Quasipotential Equation Corresponding to the Relativistic Eikonal Approximation

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A three-dimensional Lippmann-Schwinger-type equation for the elastic scattering amplitude and the corresponding homogeneous Schrödinger equation for the two-particle bound states are studied. The potential is defined as an infinite power series in the coupling constant which fits the perturbative expansion of the on-energy-shell scattering amplitude. The approximate equation obtained by keeping only the lowest-order term in the potential is local and has the following properties: (i) The scattering amplitude yields the relativistic eikonal approximation for large energies or small exchanged mass and momentum transfer; (ii) for the Coulomb problem the approximate equation is exactly soluble and leads to a relativistic Balmer formula including the fine-structure splitting.

I. INTRODUCTION

The possibility of fitting the exact relativistic scattering amplitude by an energy-dependent potential has been recognized for many years.[1] A quasipotential approach to the relativistic two-body problem was developed in the work of Logunov and Tavkhelidze.[2] It can be related to the old-fashioned off-energy-shell perturbation theory in the same way as the Bethe-Salpeter equation is related to the off-mass-shell Feynman rules.[3,4]

The choice of a quasipotential equation which fits the on-shell scattering amplitude and satisfies some general requirements listed in Sec. II is not unique.[5] (There is freedom both in the choice of the two-particle Green's function and in the off-energy-shell extrapolation of the scattering amplitude.[6]) This nonuniqueness is exploited in Sec. II to write down a new quasipotential equation which appears to be simpler than those studied previously. As usual, for a given perturbation expansion of the scattering amplitude the "potential" V is also defined as an infinite power series in the coupling constant g. Section III is devoted to considering the quasipotential equation with V replaced by its Born approximation. We show that the solution of this approximate equation yields the relativistic eikonal approximation[7,8] in the domain of its validity. In Secs. III and IV we also obtain relativistic extensions of the Balmer formula in two different cases: (a) massive scalar particles interacting via a massless scalar boson; (b) electromagnetic interaction of two scalar charged particles.

The relativistic Balmer formula so obtained includes all recoil effects up to order α^4 as well as the fine-structure splitting which appears in case (b). It does not include radiative corrections (Lamb shift) which should be expected to come from the next term in the expansion of the potential. In contrast with the treatment of the two-body problem in Refs. 9 and 10, our equation is symmetric with respect to the two particles.[11]

II. RELATIVISTIC LIPPMANN-SCHWINGER AND SCHRÖDINGER EQUATIONS

Consider the off-energy-shell elastic scattering amplitude T_w for two particles of masses m_1 and m_2 and initial (final) momenta $\mathbf{q}_1$ and $\mathbf{q}_2$ ($\mathbf{p}_1$ and $\mathbf{p}_2$). We work in the c.m. frame in which

$$\mathbf{p}_1=-\mathbf{p}_2=\mathbf{p},\quad \mathbf{q}_1=-\mathbf{q}_2=\mathbf{q}. \tag{2.1}$$

The masses of the two-particle bound states are defined by the eigenvalues of the total energy w. On the energy-

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[1] See, e.g., A. Martin and Gy. Targonski, Nuovo Cimento **20**, 1182 (1961), and references therein.

[2] A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963); A. A. Logunov *et al.*, *ibid.* **30**, 134 (1963); R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966); V. A. Matveev, R. M. Muradyan, and A. N. Tavkhelidze, Joint Institute for Nuclear Research, Dubna, USSR, Report No. E2-3498, 1967 (unpublished); P. N. Bogoliubov, Theoret. Math. Phys. (Moscow) **5**, 244 (1970). For various applications of the quasipotential equation see, e.g., R. N. Faustov, Nucl. Phys. **75**, 669 (1966); H. Grotch and D. R. Yennie, Rev. Mod. Phys. **41**, 350 (1969). A similar technique was developed in R. Fong and J. Sucher, J. Math. Phys. **5**, 456 (1964); Nguyen D. Son and J. Sucher, Phys. Rev. **153**, 1496 (1967).

[3] V. G. Kadyshevsky, Nucl. Phys. **B6**, 125 (1968); V. G. Kadyshevsky and M. D. Mateev, Nuovo Cimento **55A**, 275 (1968).

[4] C. Itzykson, V. G. Kadyshevsky, and I. T. Todorov, Phys. Rev. D **1**, 2823 (1970); I. T. Todorov, in *Lectures in Theoretcial Physics*, edited by W. Brittin and K. Mahanthappa (Gordon and Breach, New York, to be published), Vol. XII.

