Particle Theory of the Lamb Shift without Divergences or Cutoffs*†

JOHN E. KRIZAN
Department of Physics, The University of Vermont, Burlington, Vermont 05401
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Values of the Lamb shift for the transitions $1S$-$2P$, $2S$-$2P$, and $3S$-$3P$ in hydrogen are compared in a nonrelativistic particle theory with no divergences or cutoffs. The theory is based on the extension of quantum-mechanical equations to the radiative case by arguing the necessity of time symmetry and the vanishing of self-interaction effects in the absence of external forces. The characteristic radiation time constant $\epsilon = Ke^2/mc^3$ is used. When $K=1$, fairly good agreement for all three shifts is obtained. However, if K is adjusted to obtain the value for the $2S$-$2P$ level, the other, correspondingly adjusted levels give good agreement with the experimental values for $1S$-$2P$ and $3S$-$3P$ transitions. Vacuum polarization effects are included (for comparison with Jaynes and Crisp's results); however, these effects are probably more legitimately treated within a relativistic framework.

I. INTRODUCTION

THE aim of this paper is to obtain radiative corrections to the motion of a charge in an external field. It follows an earlier paper[1] in which semiquantitative comparison was made; here the attempt is also made within a field-free formalism. However, whereas earlier we took a stochastic approach[2] in combination with classical electrodynamic analogy, here a purely quantum-mechanical treatment is sought. The principal earlier premise, that the theory possesses a time-symmetric basis, is retained in this paper.[3]

It is not our intention to imply that any success in a Lamb-shift calculation means that the field-theoretic formulation of quantum electrodynamics has found a divergence-free substitute. It merely augurs well for the straightforward nonrelativistic theory given here; more tests would obviously be needed, to say nothing of the question of an appropriate relativistic generalization.

Recent reservations as to the adequacy of quantum electrodynamics have been expressed in an interesting paper by Crisp and Jaynes.[4] Theirs is a semiclassical treatment of radiative corrections, with the calculation of the Lamb shift proceeding from considerations of self-interaction of the classical atomic field. While we share the dissatisfaction of the authors of Ref. 4 with traditional renormalization procedures, we feel that no theory based on independent field degrees of freedom can avoid the divergences (classical or quantum) inherent in field theory.

For the calculations involving the Lamb shift we include the self-interaction correction to terms of order $Z^4\alpha^3$, where α is the fine-structure constant.[5] The comparisons for the cases given in Ref. 4 are made. Good agreement is obtained with experimental values; in fact, the agreement for the $1S$-$2P$ and $3S$-$3P$ transitions compares favorably to that given by quantum electrodynamics. In spite of this, we would repeat that the agreement may be fortuitous and the attempt is still to be considered exploratory. Nevertheless the absence of any renormalization or cutoff procedure makes it worthwhile, we feel, to persist in trying to find a different basis for the effects treated traditionally by renormalized field theories.

II. DERIVATION OF FUNDAMENTAL EQUATIONS

Suppose we accept the linear time-dependent Schrödinger equation as a limiting, nonradiative approximation:

$$ih\partial\psi/\partial t = H\psi\,, \tag{2.1}$$

where

$$H = -(\hbar^2/2m)\nabla^2 + V\,. \tag{2.2}$$

Substituting

$$\psi = e^{R+iS} \tag{2.3}$$

into (2.1), where R and S are real functions, results in the equations[1] (these are the WKBJ equations before approximation)[6]

$$\mathfrak{L}\mathbf{v} = \mathbf{f} + \mathcal{O}\mathbf{u}\,, \tag{2.4a}$$

$$\partial\mathbf{u}/\partial t = -(h/2m)\nabla(\nabla\cdot\mathbf{v}) - \nabla(\mathbf{v}\cdot\mathbf{u})\,; \tag{2.4b}$$

here $\mathbf{v}$, $\mathbf{u}$ are defined by

$$\mathbf{v} \equiv (h/m)\nabla S\,, \tag{2.5a}$$

$$\mathbf{u} \equiv (h/m)\nabla R\,, \tag{2.5b}$$

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[1] J. Krizan, Phys. Rev. **165**, 1725 (1968).

[2] E. Nelson, Phys. Rev. **150**, 1079 (1966).

