PHYSICAL REVIEW D VOLUME 3, NUMBER 10 15 MAY 1971

Lorentz-Invariant Localization for Elementary Systems. III. Zero-Mass Systems

A. J. KÁLNAY*

International Atomic Energy Agency, International Centre for Theoretical Physics, Trieste, Italy
and
Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela

(Received 10 November 1969)

The present series of papers is devoted to the localization problem in relativistic quantum mechanics. The consequences of imposing the condition that the descriptions of a localized state seen from two different frames of reference should be physically consistent were investigated in Paper II. Only instantaneous localization was studied. Considering elementary systems of nonzero mass and spin 0 and $\frac{1}{2}$, the following results were found. (a) As regards the localization problem, one cannot avoid accepting at least one of the following strong departures from the usual ideas: (i) Position has no meaning; (ii) it violates the physical equivalence of inertial frames; (iii) it is the only quantum variable which cannot be represented by an operator; (iv) it is non-Hermitian; (v) some unusual interaction effects do not disappear when the interaction is switched off. (b) If a component of position is to be precisely measurable as a point, then for spin 0 there is only one possible position operator; for spin $\frac{1}{2}$ the operator is unique up to a parameter; in both cases the operators are non-Hermitian with respect to the relativistically invariant scalar product, but in spite of this the eigenvalues of all components are real. The components of position are compatible with each other for spin 0 and incompatible for spin $\frac{1}{2}$. The commutation relations of position with linear momentum are the standard ones. The velocity operator, which is Hermitian, has the expected form. Several authors have stated that the localization problem can have a solution for mass greater than zero, but for the zero-mass case a solution does not exist. In this paper the localization problem is studied for zero-mass elementary systems by only imposing, as in Paper II, the above requirement of physical consistency. We consider systems with spin 0, $\frac{1}{2}$, and 1. We prove that the consequences are essentially the same as those obtained for systems of mass greater than zero. The parameter is no longer arbitrary.

I. INTRODUCTION

A. General

THE present series of papers studies the problem of localizability of elementary systems in relativistic quantum mechanics. In Paper I[1] Philips's[2] results are discussed. In Paper II[3] (hereafter called II) we derived the general consequences of imposing *Lorentz invariance of localization*, i.e., the physical consistency of the description of the localization by observers in different inertial frames, and applied them to nonzero-mass systems of spin 0 and $\frac{1}{2}$. The present paper is directly related to II; the consequences of the Lorentz invariance of localization are applied to the zero-mass systems.

One of the basic axioms used in the classic paper by Newton and Wigner[4] can be expressed as the requirement of physical consistency of the description of localization by observers in frames related by a three-dimensional rotation. However, in a relativistic theory a similar requirement should be satisfied in all inertial frames. The purpose of finding a solution which agrees as well as possible with Lorentz invariance of localization was developed in the papers of Wigner, Philips, and this author.[1–3,5]

A set of axioms which expresses the Lorentz invariance of localization for elementary systems in the most general way was stated in II. We restricted ourselves to systems for which the localization with respect to the k axis has sense at a point[6–10] at a given time. In order not to introduce avoidable hypotheses, we neither used nor rejected standard assumptions. Let us, for example, recall[3] that (a) *Lorentz invariance of localization* can be achieved without imposing manifest formal covariance (see II and the references therein); (b) *non-Hermitian operators* can have a legitimate use in quantum mechanics,[7–11]; and (c) the possibility that components of the position do not commute cannot be rejected in an absolute way (as for instance in the case of the angular momentum), so it is not necessary to impose the existence of three-localized states[3] (i.e., simultaneous eigenstates of the three components of position); instead, it is sufficient to consider one-localized states[3] (i.e., eigenstates of only one component).

Indeed, it was proved in II that (at least for mass m greater than zero and spin s equal to 0 and $\frac{1}{2}$) the

* Present address: IVIC, Sección Física, Apartado 1827, Caracas, Venezuela.

[1] J. C. Gallardo, A. J. Kálnay, and S. H. Risemberg, Phys. Rev. **158**, 1484 (1967).

[2] T. O. Philips, Phys. Rev. **136**, B893 (1964).

[3] A. J. Kálnay, Phys. Rev. D **1**, 1092 (1970).

[4] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

[5] T. O. Philips, thesis, Princeton University, 1963 (unpublished); T. O. Philips and E. P. Wigner, in *Group Theory and Its Application*, edited by E. M. Loebl (Academic, New York, 1968).

[6] This means that the *extended-type position* (see Refs. 7–9) is not considered in II or in the present paper, but that the limiting case of that position (Ref. 10) is consistent with the results of both papers (cf. II).

[7] A. J. Kálnay and B. P. Toledo, Nuovo Cimento **48**, 997 (1967).

[8] J. A. Gallardo, A. J. Kálnay, B. A. Stec, and B. P. Toledo, Nuovo Cimento **48**, 1008 (1967).

[9] M. Baldo and E. Recami, Nuovo Cimento Letters **2**, 643 (1969).