[5] For instance, the equations studied in Refs. 2–4 are not identical. Another nonequivalent equation was explored by C. Itzykson and I. T. Todorov, in *Proceedings of the First Coral Gables Conference on Fundamental Interactions at High Energy, University of Miami, 1969*, edited by T. Gudehus *et al.* (Gordon and Breach, New York, 1969), and I. T. Todorov, in *Proceedings of the Battelle-Seattle Rencontres in Mathematics and Physics, 1969* (Springer, Berlin, 1970) pp. 254–278.

[6] We are using the noncovariant normalization in which the amplitude contains as factors the external-particle wave functions. In this normalization T is related to the S matrix by

$$\langle p_1p_2|S|q_1q_2\rangle=\delta(\mathbf{p}_1-\mathbf{q}_1)\delta(\mathbf{p}_2-\mathbf{q}_2)+2\pi i T_w(\mathbf{p},\mathbf{q})\delta(p_1+p_2-q_1-q_2).$$

[7] H. D. I. Abarbanel and C. Itzykson, Phys. Rev. Letters **23**, 53 (1969); M. Lévy and J. Sucher, Phys. Rev. **186**, 1656 (1969); B. M. Barbashov *et al.*, Theoret. Math. Phys. (Moscow) **3**, 342 (1970); *ibid.* **5**, 330 (1970); G. Tiktopoulos and S. B. Treiman, Phys. Rev. D **2**, 805 (1970).

[8] E. Brezin, C. Itzykson, and J. Zinn-Justin, Phys. Rev. D **1**, 2349 (1970).

[9] A. O. Barut and A. Baiquni, Phys. Rev. **184**, 1342 (1969); Phys. Letters **30A**, 352 (1969).

[10] C. Fronsdal and L.-E. Lundberg, Phys. Rev. D **1**, 3247 (1970).

shell we would have

$$p_1^0+p_2^0=w=q_1^0+q_2^0 \quad \text{or} \quad \mathbf{p}^2=\mathbf{q}^2=b^2(w), \tag{2.2}$$

where

$$4w^2b^2(w)=\Delta(w^2,m_1^2,m_2^2) =w^4-2(m_1^2+m_2^2)w^2+(m_1^2-m_2^2)^2. \tag{2.3}$$

The quasipotential equation is a linear integral equation for $T_w(\mathbf{p},\mathbf{q})$ of the Lippmann-Schwinger type

$$T+V+VGT\equiv T_w(\mathbf{p},\mathbf{q})+V_w(\mathbf{p},\mathbf{q}) +\int V_w(\mathbf{p},\mathbf{k})G_w(\mathbf{k})T_w(\mathbf{k},\mathbf{q})d^3k=0. \tag{2.4}$$

This equation is assumed to satisfy the following general properties.

(i) For a Hermitian potential ($V=V^*$), Eq. (2.4) implies the on-shell elastic unitarity condition for all energies. Writing (formally) the solution of (2.4) as

$$T=-\frac{1}{1+VG}V=-V\frac{1}{1+GV},$$

we obtain (for $V=V^*$)

$$T-T^*=T^*(G-G^*)T. \tag{2.5}$$

Thus our first requirement fixes the discontinuity of the Green's function G.

(ii) Equation (2.4) is consistent with the perturbation of T in quantum field theory. In particular, if G is independent of the coupling constant g and $T=\sum T_{2n}g^{2n}$, then $V=\sum V_{2n}g^{2n}$, with

$$V_2=-T_2, \quad V_4=-T_4+T_2GT_2, \ldots . \tag{2.6}$$

These basic requirements obviously do not fix Eq. (2.4) uniquely. In particular, they are satisfied by any of the four different equations considered in Refs. 2–5. Hence, we supplement them by the following assumptions.

(iii) For spinless particles $G_w^{-1}(\mathbf{k})$ is a linear function of $\mathbf{k}^2$ (just as in the nonrelativistic Lippmann-Schwinger equation).

Requirements (i) and (iii) fix the Green's function completely. For spinless particles the on-shell elastic unitarity condition is given by[6]

$$T_w(\mathbf{p},\mathbf{q})-T_w^*(\mathbf{p},\mathbf{q})=\frac{4\pi iE_1E_2}{w}\int T_w^*(\mathbf{p},\mathbf{k})T_w(\mathbf{k},\mathbf{q}) \times\delta(\mathbf{k}^2-b^2)d^3k \quad (\mathbf{p}^2=\mathbf{q}^2=b^2) \tag{2.7}$$

and according to (i) and (iii)

$$G_w(\mathbf{k})=2\frac{E_1E_2}{w}(\mathbf{k}^2-b^2-i0)^{-1}. \tag{2.8}$$

(iv) The off-energy-shell extrapolation of the scattering amplitude is obtained from the Feynman perturbation expansion by the substitution

$$p_1^0,q_1^0\to E_1=\frac{1}{2w}(w^2+m_1^2-m_2^2), \qquad p_2^0,q_2^0\to E_2=\frac{1}{2w}(w^2+m_2^2-m_1^2). \tag{2.9}$$

(Note that on the energy shell $p_1^0=q_1^0=E_1$, $p_2^0=q_2^0=E_2$.)