[3] J. Krizan, Phys. Rev. D **1**, 2772 (1970). This paper argues for a relativistic time-symmetric theory which gives rise to the usual quantum-electrodynamic results; the treatment follows especially from consideration of the limiting classical case. While results which do not require renormalization follow from this particle approach, the nonrelativistic theory in the paper here does not require any cutoffs.

[4] M. D. Crisp and E. T. Jaynes, Phys. Rev. **179**, 1253 (1969); **185**, 2046 (E) (1969). See reference to earlier work by Jaynes and colleagues here.

[5] In Ref. 1, all terms were not considered in the semiquantitative comparison of the theory (as was stated in the paper). The basic time-symmetric motivation for the equations, although not the stochastic derivation, remains here. The equations used there were given in terms of $\mathcal{O}(\mathbf{u})$ rather than $\mathcal{O}(-\mathbf{u})$, for $\mathfrak{R}$. A stochastic derivation similar to that in Ref. 1 was recently given by L. de la Peña-Auerbach and A. M. Cetto, Phys. Rev. D **3**, 795 (1971). These authors evaluated a term previously neglected in Ref. 1. It was suggested in Ref. 1, following a sign error (noted in proof) that this was a possible improvement.

[6] Again here we have proceeded from the more familiar Schrödinger equation to the fundamental set. Clearly one could have also started with the set without recourse to the nonradiative approximation in terms of the wave equation.

while

$$\mathfrak{L}\equiv\partial/\partial t+\mathbf{v}\cdot\nabla \tag{2.6a}$$

and

$$\mathfrak{O}\equiv\mathbf{u}\cdot\nabla+(h/2m)\nabla^2. \tag{2.6b}$$

Above also

$$\mathbf{f}\equiv-\nabla V/m. \tag{2.7}$$

Thus the nonrelativistic Schrödinger equation is equivalent to the set of equations (2.4). The question arises as to the extension of these when self-interaction effects are considered.

To modify these equations when self-interaction is considered, we are guided by the following requirements: (1) When there is no external force acting, we require that the situation reduce to that of a free particle with no self-interaction. (2) The additional reaction force must be time symmetric.

The first requirement suggests that (2.4a) be generalized to

$$\mathfrak{L}\mathbf{v}-\mathfrak{O}\mathbf{u}=\mathbf{f}+\epsilon\mathfrak{R}(\mathfrak{L}\mathbf{v}-\mathfrak{O}\mathbf{u}), \tag{2.8}$$

where the vanishing of the left-hand side when $\mathbf{f}=0$ is assured by the form of the reactive term, in which $\mathfrak{R}$ is an operator which is to be specified; ϵ is a coupling which is a characteristic time parameter (the parameter which naturally occurs in consideration of the self-interaction of an electron is e^2/mc^3). The form of the reactive term, particularly the dependence on the above parameters, may also be inferred from the classical analogy (as given elsewhere[1]).

Thus when the particle is free, the real part R is constant in (2.3), and (2.5b) leads to $\mathbf{u}=0$, so that (2.8) leads simply to $\mathfrak{L}(\mathbf{v})=0$. The solution of this is, of course, $\mathbf{v}=$const which leads immediately to the wave function $\psi=A(t)e^{i\mathbf{k}\cdot\mathbf{x}}$ in (2.3), where $\mathbf{k}=(m/h)\mathbf{v}$. Further, if we are dealing with an approximation in which self-interaction is neglected (or, that a time scale of the order of ϵ is ignored), but $\mathbf{f}\neq0$, then (2.8) reduces to (2.4a), which in turn can be put into the time-dependent form (2.1).

When $\mathbf{v}=0$, or in the stationary limit, one gets (for $\epsilon=0$):

$$\mathfrak{O}\mathbf{u}+\mathbf{f}=0. \tag{2.9}$$

This can be put into the form of the ordinary time-independent Schrödinger equation, using (2.5), (2.4b), and (2.3) [in using (2.4b), it is seen that $\partial\mathbf{u}/\partial t=0$ when $\mathbf{v}=0$].