[10] J. A. Gallardo, A. J. Kálnay, B. A. Stec, and B. P. Toledo, Nuovo Cimento **49**, 393 (1967).

[11] E. C. Kemble, *The Fundamental Principles of Quantum Mechanics with Elementary Applications* (Dover, New York, 1958); W. E. Brittin, Am. J. Phys. **34**, 957 (1966); see also Sec. 3.1 of Ref. 7.

Lorentz invariance of the localization denies (a′) the formal covariance,[12] (b′) the Hermiticity of the position operator, and (c′) the compatibility of the components of position for spin $\frac{1}{2}$.

Let **X** be the *position three-vector* and $\varphi_{ab\nu}$ be a *one-localized state*,

$$X^3\varphi_{ab\nu}=(a^3+ib^3)\varphi_{ab\nu}\,,\quad a\equiv a^3\,,\quad b\equiv b^3\,,\qquad(1.1)$$

where a and b are real numbers and ν stands for the remaining degeneracy; in order to treat a more general case, we assume complex eigenvalues.[3,7–9,13] As in II, all one-localized states can be obtained from $\varphi_{ab\nu}$ by the induced action of the Poincaré group, so it is sufficient to find $\varphi_{0b\nu}$. Let Λ be a projection operator into the subspace of the allowed wave functions φ,

$$\Lambda^2=\Lambda\,,\quad \Lambda\varphi=\varphi\,.\qquad(1.2)$$

It was shown in II that if one finds the three-vector operators **X**, **R**, the functions $\varphi_{0b\nu}(\mathbf{p})$, and $b_\lambda{}^3$ related by

$$X^k=\Lambda(i\partial_k+R^k)\Lambda\qquad(1.3)$$

and

$$e^{i\lambda M_{03}}\Lambda(L^{03}-p_0R^3+ip_0b_\lambda{}^3)\Lambda e^{-i\lambda M_{03}}\varphi_{0b\nu}=0\,,\qquad(1.4\text{a})$$

then the Lorentz-invariant localization problem is solved. Here **R** is a three-vector which only depends on **p** and on the matrices of the theory; λ is a real parameter; $b_\lambda{}^3$ is a function of λ such that

$$b_0{}^3=b\,.\qquad(1.4\text{b})$$

The operator

$$L^{03}=-ip^0\partial_3\qquad(1.5\text{a})$$

is a component of the orbital angular momentum tensor $L^{\mu\nu}$, and

$$M^{\mu\nu}=L^{\mu\nu}+S^{\mu\nu}\qquad(1.5\text{b})$$

is the total angular momentum tensor.

B. Plan

In Sec. II the notations, the conventions, and some auxiliary formulas are stated. In Secs. III, IV, and V the Lorentz-invariant localization problems for the scalar (and pseudoscalar) case, the neutrinos, and the photon, respectively, are solved. The results are discussed in Sec. VI.

C. Remark on Procedure Used

We find an essential identity between the results for the previous paper (mass $m>0$) and the present one ($m=0$). This arouses the suspicion that perhaps the way in which the consequences of Eqs. (1.3) and (1.4) were found in II for $m>0$ can also be used for $m=0$. However, this is not so for the following reasons (among others).

(1) In II, for the case of spin $s=\frac{1}{2}$, m was used several times as a divisor.

(2) In II, for the cases $s=0$ and $s=\frac{1}{2}$, the calculations were simplified several times by propositions of the type

$$(p_1{}^2+p_2{}^2+m^2)A(\mathbf{p})\varphi_{0b\nu}(\mathbf{p})=0\ (\text{for all }\mathbf{p})\Rightarrow A(\mathbf{p})=0\ (\text{for all }\mathbf{p})\,,$$

where A is an ordinary function. (Here, ordinary means a function $R_3\rightarrow C$ rather than an operator-valued function.)

This is not so for $m=0$, because in this case we can have, in principle,

$$A(\mathbf{p})\neq0\,,\quad \varphi_{0b\nu}(\mathbf{p})=B\delta((p_1{}^2+p_2{}^2)^{1/2})\,,$$

where δ is Dirac's delta function.

D. Note on Time Component of Position

We shall complement here a study done in II on the time component of position. Let us first notice that in II (as well as in the present paper), the only departure from standard quantum-mechanical procedures is the fact that Hermitian operators were neither imposed nor rejected. In particular, time is a c number, which is the usual assumption in quantum mechanics (relativistic case included).[14] It follows that time is compatible with any observable, so that the value of a component of the position vector can be known at any given time. This means that we assume instantaneous localization, and this hypothesis should be stated explicitly.

Let us now discuss two possibilities of noninstantaneous localization.