The equation thus obtained,

$$T_w(\mathbf{p},\mathbf{q})+V_w(\mathbf{p},\mathbf{q}) +\frac{2E_1E_2}{w}\int V_w(\mathbf{p},\mathbf{k})(\mathbf{k}^2-b^2-i0)^{-1} \times T_w(\mathbf{k},\mathbf{q})d^3k=0, \tag{2.10}$$

is very close to the original Logunov-Tavkhelidze quasipotential equation.[2]

The simplification in Eq. (2.10) (consisting in replacing a p-dependent factor in front of the integral term in Ref. 2 by w^{-1}) allows us to find an exactly soluble model for the relativistic Coulomb problem.

In order to get an idea of the content of Eq. (2.10), we first compute the second iteration $T^{(2)}$ with the Yukawa potential

$$(2\pi)^3V=\frac{1}{4E_1E_2}\frac{-g^2}{\mu^2+(\mathbf{p}-\mathbf{q})^2}. \tag{2.11}$$

For the on-shell amplitude $\mathbf{T}^{(2)}$ in the forward direction, we obtain

$$\mathbf{T}^{(2)}(\mathbf{p},\mathbf{p})=\frac{g^4}{(4\pi)^4w\mu^2(\mu^2+4p^2)E_1E_2} \times\left[ip+\mu\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{p}{\mu}\right)\right]. \tag{2.12}$$

We see that not only the imaginary part of $\mathbf{T}^{(2)}$ is exact (which is in fact an input) but that $\mathbf{T}^{(2)}$ also coincides with the sum of the contributions of the box and crossed-box Feynman diagrams (cf. Ref. 4) in the limits $p\to\infty$ or $\mu\to 0$.

Following Ref. 4, we can derive formally from Eq. (2.4) the corresponding homogeneous equation for the bound-state wave function ϕ:

$$G_w^{-1}(\mathbf{p})\phi(\mathbf{p})+\int V_w(\mathbf{p},\mathbf{k})\phi(\mathbf{k})d^3k=0. \tag{2.13}$$

In analogy with the nonrelativistic Schrödinger equa-

[11] We are not discussing here another three-dimensional approach to the two-body problem in which one particle is considered off its mass shell; see F. Gross, Phys. Rev. **186**, 1448 (1969).

tion, Eq. (2.13) can also be used (with appropriate boundary conditions) to describe the scattering states. The corresponding inhomogeneous equation which incorporates the boundary condition is

$$\phi_{\mathbf{q}}(\mathbf{p})=\delta(\mathbf{p}-\mathbf{q})-G_q(\mathbf{p})\int V_q(\mathbf{p},\mathbf{k})\phi_{\mathbf{q}}(\mathbf{k})d^3k\,, \quad (2.14)$$

where the subscript q indicates that $w_q=q_1{}^0+q_2{}^0$ $[b(w)=q]$. Setting

$$T_q(\mathbf{p},\mathbf{q})=-\int V_q(\mathbf{p},\mathbf{k})\phi_{\mathbf{q}}(\mathbf{k})d^3k\,, \quad (2.15)$$

we reobtain the Lippmann-Schwinger equation (2.4) for T_w.

The quasipotential equation (2.10) can also be written in a covariant form. To do this we set

$$\begin{aligned}&P=\frac{w}{w_p}(p_1+p_2)=\frac{w}{w_q}(q_1+q_2)\,,\\&2p=p_1-p_2-(m_1{}^2-m_2{}^2)w_p{}^{-2}(p_1+p_2)\,,\\&2q=q_1-q_2-(m_1{}^2-m_2{}^2)w_q{}^{-2}(q_1+q_2)\,,\\&w_p=[(p_1+p_2)^2]^{1/2},\quad w_q=[(q_1+q_2)^2]^{1/2},\end{aligned} \quad (2.16)$$

and introduce the covariant amplitude $\mathfrak{M}$ and "potential" K:

$$\begin{aligned}\mathfrak{M}(P;\,p,q)&=2E_1E_2T_w(\mathbf{p},\mathbf{q})\,,\\K(P;\,p,q)&=2E_1E_2V_w(\mathbf{p},\mathbf{q})\,.\end{aligned} \quad (2.17)$$