The second requirement of time symmetry leads to the following selection for the operator $\mathfrak{R}$ in (2.8). Time symmetry is manifested by symmetry under time inversion. Under this operation, $\mathbf{v}\rightarrow-\mathbf{v}$ and $\mathbf{u}\rightarrow\mathbf{u}$ [this follows from the time symmetry of (2.1) under $t\rightarrow-t$ and complex conjugation]. Therefore, we introduce a time-symmetric operator $\mathfrak{O}^\dagger\equiv\mathfrak{O}(-\mathbf{u})$, where $\mathfrak{O}(\mathbf{u})$ is given by (2.6b).[7]

[7] The choice of $\mathfrak{O}(-\mathbf{u})$ rather than $\mathfrak{O}(\mathbf{u})$ for $\mathfrak{R}$ perhaps has justification in a symmetry requirement for the fundamental set. Here as in Ref. 1, the basis for the choice is that the operator be time symmetric and the immediate justification is the pragmatic one that reasonable results follow.

The time-symmetric equation of motion is then

$$\mathfrak{L}\mathbf{v}-\mathfrak{O}\mathbf{u}=\mathbf{f}+\epsilon\mathfrak{O}^\dagger(\mathfrak{L}\mathbf{v}-\mathfrak{O}\mathbf{u}). \tag{2.10}$$

The second equation, (2.4b), is unchanged from the $\epsilon=0$ case since it expresses conservation of probability, which we assume to hold also for the more general situation. Thus if we make the usual identification of $\rho=\psi^*\psi$, Eq. (2.4b) is transformed with the help of (2.5) into

$$\partial\rho/\partial t+\nabla\cdot(\rho\mathbf{v})=0. \tag{2.11}$$

Before proceeding to the calculation of radiative effects we put (2.10) in a slightly different form. We write

$$(1-\epsilon\mathfrak{O}^\dagger)\mathfrak{L}\mathbf{v}=\mathbf{f}+(1-\epsilon\mathfrak{O}^\dagger)\mathbf{f}_B, \tag{2.12}$$

where

$$\mathbf{f}_B\equiv-(h/m)\nabla\varphi_B \tag{2.13}$$

and

$$\varphi_B\equiv-(h/4m)\{(\nabla^2\psi/\psi)+(\nabla^2\psi^*/\psi^*)-\tfrac{1}{2}[(\nabla\psi/\psi)-(\nabla\psi^*/\psi^*)]^2\}. \tag{2.14}$$

The latter-defined quantities are the quantum-mechanical force and potential.[8]

III. SELF-INTERACTION ENERGY

In investigating the notion of an electron in an external field in which self-interaction corrections are interpreted as the Lamb shift, we are dealing with a stationary system. This means that the calculations are carried out for a time-independent, energy-conserving system; equivalently $\mathbf{v}=0$ in Eqs. (2.12) and (2.4b). Thus (2.12) becomes

$$\mathbf{f}+\mathbf{f}_B-\epsilon\mathfrak{O}^\dagger\mathbf{f}_B=0. \tag{3.1}$$

If we integrate this and take the expectation (denoted by brackets), the average energy of the electron in the external field may be obtained:

$$\left\langle\int\mathbf{f}\cdot d\mathbf{x}\right\rangle+\left\langle\int\mathbf{f}_B\cdot d\mathbf{x}\right\rangle-\epsilon\left\langle\int\mathfrak{O}^\dagger\mathbf{f}_B\cdot d\mathbf{x}\right\rangle=\text{const.} \tag{3.2}$$

The expectation is taken, of course, with the probability density ρ. The terms in (3.2) have a straightforward physical explanation in that the first two represent the work done against the external field and quantum-mechanical forces, respectively, while the last is the self-interaction work. If the wave functions of the unperturbed problem are used (we denote this by the subscript on the brackets) where (2.7) is used, the first two terms are merely equal to $-E/m$. The last term gives

[8] These terms have been used by D. Bohm and J. P. Vigier, Phys. Rev. 96, 208 (1954), although we wish to stress that the result here does follow from the usual quantum-mechanical formalism.

the self-energy correction as

$$E_{\text{self}}=-\epsilon\hbar\left\langle\int\left(-\mathbf{u}\cdot\nabla+\frac{\hbar}{2m}\nabla^2\right)\nabla\varphi_B\cdot d\mathbf{x}\right\rangle_0$$
$$=\epsilon\left\langle\int\left(-\mathbf{u}\cdot\nabla+\frac{\hbar}{2m}\nabla^2\right)\nabla V\cdot d\mathbf{x}\right\rangle_0. \quad (3.3)$$

In the preceding, use is made of (2.13) and (2.14). From (2.14) it is seen that $\varphi_B=-(\hbar/2m)(\nabla^2\psi/\psi)$ when $\mathbf{v}=0$. With unperturbed wave functions then, $\nabla\varphi_B=-\nabla V/\hbar$.