(1) The first is the use of general spacelike hypersurfaces on which the data that specify a state are given. This generalization is not a departure from standard quantum mechanics and gives a general frame in which the localization problem can be discussed. However, in this paper (as in II) we are only concerned with flat spacelike hyperplanes, because our basic purpose is to deduce the form of the position operator. We prove (cf. II and Secs. III–V of the present paper) that in order to accomplish this it is sufficient to consider states localized in such hyperplanes. Moreover, here (as in II) we restrict ourselves to the particular

[12] This means that the only position operator X^k which is consistent with Lorentz invariance of localization is not the space part of a four-vector operator. However, it was proved in II that an associated four-vector can exist (like Bunge's) which plays a role in a manifestly covariant description of localization. See M. Bunge, Nuovo Cimento **1**, 977 (1955); M. Bunge and A. J. Kálnay, Progr. Theoret. Phys. (Kyoto) **42**, 1445 (1969).

[13] A. Das, J. Math. Phys. **7**, 45 (1966); **7**, 52 (1966); **7**, 61 (1966); E. E. Shin, *ibid.* **7**, 174 (1966).

[14] C. Møller, Comm. Dublin Inst. Advan. Studies, Ser. A, **5**, 1 (1949). See also W. Pauli, in *Handbuch der Physik*, edited by H. Geiger and K. Scheel (Springer, Berlin, 1933), Vol. XXIV: 1, p. 140. We acknowledge Professor W. Hanus for calling our attention to Pauli's discussion of the problem. As reviewed by Hanus (see Ref. 16), "objections against the use of the time operator have been raised many years ago by Pauli who has shown that it is impossible to introduce this operator, at least without some considerable modifications in the basic assumptions underlying the formalism of quantum mechanics."

case of constant-time hyperplanes, again because the study of localization on them is sufficient in order to obtain the position operator. (For a similar reason, in II we only used transformations which leave the region of localization invariant. General transformations will be considered in a future paper; they do not overdetermine the problem.) Once localized states are well defined on constant-time hyperplanes, they are also well defined (through a Lorentz transformation) on arbitrary flat spacelike surfaces.

(2) A second possibility of noninstantaneous localization is to consider time as a q number. This is a departure from standard mechanics.[14] We are not concerned here (nor were we in II) with this possibility. This is not because we prejudge that standard uses in quantum mechanics will remain accepted in the future without change, but because in the present paper (as in II) we explore the results that can be deduced without changing the quantum-mechanical representatives of any of those physical variables that have widely accepted ones (as four-momentum, angular momentum, *and time*). However, the possibility that widely accepted rules are wrong must remain open.

Johnson[15] proposed a very interesting reformulation of quantum mechanics in which proper time (but not ordinary time) is a c number. Several consequences of Johnson's theory seem more natural than those of the standard form of quantum mechanics. His work offers an interesting possibility.[15a]

Let us conclude by citing (without pretending to completeness) another nice reformulation, constructed by Hanus,[16] who obtains a manifestly covariant Hamiltonian formalism of relativistic quantum mechanics such that the timelike position component remains a c number without altering the operator character of the spatial components.

II. CONVENTIONS, NOTATIONS, AND AUXILIARY FORMULAS

A. Conventions and Notations

We use $\hbar=c=1$ and the same conventions and notations for vectors, metric in space-time, and tensor indices as in II. The sum convention will only be used for tensor and spinor indices. The dimension of a magnitude A will be indicated by $[A]$.

We shall only work in the p representation and in the Heisenberg picture of the one-particle states. The particular formalisms used for a fixed spin will be shown in Secs. III–V. In all cases the scalar product will be (as in II) the invariant one, which for spin s takes the form[17]

$$(\psi,\varphi)=\int p_0^{-2s-1}\psi^\dagger(\mathbf{p})\varphi(\mathbf{p})d^3p\,, \tag{2.1}$$

where

$$p_0=+(p^kp^k)^{1/2}\,. \tag{2.2}$$

As in II, we introduce the variables $K\equiv p_H$, θ, and ϕ such that[18]

$$K\equiv p_H=+[(p^1)^2+(p^2)^2]^{1/2}\,,\quad \tanh\theta=p^3/p_0\,, \tag{2.3a}$$

$$p_0=K\cosh\theta\,,\qquad p^3=K\sinh\theta\,, \tag{2.3b}$$

$$p^1\pm ip^2=p_He^{\pm i\phi}\,, \tag{2.3c}$$

and a function $\omega_y(x)$ which depends apparently on two variables,

$$\omega_y(x)=b_{x+y}{}^3\,. \tag{2.4a}$$

The decomposition

$$\begin{aligned}\omega_y(x)&=\omega_y{}^+(x)+\omega_y{}^-(x)\,,\\ \omega_y{}^\pm(x)&=\tfrac{1}{2}[\omega_y(x)\pm\omega_y(-x)]\end{aligned} \tag{2.4b}$$

is such that

$$b_\lambda{}^3=\omega_\theta{}^+(\lambda-\theta)+\omega_\theta{}^-(\lambda-\theta) \tag{2.4c}$$

and

$$\omega_y{}^\pm(x)=\pm\omega_y{}^\pm(-x)\,. \tag{2.4d}$$

B. Properties Related to Generators of Poincaré Group

The well-known commutation relations of p^μ and $M^{\mu\nu}$ imply that if λ is a parameter, then

