, but we believe that even if our hypothesis is wrong, the question of infinitely composite particles deserves a great deal more study.

[25] It is, of course, possible that none of the equations is an identity by itself, but that they are not all independent. We would like to believe, however, that we will get the same bootstrap conditions no matter which subset of the particles we initially regard as elementary. It is hard to see how this could happen if we always had exactly one equation too many.

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APPENDIX

We wish to verify the solution of Eq. (16). Clearly $Z_1=1$, and we assume that (17) holds for $n=1, 2, \ldots, N-1$. Then (16) gives

$$Z_N=1-\sum_{r=1}^{N-1}\frac{\gamma_{N-1}{}^2\cdots\gamma_{N-r}{}^2}{r!}I^r-\sum_{r=1}^{N-2}\sum_{s=1}^{N-r-1}(-)^s\frac{\gamma_{N-1}{}^2\cdots\gamma_{N-r-s}{}^2}{r!s!}I^{r+s}. \quad \text{(A1)}$$

Putting $r+s=\mu$ in the last term, this double sum becomes

$$-\sum_{r=1}^{N-2}\sum_{\mu=r+1}^{N-1}\frac{\gamma_{N-1}{}^2\cdots\gamma_{N-\mu}{}^2}{r!(\mu-r)!}(-)^{r-\mu}I^\mu$$

$$=-\sum_{\mu=2}^{N-1}\gamma_{N-1}{}^2\cdots\gamma_{N-\mu}{}^2(-)^\mu I^\mu\sum_{r=1}^{\mu-1}\frac{(-)^r}{r!(\mu-r)!}$$

$$=\sum_{\mu=2}^{N-1}\frac{\gamma_{N-1}{}^2\cdots\gamma_{N-\mu}{}^2}{u!}(-)^\mu I^\mu[1+(-)^\mu], \quad \text{(A2)}$$

where the last line follows from the binomial theorem. Putting this back into (A1), the desired result follows by induction.