In these variables

$$P\cdot p=P\cdot q=0,\quad P^2=w^2,\quad p^2=-\mathbf{p}^2,\quad q^2=-\mathbf{q}^2,$$

and Eq. (2.10) assumes the form

$$\mathfrak{M}(P;p,q)+K(P;p,q)=\int K(P;p,k)\frac{\delta(Pk)}{k^2+b^2+i0}\mathfrak{M}(P;k,q)d^4k \quad (2.18)$$

(cf. Matveev *et al.*[2]). The homogeneous equation (2.13) can also be written in a similar form.

III. RELATION OF QUASIPOTENTIAL EQUATION TO NONRELATIVISTIC SCHRÖDINGER EQUATION AND TO RELATIVISTIC EIKONAL APPROXIMATION

We start with the model of two complex scalar fields ψ_1 and ψ_2 of masses m_1 and m_2 interacting via a neutral scalar field φ of mass μ with the interaction Lagrangian

$$L_1(x)=[g_1{:}\psi_1{}^*(x)\psi_1(x){:}+g_2{:}\psi_2{}^*(x)\psi_2(x){:}]\varphi(x)\,. \quad (3.1)$$

In this model, the coupling constants g_i have the dimension of mass and we set

$$g_1g_2=16\pi m_1m_2\alpha\,, \quad (3.2)$$

where α is dimensionless. The lowest-order approximation to the potential is given by (2.11), i.e., by the Fourier transform of the Yukawa potential

$$\frac{e^{-\mu r}}{r}=\frac{1}{2\pi^2}\int\frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mu^2+p^2}d^3p\quad(r=|\mathbf{r}|)\,. \quad (3.3)$$

Substituting (2.8) and (2.11) in the Fourier transform of the Schrödinger equation (2.13) and taking (3.2) and (3.3) into account we obtain

$$-\left(\nabla^2+b^2+2\frac{m_1m_2}{w}\frac{\alpha}{r}e^{-\mu r}\right)\phi(\mathbf{r})=0\,. \quad (3.4)$$

Equation (3.4) differs from the nonrelativistic Schrödinger equation in a Yukawa potential only in the constant kinematical factors

$$\begin{aligned}&\text{(a)}\quad \text{reduced mass } m_R=\frac{m_1m_2}{m_1+m_2}\to m_w=\frac{m_1m_2}{w}\,,\\&\text{(b)}\quad \text{binding energy } B\to\frac{b^2}{2m_w}\,.\end{aligned} \quad (3.5)$$

This observation allows us to write down the (exact or approximate) solution of our quasipotential equation whenever such a solution is available for the corresponding nonrelativistic problem. In particular, Eq. (3.4) yields the relativistic eikonal approximation. Indeed, according to Ref. 8, the small-angle eikonal behavior of the on-shell nonrelativistic amplitude $T_{\rm NR}$ is given by

$$T_{\rm NR}=(\mathbf{p},\mathbf{q})=\frac{-i}{(2\pi)^3}p\int d^2y\,\exp[i(\mathbf{p}-\mathbf{q})\cdot\mathbf{y}]\times\left\{\exp\left[\frac{2im_R}{p}\alpha K_0(\mu y)\right]-1\right\}. \quad (3.6)$$

The integration in $\mathbf{y}$ is performed in the two-plane orthogonal to $\mathbf{p}+\mathbf{q}$. The representation (3.6) is valid in any one of the two limits:

$$\mu\to 0,\quad t=-(\mathbf{p}-\mathbf{q})^2\to 0 \quad (3.7\text{a})$$

or

$$2m_RB=p^2\to\infty\ (\text{or } w^2\to\infty),\quad \mu,\,t\text{ finite}. \quad (3.7\text{b})$$

Using the substitution (3.5), we obtain from here the relativsitic (on-shell) eikonal formula[7,8] [in which m_R is replaced by m_w and p by $b(w)$ and the whole amplitude is multiplied by w/E_1E_2]. It is a valid approximation in any of the two limits (3.7).