$$\begin{aligned}e^{i\lambda M_{03}}p^0e^{-i\lambda M_{03}}&=K\cosh(\lambda-\theta)\,,\\ e^{i\lambda M_{03}}p^1e^{-i\lambda M_{03}}&=p^1\,,\\ e^{i\lambda M_{03}}p^2e^{-i\lambda M_{03}}&=p^2\,,\\ e^{i\lambda M_{03}}p^3e^{-i\lambda M_{03}}&=-K\sinh(\lambda-\theta)\,,\end{aligned} \tag{2.5a}$$

and

$$e^{i\lambda M_{03}}T^{03}e^{-i\lambda M_{03}}=T^{03}\,, \tag{2.5b}$$

where $T^{\mu\nu}$ is an arbitrary second-rank antisymmetric four-tensor operator.

We shall also need

$$[\Lambda,p^\mu]=0\,,\quad [\Lambda,M^{\mu\nu}]\Lambda=0\,, \tag{2.6}$$

which express the relativistic invariance of the formalism.

C. Lemma

Let us call $\Lambda\mathbf{V}_{(1)}\Lambda$, ..., $\Lambda\mathbf{V}_{(n)}\Lambda$ a maximal set of linearly independent three-vectors which are functions of $\mathbf{p}$ and of the matrices of the theory.[19]

---

[15] J. E. Johnson, Phys. Rev. **181**, 1755 (1969). See also the references quoted therein.

[15a] *Note added in proof.* See also, A. A. Broyles, Phys. Rev. D **1** 979 (1970); University of Florida report (unpublished); D. M. Rosenbaum, J. Math. Phys. **10**, 1127 (1969).

[16] W. Hanus, Instytut Fisyki Uniwersytetu Mikołaja Kopernika, Toruń, Poland, Report No. 109, 1970 (unpublished).

[17] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **34**, 211 (1948).

[18] Two symbols K and p_H are introduced to represent the same entity in order to make easier the comparison of the formulas for $m=0$ with the corresponding ones for $m\neq0$, because in the latter case $K\neq p_H$ (cf. II).

[19] All physical operators are of the form $\Lambda\Omega\Lambda$ (cf. II).

(1) Then the vector **R** [cf. Eq. (1.3)] is of the form

$$\mathbf{R}=\Lambda[\sum_{i=1}^{n} g_{(i)}(p_0)\mathbf{V}_{(i)}]\Lambda, \tag{2.7a}$$

where the $g_{(i)}$ are ordinary functions.

(2) Let us denote by α_i the numbers which satisfy

$$[g_{(i)}(p_0)]=[p_0]^{-\alpha_i}. \tag{2.7b}$$

Then a set of constants C_i exists such that

$$[C_i]=1 \quad \text{for all } i \tag{2.7c}$$

and

$$g_{(i)}(p_0)=C_i p_0^{-\alpha_i} \quad \text{for all } i. \tag{2.7d}$$

Proof. Part 1 is obvious. The proof of part 2 follows from the fact that $g_{(i)}$ is to be constructed from the ingredients of the theory. Since $m=0$ and **X** cannot depend on dimensional constants (like a^3) which are state dependent, a constant M, such that $[M]=[p_0]$, and which could be used to construct **R**, cannot be introduced. □

III. SCALAR AND PSEUDOSCALAR ELEMENTARY SYSTEMS

For the scalar and pseudoscalar elementary systems the wave function $\varphi(\mathbf{p})$ has one component; the projector Λ is the identity. Lemma (2.7) implies

$$\mathbf{R}=Cp_0^{-2}\mathbf{p}, \quad [C]=1. \tag{3.1}$$

By using Eqs. (1.4a), (2.4c), and (2.5) we derive

$$[L^{03}+C\tanh(\lambda-\theta)+iK\cosh(\lambda-\theta)\,\omega_\theta^+(\lambda-\theta)+iK\cosh(\lambda-\theta)\,\omega_\theta^-(\lambda-\theta)]\varphi_{0b\nu}=0, \tag{3.2}$$

which, taking into account Eq. (2.4d), implies

$$[L^{03}+iK\cosh(\lambda-\theta)\,\omega_\theta^+(\lambda-\theta)]\varphi_{0b\nu}=0 \tag{3.3}$$

and

$$[C\tanh(\lambda-\theta)+iK\cosh(\lambda-\theta)\,\omega_\theta^-(\lambda-\theta)]\varphi_{0b\nu}=0. \tag{3.4}$$

From Eq. (3.3) it follows that

$$[L^{03}+iK\omega_\theta^+(0)]\varphi_{0b\nu}=0 \tag{3.5}$$

and

$$K[\cosh(\lambda-\theta)\omega_\theta^+(\lambda-\theta)-\omega_\theta^+(0)]\varphi_{0b\nu}=0, \tag{3.6}$$

so that

$$\omega_\theta^+(\lambda-\theta)=\omega_\theta^+(0)/\cosh(\lambda-\theta). \tag{3.7}$$