For $\mu=0$, Eq. (3.4) corresponds to the Schrödinger equation for a particle of mass m_w in a Coulomb potential and therefore is exactly soluble. The substitution (3.5) in the nonrelativistic Balmer formula

$$B_n=-m_R\alpha^2/2n^2$$

gives the following relativsitic formula:

$$w_n^2=m_1^2+m_2^2+2m_1m_2(1-\alpha^2/n^2)^{1/2}. \quad (3.8)$$

This formula is a known consequence of the eikonal approximation.[12]

IV. ELECTROMAGNETIC INTERACTION OF SCALAR CHARGED PARTICLES

A. Modified Feshbach-Villars Formalism

In Sec. III we reduced (in a first approximation) the relativistic two-body problem to the problem of a single "reduced" particle of mass m_w (3.5) in an external field. The energy parameter of this fictitious particle is given by

$$E=[m_w^2+b^2(w)]^{1/2}=\frac{1}{2w}(w^2-m_1^2-m_2^2). \quad (4.1)$$

The Klein-Gordon eigenvalue equation with a "minimal" electromagnetic interaction should be

$$[(E-eA_0)^2-(\mathbf{p}-e\mathbf{A})^2-m_w^2]\psi_w=0. \quad (4.2)$$

In order to reconstruct the 4-potential A_μ from perturbation expansion of the scattering amplitude, we use a modification of the Feshbach-Villars formalism,[13] which enables us to write down a Lippmann-Schwinger type of equation for the scattering amplitude in this case.

It is convenient to linearize Eq. (4.2) in terms of the following system of four first-order equations:

$$(H_1+H_2-w)\Psi=0, \quad (4.3)$$

where

$$\begin{aligned}H_1&=m_1\tau_3\otimes\mathbf{1}\\&\quad+\frac{1}{2m_1}[(\mathbf{p}-e\mathbf{A})^2+2EeA_0-e^2A_0^2]\tau\otimes\mathbf{1},\\H_2&=m_2\mathbf{1}\otimes\tau_3\\&\quad+\frac{1}{2m_2}[(\mathbf{p}-e\mathbf{A})^2+2EeA_0-e^2A_0^2]\mathbf{1}\otimes\tau.\end{aligned} \quad (4.4)$$

Here $\mathbf{1}$ is the 2×2 unit matric, τ_j are the Pauli matrices, and

$$\tau=\tau_3+i\tau_2=\begin{pmatrix}1&1\\-1&-1\end{pmatrix},\quad(\tau^2=0). \quad (4.5)$$

The "Hamiltonian" (4.4) is Hermitian with respect to the scalar product defined by the metric tensor $\tau_3\otimes\tau_3$. In other words,

$$(H_1^*+H_2^*)\tau_3\otimes\tau_3=\tau_3\otimes\tau_3(H_1+H_2). \quad (4.6)$$

On the other hand, using the fact that $[H_1,H_2]=0$, we obtain

$$\begin{aligned}(w+H_1+H_2)(w+H_1-H_2)(w-H_1+H_2)(w-H_1-H_2)\\=4w^2[(E-eA_0)^2-(\mathbf{p}-e\mathbf{A})^2-m_w^2],\end{aligned} \quad (4.7)$$

so that Eq. (4.2) is indeed a consequence of Eq. (4.3).

The different components of Ψ are related to the positive and negative energy states of the two particles. This is clearly exhibited in the Feshbach-Villars representation, which is obtained through the unitary transformation

$$\Phi=U\Psi,\quad H_\Phi=U(H_1+H_2)U^{-1}, \quad (4.8)$$

where

$$\begin{aligned}U\Psi(\mathbf{p})=\frac{1}{4(m_1p_1^0m_2p_2^0)^{1/2}}&[(p_1^0+m_1)\mathbf{1}\times(p_1^0-m_1)\tau_1]\\&\otimes[(p_2^0+m_2)\mathbf{1}+(p_2^0-m_2)\tau_1]\Psi(\mathbf{p})\end{aligned} \quad (4.9)$$

$[p_i^0=(m_i^2+\mathbf{p}^2)^{1/2},\ i=1,\ 2]$. Taking into account that A_0 and $\mathbf{A}$ are integral operators in momentum space, we obtain

$$H_\Phi=p_1^0\tau_3\otimes\mathbf{1}+p_2^0\mathbf{1}\otimes\tau_3+\int V_\Phi(\mathbf{p},\mathbf{q})d^3q, \quad (4.10)$$

where

$$\begin{aligned}V_\Phi(\mathbf{p},\mathbf{q})=&[2Ee\mathfrak{A}_0(\mathbf{p},\mathbf{q})-(\mathbf{p}+\mathbf{q})e\boldsymbol{\mathfrak{A}}(\mathbf{p},\mathbf{q})-e^2\mathfrak{A}^2(\mathbf{p},\mathbf{q})]\\&\times(16p_1^0p_2^0q_1^0q_2^0)^{1/2}\{[(p_1^0+q_1^0)\mathbf{1}+(p_1^0-q_1^0)\tau_1]\\&\otimes\tau+\tau\otimes[(p_2^0+q_2^0)\mathbf{1}+(p_2^0-q_2^0)\tau_1]\}.\end{aligned} \quad (4.11)$$

Here $\mathfrak{A}_0(\mathbf{p},\mathbf{q})$ and $\boldsymbol{\mathfrak{A}}(\mathbf{p},\mathbf{q})$ are the kernels of the integral operators A_0 and $\mathbf{A}$ and $\mathfrak{A}^2=\mathfrak{A}_0^2-\boldsymbol{\mathfrak{A}}^2$.

Let $\phi^{\sigma\sigma'}$ $(\sigma,\sigma'=\pm)$ be the normalized solutions of the free part of the Hamiltonian (4.10):

$$\phi^{++}=\frac{1}{\sqrt2}\begin{pmatrix}1&1\\0&0\end{pmatrix},\quad\phi^{+-}=\frac{1}{\sqrt2}\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$\phi^{-+}=\frac{1}{\sqrt2}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\phi^{--}=\frac{1}{\sqrt2}\begin{pmatrix}0&0\\1&1\end{pmatrix}.$$

Then, we have the following inhomogeneous equation for $\Phi_\mathbf{q}(\mathbf{p})$ incorporating the boundary condition for the scattering problem:

$$\begin{aligned}\Phi_\mathbf{q}^{\sigma\sigma'}(\mathbf{p})=&\phi^{\sigma\sigma'}\delta(\mathbf{p}-\mathbf{q})\\&-\frac{1}{H_0-w-i0}\int V_\Phi(\mathbf{p},\mathbf{k})\Phi_\mathbf{q}^{\sigma\sigma'}(\mathbf{k})d^3k,\end{aligned} \quad (4.12)$$

where $H_0=p_1^0\tau_3\otimes\mathbf{1}+P_2^0\mathbf{1}\otimes\tau_3$ [cf. (2.14)]. By analogy with (2.15) the scattering amplitude is defined by

$$T(\mathbf{p},\sigma_1\sigma_2;\mathbf{q},\sigma_3\sigma_4)=-\bar\phi^{\sigma_1\sigma_2}\int V_\Phi(\mathbf{p},\mathbf{k})\Phi_\mathbf{q}^{\sigma_3\sigma_4}(\mathbf{k})d^3k. \quad (4.13)$$

[The physical scattering amplitude $T_w(\mathbf{p},\mathbf{q})$ will be identified with the matrix element between positive

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[12] C. Itzykson (private communication); see H. Crater, Phys. Rev. D **2**, 1060 (1970). The same energy eigenvalues were obtained also from different equations in Refs. 9 and 10. Recently, Lamb-shift corrections to these energy levels have been calculated by C. Fronsdal and R. W. Huff, Phys. Rev. D **3**, 933 (1971), where a similar method is used.

[13] H. Feshbach and F. Villars, Rev. Mod. Phys. **30**, 24 (1958).

energy states $T(\mathbf{p},++;\mathbf{q},++)$.] Introducing (4.13) in (4.12) we obtain the following generalized Lippmann-Schwinger equation:

$$T_w(\mathbf{p},\sigma_1\sigma_2;\mathbf{q},\sigma_3\sigma_4)+V_w(\mathbf{p},\sigma_1\sigma_2;\mathbf{q},\sigma_3\sigma_4)+\int V_w(\mathbf{p},\sigma_1\sigma_2;\mathbf{k},\sigma\sigma')G_w^{\sigma\sigma'}(\mathbf{k})\times T_w(\mathbf{k},\sigma\sigma';\mathbf{q},\sigma_3\sigma_4)d^3k=0. \quad (4.14)$$

Here

$$V_w(\mathbf{p},\sigma_1\sigma_2;\mathbf{q},\sigma_3\sigma_4)=\bar{\phi}^{\sigma_1\sigma_2}V_\Phi(\mathbf{p},\mathbf{q})\phi^{\sigma_3\sigma_4},$$

$$G_w^{\sigma\sigma'}(\mathbf{k})=\bar{\phi}^{\sigma\sigma'}\frac{1}{H_0-w-i0}\phi^{\sigma\sigma'}, \quad (4.15)$$

and we have used the completeness relation

$$\sum_{\sigma\sigma'}\phi^{\sigma\sigma'}\otimes\bar{\phi}^{\sigma\sigma'}=\mathbf{1}.$$