Proof of Eq. (3.7). Assume the contrary; then (for fixed ω^+), $\varphi_{0b\nu}$ can only be of the form $\varphi_{0b\nu}=B\delta(K)$. Then from Eqs. (3.4), (3.1), and (1.3) it follows that $X^3=i\partial_3$, so there are solutions of Eq. (1.1) (with $a^3=0$) different from $\varphi_{0b\nu}=B\delta(K)$. □

Equation (3.4) implies

$$C=\omega_\theta^-(\lambda-\theta)=0, \tag{3.8}$$

substitution of which, with Eq. (3.7), into Eq. (2.4c) shows that

$$b_\lambda{}^3=\omega_\theta^+(0)/\cosh(\lambda-\theta). \tag{3.9}$$

Since λ and θ are independent variables, one can show that

$$\omega_\theta^+(\lambda-\theta)=b_\lambda{}^3=b=0. \tag{3.10}$$

Equations (3.1), (3.8), and (1.3) imply

$$X^k=i\partial_k. \tag{3.11}$$

From now on, the reasoning follows that in Sec. V of II. The conclusions are the same, and thus we do not state them here.

IV. NEUTRINOS

A. Formalism Used

In the formalism we use, the state vectors are solutions of Weyl's equation,

$$p_0\varphi=\varpi\boldsymbol{\sigma}\cdot\mathbf{p}\varphi, \quad \varpi=\pm1 \tag{4.1}$$

where the helicity ϖ is *fixed*.[20] The projection operator is

$$\Lambda=\tfrac{1}{2}(I+\varpi p_0^{-1}\boldsymbol{\sigma}\cdot\mathbf{p}). \tag{4.2}$$

The spin tensor $S^{\mu\nu}$ is such that

$$\tfrac{1}{2}\sigma^i=S^i=\tfrac{1}{2}\epsilon_{ijk}S^{jk}, \quad S^{0k}=i\varpi S^k. \tag{4.3}$$

We shall work in the standard representation of Pauli matrices. Then Λ is of the form

$$\Lambda=\frac{1}{2}\begin{pmatrix}1+\varpi p^3(p_0)^{-1} & \varpi p_H(p_0)^{-1}e^{-i\phi}\\ \varpi p_H(p_0)^{-1}e^{i\phi} & 1-\varpi p^3(p_0)^{-1}\end{pmatrix}. \tag{4.4}$$

B. Position Operator and Localized States

Equation (4.1) is not covariant under spatial reflections, so it is not a necessary requirement that **X** be a pure vector; it can also have an axial-vector term. From the Lemma (2.7) it follows then that

$$\Lambda\mathbf{R}\Lambda=\Lambda Cp_0^{-2}\mathbf{p}I\Lambda \tag{4.5}$$

because

$$\Lambda\boldsymbol{\sigma}\times\mathbf{p}\Lambda=0 \quad \text{and} \quad \Lambda\boldsymbol{\sigma}\Lambda=\varpi p_0^{-1}\mathbf{p}\Lambda.$$

In the same way as for $s=0$, but taking into account Eq. (2.6), we derive

$$\Lambda[L^{03}+C\tanh(\lambda-\theta)+iK\cosh(\lambda-\theta)\,\omega_\theta^+(\lambda-\theta)+iK\cosh(\lambda-\theta)\,\omega_\theta^-(\lambda-\theta)]\varphi_{0b\nu}=0, \tag{4.6}$$

$$\Lambda[L^{03}+iK\cosh(\lambda-\theta)\,\omega_\theta^+(\lambda-\theta)]\varphi_{0b\nu}=0, \tag{4.7}$$

$$\Lambda[C\tanh(\lambda-\theta)+iK\cosh(\lambda-\theta)\,\omega_\theta^-(\lambda-\theta)]\varphi_{0b\nu}=0, \tag{4.8}$$

$$\Lambda[L^{03}+iK\omega_\theta^+(0)]\varphi_{0b\nu}=0, \tag{4.9}$$

and

$$K\Lambda[\cosh(\lambda-\theta)\omega_\theta^+(\lambda-\theta)-\omega_\theta^+(0)]\varphi_{0b\nu}=0. \tag{4.10}$$

[20] Equations (1.3) and (1.4) were derived in the frame of the Bargmann-Wigner formalisn (cf. II), but they do not change their form when we translate them to this alternative formalism.