In agreement with (2.7),

$$\mathrm{Im}G_w^{++}(\mathbf{k})=2\pi i\frac{E_1E_2}{w}\delta(\mathbf{k}^2-b^2).$$

B. Relativistic Balmer Formula Including the Fine-Structure Splitting[14]

According to the general rules of Sec. II, the 4-potential A_μ will be determined from (4.14) and (4.11). Starting with the electromagnetic interaction Lagrangian

$$L_{\mathrm{em}}=ie[:\psi_1^*(x)\overleftrightarrow{\partial}_\mu\psi_1(x):-:\psi_2^*(x)\overleftrightarrow{\partial}_\mu\psi_2(x):]A^\mu(x)+e^2[:\psi_1^*(x)\psi_1(x):+:\psi_2^*(x)\psi_2(x):]A^\mu(x)A_\mu(x). \quad (4.16)$$

Up to second order in e, on the mass shell, we have

$$\mathbf{T}_2(\mathbf{p},\mathbf{q})=\frac{\alpha}{2\pi^2}\frac{1}{(\mathbf{p}-\mathbf{q})^2}\left[1+\frac{(\mathbf{p}+\mathbf{q})^2}{4E_1E_2}\right]=-V_2(\mathbf{p},++;\mathbf{q},++)=e[\mathfrak{A}(\mathbf{p},\mathbf{q})\cdot(\mathbf{p}+\mathbf{q})-2E\mathfrak{A}_0(\mathbf{p},\mathbf{q})]\frac{w}{2E_1E_2}. \quad (4.17)$$

(Here we have introduced the fine-structure constant $\alpha=e^2/4\pi$.) By an appropriate choice of the gauge, we can define $e\mathfrak{A}_0$ to be the Coulomb potential:

$$e\mathfrak{A}_0(\mathbf{p},\mathbf{q})=\frac{-\alpha}{2\pi^2}\frac{1}{(\mathbf{p}-\mathbf{q})^2}. \quad (4.18)$$

Using, furthermore, the fact that

$$Ew/E_1E_2=1+b^2/E_1E_2,$$

we obtain

$$e\mathfrak{A}(\mathbf{p},\mathbf{q})\cdot(\mathbf{p}+\mathbf{q})=-\alpha/4\pi^2w, \quad (4.19)$$

so that we can define

$$e\mathfrak{A}(\mathbf{p},\mathbf{q})=-\frac{\alpha}{4\pi^2w}\frac{\mathbf{p}+\mathbf{q}}{(\mathbf{p}+\mathbf{q})^2}. \quad (4.20)$$

With this choice, both $e\{\mathbf{p},\mathbf{A}\}$ and $e^2\mathbf{A}^2$ are local operators in coordinate space:

$$-e\{\mathbf{p},\mathbf{A}\}(\mathbf{r})=2\pi\frac{\alpha}{w}\delta(\mathbf{r}),\quad e^2\mathbf{A}^2(\mathbf{r})=\pi^2\frac{\alpha^2}{w^2}\frac{1}{r^4}. \quad (4.21)$$

We note that here the arbitrariness in the choice of the off-shell potential is even greater than in the case of a scalar potential (Sec. II), since it also depends on the choice of the gauge.

We proceed now to the determination of the energy eigenvalues. First, we observe that the δ-function term in Eq. (4.21) contributes only to the s-wave values $w_{n\,l=0}$, while the second term ($e^2\mathbf{A}^2$) coming from the vector potential is of order α^6. To see the latter point, it is sufficient to write down (4.21) in atomic units.[15] Therefore, we first consider the eigenvalue problem for a pure Coulomb interaction (i.e., set $e\mathbf{A}=0$); at the end we introduce the correction coming from the $e\{\mathbf{p},\mathbf{A}\}$ term treated as a perturbation.

The radial equation corresponding to a pure Coulomb interaction has the form

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+b^2(w)+2E\frac{\alpha}{r}+\frac{\alpha^2-l(l+1)}{r^2}\right]R_{wl}(r)=0. \quad (4.22)$$

It differs from the nonrelativistic radial Schrödinger equation by the replacements

$$l\to l-\epsilon_l,\quad \epsilon_l=l+\tfrac{1}{2}-[l(+\tfrac{1}{2})^2-\alpha^2]^{1/2}; \quad (4.23)$$

$$m_R\to E,\quad 2m_RB\to b^2(w). \quad (4.24)$$

Substituting in the nonrelativistic Balmer formula, we obtain the following eigenvalues[8]:

$$w_{nl}^2=m_1^2+m_2^2+2m_1m_2\left[1+\frac{\alpha^2}{(n-\epsilon_l)^2}\right]^{-1/2}, \quad (4.25)$$

where ϵ_l is given by (4.23).