The solution $\varphi_{0b_\nu}=B\delta(K)$ can be excluded by reasoning similar to that used in Sec. III. Then,

$$\omega_\theta^+(\lambda-\theta)=\omega_\theta^+(0)/\cosh(\lambda-\theta)\,. \quad (4.11)$$

Again using a procedure like that in Sec. III, we obtain

$$C=\omega_\theta^+(\lambda-\theta)=b_\lambda^3=b=0\,. \quad (4.12)$$

Then Eq. (4.9) is the only one which remains to be satisfied. In order to do so, we first introduce a vector Ξ^k which will play the same role that the $0k$ component of Hilgevoord and Wouthuysen's spin tensor $\Sigma^{\mu\nu}$ did in II[21,22]:

$$\Xi^k=i\varpi(\sigma^k+[\Lambda,\sigma^k])\,. \quad (4.13)$$

It is such that

$$\Xi^3=i\varpi\sigma^3+p_0^{-1}(p^1\sigma^2-p^2\sigma^1) \quad (4.14)$$

(with cyclic permutations of 1, 2, 3),

$$(\Xi^3)^2=-(p_0)^{-2}(p^3)^2 \quad (4.15)$$

(notice the difference from the case $m>0$), and

$$[\Xi^k,\Lambda]\Lambda=0\,. \quad (4.16)$$

Equation (4.9) can be written as

$$M^{03}\varphi_{00\nu}=\tfrac{1}{2}\varpi\Xi^3\varphi_{00\nu}\,. \quad (4.17)$$

We look for the explicit form of the solutions $\varphi_{00\nu}$ of this equation. It is useful to write first the solutions φ of

$$\varphi=\Lambda\varphi\,. \quad (4.18a)$$

One can check that they are

$$\varphi_i(\mathbf{p})=D(\mathbf{p})w_i^{\varpi}(\mathbf{p})\,, \quad (4.18b)$$

where D is an arbitrary function and[23]

$$w^{\varpi}(\mathbf{p})=\begin{pmatrix}(\varpi p^0+p^3)e^{-i\phi/2}\\ p_H e^{i\phi/2}\end{pmatrix}. \quad (4.18c)$$

Incidentally, Eq. (4.14) shows that φ_i has only one independent component, which is right because the helicity ϖ is fixed (cf., e.g., Ref. 24).

The one-localized states $\varphi_{00\nu}$ must be of the form (4.18). Using this fact and the explicit representations of the operators involved in Eq. (4.17), a straightforward calculation shows

$$\varphi_{00\nu,i}(\mathbf{p})=f_\nu(p_H,\phi)(p^0+\varpi p^3)^{-1/2}(p_0)^{-1/2}w_i^{\varpi}(\mathbf{p})\,, \quad (4.19)$$

where f_ν is an arbitrary function.

[21] The expressions will be written in a form which makes easier the comparison with the corresponding expressions for the electron (cf. II).

[22] J. Hilgevoord and S. A. Wouthuysen, Nucl. Phys. **40**, 1 (1963).

[23] This wave function can be expressed in several forms which are only apparently different.

[24] D. Korff, J. Math. Phys. **5**, 869 (1964).

As in II, here we have

$$\varphi_{a^30\nu}=\varphi_{00\nu}e^{-ia^3p^3} \quad (4.20)$$

and the partial eigendifferential

$$\varphi_{a^30\nu}{}^{\delta a^3}=2C^{\delta a^3}(p^3)^{-1}[\sin(\tfrac{1}{2}p^3\delta a^3)]\varphi_{a^30\nu}\,, \quad (4.21)$$

where $C^{\delta a^3}$ is a normalization constant.

The one-localized states should be non-normalizable to unity, but one should have

$$\|\varphi_{a^30\nu}{}^{\delta a_3}\|=1\,, \quad (4.22)$$

because the spectrum of position is continuous. This indeed happens provided that the f_ν satisfy normalization requirements.

From Eqs. (1.3), (4.5), and (4.13) it follows that

$$X^k=(p_0)^{-1}(\tfrac{1}{2}\Xi^k-M^{0k})\Lambda\,. \quad (4.23)$$

The properties of the position operator and the localized states are the same as those of the electron (cf. II) but the parameter G takes the value zero.

(1) The position operator is non-Hermitian and non-normal and can be considered as a limiting case of the extended-type position. This does not exclude its having a physical meaning. (Cf. Refs. 7–11.)

(2) The velocity is Hermitian and is as expected:

$$(d\mathbf{X}/dt)\varphi=(\mathbf{p}/p_0)\varphi \quad \text{for all } \varphi=\Lambda\varphi\,. \quad (4.24)$$

(3) There are no three-localized states: No φ exists such that

$$X^k\varphi=0\,,\quad k=1,2,3.$$

The proof of this statement can be given in the same way as the corresponding proof in II. To do so, one notes that because of Eqs. (4.3) and (4.13), Ξ^k is the $0k$ part of a four-tensor.

Notice, however, that in contrast with the $m\neq 0$ case, there is no possibility for $\mathbf{X}$ to be one of the possible quantum versions of Pryce's[25] definition (3) of the classical center of mass (cf. II).

V. PHOTON

A. Formalism Used

Let $\mathbf{e}(\mathbf{p})$ and $\mathbf{h}(\mathbf{p})$ be the electric and the magnetic fields, respectively, in p representation. We use the formalism[26] in which the wave function is

$$\boldsymbol{\varphi}(\mathbf{p})=2^{-1/2}[\mathbf{h}(\mathbf{p})-i\varpi\mathbf{e}(\mathbf{p})]\,,\quad \varpi=\pm 1 \quad (5.1)$$

where ϖ is the helicity; the wave equation is

$$-i\varpi p_0\boldsymbol{\varphi}=\mathbf{p}\times\boldsymbol{\varphi}\,, \quad (5.2)$$

so that

$$\mathbf{p}\cdot\boldsymbol{\varphi}=0\,. \quad (5.3)$$

[25] M. H. L. Pryce, Proc. Roy. Soc. (London) **195A**, 62 (1948).

[26] R. E. Marshak and E. C. G. Sudarshan, *Introduction to Elementary Particle Physics* (Interscience, New York, 1961).

The scalar product[17,27] equals[28]

$$(\psi,\varphi)=2\int(p_0)^{-3}\psi^{\mathrm{T}}(\mathbf{p})\cdot\varphi(\mathbf{p})d^3p. \tag{5.4}$$

The projector is

$$\Lambda_{ij}(\mathbf{p})=\tfrac{1}{2}[\delta_{ij}-p^ip^j(p_0)^{-2}-i\varpi e_{ijk}p^k(p_0)^{-1}]. \tag{5.5}$$

We write the well-known spin matrices

$$(S^i)_{ij}=-ie_{ijk}, \tag{5.6a}$$

$$S^{rs}=e_{rsi}S^i,\quad S^{0k}=+i\varpi S^k, \tag{5.6b}$$

so that

$$(\mathbf{S}\cdot\mathbf{p})\varphi=\varpi p_0\varphi, \tag{5.7}$$

as it should be.

B. Position Operator and Localized States

Maxwell equations are covariant under reflections, so the argument we used in Sec. III in order to show that the neutrino's position can have an axial-vector term does not apply. However, it was proved by Korff[24] that photon states with different helicities are incoherent, so that *in Eq. (5.1) we must choose and fix the value of* ϖ. This allows **X** to have an axial-vector term again. Then it follows from the Lemma (2.7) that[21]

$$\Lambda\mathbf{R}\Lambda=\Lambda Cp_0^{-2}\mathbf{p}I\Lambda, \tag{5.8}$$

which has the same form as Eq. (4.5). The calculations are exactly the same as those done for the neutrino if we replace $\boldsymbol{\sigma}$ by 2**S**, so we give only the results:

$$C=b_\lambda{}^3=b=0, \tag{5.9}$$

$$\varphi_{00\nu}(\mathbf{p})=f_\nu(p_H,\phi)\mathbf{w}^{\varpi}(\mathbf{p}), \tag{5.10}$$

where f_ν is a function which is arbitrary up to normalization requirements,

$$\mathbf{w}^{\varpi}(\mathbf{p})=p_H^{-1}\begin{pmatrix}-p^2-(i\varpi p^1p^3/p^0)\\ p^1-(i\varpi p^2p^3/p^0)\\ i\varpi(p_H)^2/p_0\end{pmatrix}. \tag{5.11}$$

The normalization properties are as for the neutrino. The position operator is

$$X^k=(p_0)^{-1}(\tfrac{1}{2}\Xi^k-M^{0k})\Lambda, \tag{5.12}$$

where[21]

$$\Xi^k=2\varpi i(S^k+[\Lambda,S^k]) \tag{5.13}$$

has the representation

$$\Xi^3=\varpi\begin{pmatrix}+2q_1q_2 & 2-(q_1)^2+(q_2)^2 & q_2q_3+i\varpi q_1\\ -2-(q_1)^2+(q_2)^2 & -2q_1q_2 & -q_1q_3+i\varpi q_2\\ q_2q_3-i\varpi q_1 & -q_1q_3-i\varpi q_2 & 0\end{pmatrix}. \tag{5.14a}$$

The other components can be obtained from this by cyclic permutations of 1, 2, 3; and

$$q_k=p^k/p^0. \tag{5.14b}$$

The properties of the localized states and the position operator are the same as those of the neutrino.

VI. DISCUSSION

For discussions of the zero-mass case see Refs. 24, 25, and 29–31. Let us assume instantaneous localization and quantum mechanics (see Sec. I D).

Using as the only basic assumption the consistency of the description of localization from different inertial frames, we have succeeded in showing that a solution of the localization problem for $m=0$ *exists, and that its properties are essentially the same as those obtained in II for* $m>0$.

(1) There is no Hermitian position operator. This implies, as in II, that it is almost[3,32] certain that one of the following statements is right:

(a) Position has no meaning at the quantum relativistic level. (But then, how does the macroscopic localizability arise?)

(b) The description of localization in different inertial frames is inconsistent.

(c) Position is the only observable which cannot be represented by an operator.

(d) The position operator exists, but it is non-Hermitian.

(2) One should try to look for the consequences of some of these surprising possibilities. It is possible that the hypothesis (a) is the right one, but then it is difficult to understand the standard evidence of approximate localization as well as the macroscopic and the quantum nonrelativistic localizability (cf., e.g., Ref. 31). Possibility (b) may also be the right one, but this would imply a departure from relativity theory because a relativistic self-consistency requirement cannot be violated in a relativistic theory. We consider cases (c) and (d) to be of interest.