The lowest nonvanishing contribution of the term $-e\{\mathbf{p},\mathbf{A}\}$ (4.21) is of order α^4. It is

$$\frac{\pi\alpha}{m_1m_2}|\psi_{nl\zeta}(0)|^2=\delta_{l0}\frac{m_R^2}{m_1+m_2}\frac{\alpha^4}{n^3} \quad (4.26)$$

[14] The results of this subsection are obtained in collaboration with V. Rizov. A more extensive survey of the quasipotential approach to electromagnetic interactions is in preparation.

[15] See, e.g., H. Bethe and E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, Berlin, 1957).

[$\psi_{nl\zeta}(\mathbf{r})$ are the nonrelativistic hydrogen wave functions]. Thus, the complete expression for w_{nl} up to order α^4 is[16]

$$w_{nl}=m_1+m_2-\frac{m_R\alpha^2}{2n^2}-\frac{m_R\alpha^4}{n^3}\times\left[\frac{1}{2l+1}-\frac{3}{8n}+\frac{m_R}{8(m_1+m_2)n}\right]+\frac{m_R^2}{m_1+m_2}\frac{\alpha^4}{n^3}\delta_{l0}+o(\alpha^4). \quad (4.27)$$

We observe that the eigenvalues (4.27) coincide with those calculated from the more conventional Breit equation[15]

$$\left[m_1+m_2+\frac{1}{2m_R}\mathbf{p}^2-\frac{1}{8m_R{}^3}\left(1-3\frac{m_R}{m_1+m_2}\right)\mathbf{p}^4-w_{nl}\right]\psi_{nl\zeta}(\mathbf{p})=\frac{\alpha}{2\pi^2}\int\left\{\frac{1}{(\mathbf{p}-\mathbf{q})^2}-\frac{1}{4m_1m_2}\times\left[1-2\frac{\mathbf{p}^2+\mathbf{q}^2}{(\mathbf{p}-\mathbf{q})^2}+\left(\frac{\mathbf{p}^2-\mathbf{q}^2}{(\mathbf{p}-\mathbf{q})^2}\right)^2\right]\right\}\psi_{nl\zeta}(\mathbf{q})d^3q \quad (4.28)$$

[although our starting point—Eq. (4.2) with the potential (4.18) and (4.20)—seems, at first sight, to have little in common with Eq. (4.28)]. The new calculation is not only technically simpler; in contrast with the Breit equation (4.28), which only makes sense in perturbation theory, the quasipotential equation (4.2) is a *bona fide* equation. Furthermore, as explained in Sec. II, it is readily adapted for the calculation of higher-order corrections: One just has to take the next terms in the potential from Eq. (2.6).

V. CONCLUDING REMARKS

We single out some properties of the approximate quasipotential equations (3.4) and (4.2) (corresponding to different quantum-field-theoretic Lagrangians) which distinguish them favorably from the ladder approximation in the Bethe-Salpeter equation in the same cases.[17]

(a) For small momenta and weak coupling one can write

$$w=m_1+m_2+B\,, \quad (5.1)$$

where the binding energy B is small as compared with the rest masses. Then either of Eqs. (3.4) and (4.2) goes into the corresponding nonrelativistic equation. In particular, the relativistic effective mass m_w (3.5) goes into the nonrelativistic reduced mass m_R.

(b) If one of the masses (say, m_2) tends to infinity, then the quasipotential equation goes into the Klein-Gordon equation for particle 1 in an external field. (We note that $w/m_2 \to 1$ for $m_2 \to \infty$.)

(c) In terms of Feynman diagrams, the quasipotential equation sums up all crossed-ladder diagrams in any one of the two limits (3.7) in which the relativistic eikonal approximation is valid.

Properties (b) and (c) are actually related. It is just the lack of the latter property in the ladder approximation of the Bethe-Salpeter equation which does not allow the proper static limit to be obtained from it when the proton mass goes to infinity. It may also be the origin of the unwanted solutions in the Wick-Cutkosky model.[17]

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[16] The appearance of the term (4.26) in Eq. (4.27) shows that the conjecture made in Ref. 8, that (4.25) includes all recoil effects, is not correct.

[17] G. C. Wick, Phys. Rev. **96**, 1124 (1954); R. E. Cutkosky, *ibid.* **96**, 1135 (1954).