It was suggested by Werle[33] that if case (c) is the right one then an equal footing of space and time which

[27] L. Gross, J. Math. Phys. 5, 687 (1964).

[28] The electromagnetic field and the potential can both be used as state vectors of the photon, but the form of the scalar product changes when the fields are replaced by the potentials (cf. Ref. 27).

[29] R. Acharya and E. C. G. Sudarshan, J. Math. Phys. 1, 532 (1960).

[30] C. Fronsdal, Phys. Rev. 113, 1367 (1959).

[31] G. Fleming, Phys. Rev. 139, B963 (1965).

[32] The word "almost" allows for the possibility of nonzero effects of the interaction in the limit in which the interaction is switched off. This possibility exists but, if it were realized, position would again be an exceptional member of the set of physical observables (cf. Sec. VII of II).

[33] J. Werle (private communication).

quantum mechanics destroyed would be reestablished, because in (relativistic or nonrelativistic) quantum mechanics the time is a c number, while the three-dimensional space coordinates are q numbers (cf. Ref. 14). The first step in a study of case (c) should be to explain Heisenberg's uncertainty relation without the position operator.

We have considered in the present paper possibility (d) for $m=0$, as we did in II for $m>0$. The legitimacy of a non-Hermitian position operator is discussed in several of the references. The strongest indication that (d) may be right seems to be that, by the use of a relativistic self-consistency requirement only, we can deduce in case (d) a position operator that is unique.

With respect to case (d) for $m=0$, we make the following remarks.

(i) We had no need of Axiom 6 of II.

(ii) The eigenvalues of X^k are real in spite of the non-Hermiticity of the operator (cf. Secs. III–V of this paper and Refs. 22 and 24 of II).

(iii) The components of position are compatible with each other for spin 0 and incompatible for spin $\frac{1}{2}$ and 1.

(iv) The commutation relations of position and the linear momentum are the expected ones (cf. II).

(v) The velocity operator is Hermitian and has the expected form.

ACKNOWLEDGMENTS

The author is grateful to Professor Abdus Salam and Professor P. Budini for hospitality at the International Centre for Theoretical Physics, Trieste, and to the International Atomic Energy Agency for an associateship at the Centre. The author is indebted to Professor Mario Bunge for an enlightening remark and to Miss K. Kademova for useful discussions and for a very careful critical revision of this paper.

PHYSICAL REVIEW D VOLUME 3, NUMBER 10 15 MAY 1971

Renormalizability of Massive Spin-One Theory with Pseudovector Interaction

Q. Shafi
Physics Department, Imperial College, London SW7, England
(Received 11 September 1970)

The conventionally nonrenormalizable theory of massive neutral pseudovector mesons interacting with nucleons is treated in the Stückelberg formalism using some recently developed nonlinear techniques. The theory is split into two parts: a renormalizable interaction of mesons with nucleons, which can be treated by standard methods, and an exponential interaction of pseudoscalar mesons with nucleons, which is dealt with using nonlinear methods. It is shown that all the infinities of the theory may be eliminated by adding suitable counterterms. Thus the complete theory becomes renormalizable. Similar considerations also apply if a parity-violating term is introduced in the original pseudovector interaction.

I. INTRODUCTION

In this paper we consider the renormalizability of massive neutral pseudovector-meson theory coupled to a *nonconserved* nucleon current. It is well known that such a theory is not renormalizable in the conventional sense, since a formal power-series expansion of any transition amplitude in terms of the coupling constant leads to an expression in which every higher-order term becomes increasingly more singular. The apparent unrenormalizable character of the theory stems from the fact that the meson propagator is $(g_{\mu\nu}-k_\mu k_\nu/\mu^2)\times(k^2-\mu^2)^{-1}$, rather than, say, $(g_{\mu\nu}-k_\mu k_\nu/k^2)(k^2-\mu^2)^{-1}$ as for the photon propagator in quantum electrodynamics (QED), which theory is known to be renormalizable. The degree of divergence for such a theory using the well-known power-counting arguments is given by

$$D=4-\tfrac{3}{2}E_N-2E_A+N\,,$$

where E_N is the number of external nucleon lines, E_A is the number of external pseudovector-meson lines, and N is the order of perturbation expansion. From this expression two interesting points emerge: (a) The degree of divergence associated with any given primitively divergent graph increases with the order of perturbation expansion in the coupling constant, and (b) all graphs are primitively divergent for sufficiently high orders of perturbation theory.

However, such power-counting arguments could be misleading. For instance, the above expression for the degree of divergence applies also to the theory of a massive vector meson coupled to a *conserved* nucleon current. Thus, on power-counting arguments alone, one would expect this theory to be unrenormalizable when a perturbation expansion in the coupling constant is carried out. But it is well known that for such a theory the $k_\mu k_\nu/\mu^2$ term in the meson propagator may be treated in a nonperturbative manner and only contributes to wave-function renormalizations. The rest of the theory is then renormalizable and may be treated