IV. CALCULATION OF LAMB SHIFT

The calculation is carried out for the $2S$-$2P$ and $3S$-$3P$ transitions. Experimental data are available for these, and field-theoretic quantum-electrodynamic calculations have been made. The agreement for the $2S$-$2P$ shift is excellent, as is well known. For the other two shifts it is not as good. Indeed, if the $2S$-$2P$ discrepancy were comparable to the others, it is probable that confidence in the quantitative accuracy of the conventional treatment would have long since eroded.

We write

$$E_{\text{self}}=E_1+E_2, \quad (4.1)$$

where

$$E_1\equiv-\frac{\epsilon\hbar}{m}\left\langle\int(\nabla\ln\psi\cdot\nabla)\nabla V\cdot d\mathbf{x}\right\rangle_0 \quad (4.2)$$

and

$$E_2\equiv\frac{\epsilon\hbar}{m}\left\langle\int\nabla^2(\nabla V)\cdot d\mathbf{x}\right\rangle_0. \quad (4.3)$$

The term E_2 in (4.3) has contributions from S states alone. This is seen when the line integration is carried out to yield ∇^2V and subsequent expectation integration yields, since $V=V(r)=-Ze^2/r$,

$$E_2=\frac{4\pi\epsilon\hbar Ze^2}{2m}\int|\psi_{nl}(r)|^2\delta(\mathbf{r})d^3r$$
$$=\frac{4\pi\epsilon\hbar Ze^2}{2m}|\psi_{nl}(0)|^2. \quad (4.4)$$

Since all states except S states vanish at the origin, only these contribute.

Turning to the evaluation of E_1, we must carry out separate integrations for each state, unlike the calculation in (4.4). The $2S$ line integration is made first. From (4.2) we get, since $\psi=A[2-(Zr/a_0)]e^{-Zr/2a_0}$, where A is constant and a_0 is the Bohr radius:

$$-\frac{\epsilon\hbar Z^2e^2}{2ma_0}\int\frac{4a_0Z^{-1}-r}{2a_0Z^{-1}-r}\frac{d}{dr}\left(\frac{1}{r^2}\right)dr.$$

This integrates directly to

$$\frac{2\epsilon\hbar Z^4e^2}{(2a_0)^3m}\left[\ln\frac{r-2a_0Z^{-1}}{r}+\left(\frac{rZ}{2a_0}\right)^{-2}+\left(\frac{rZ}{2a_0}\right)^{-1}\right],$$
$$r>2a_0Z^{-1}, \quad (4.5a)$$

and

$$\frac{2\epsilon\hbar Z^4e^2}{(2a_0)^3m}\left[\ln\frac{2a_0Z^{-1}-r}{r}+\left(\frac{rZ}{2a_0}\right)^{-2}+\left(\frac{rZ}{2a_0}\right)^{-1}\right],$$
$$r<2a_0Z^{-1}. \quad (4.5b)$$

For the corresponding $2P$ integration, we have from (4.2)

$$\ln\psi_{2Pz}\propto\ln(Zr/a_0)-(rZ/2a_0)+\ln(\cos\theta)$$

and

$$-\frac{\epsilon\hbar Z^2e^2}{2ma_0}\int\frac{2a_0Z^{-1}-r}{r}\frac{d}{dr}\left(\frac{1}{r^2}\right)dr$$

yields directly

$$\frac{2\epsilon\hbar Z^4e^2}{(2a_0)^3m}\left[\frac{1}{3}\left(\frac{rZ}{2a_0}\right)^{-3}-2^{-1}\left(\frac{rZ}{2a_0}\right)^{-2}\right]. \quad (4.6)$$

The corresponding expectation integrations are carried out in Appendix A for the $2S$ and $2P$ states. The results are

$$E_{1,2S}=0.522KZ^4\alpha^3\ \text{Ry}$$

and

$$E_{1,2P}=-0.0278KZ^4\alpha^3\ \text{Ry}.$$

In the above we have put, following an earlier statement, $\epsilon=Ke^2/mc^3$ where K is given as an adjustable constant of order unity. (Note that if an exact correspondence of the coefficient with that of the classical Lorentz equation is made, $K=\frac{2}{3}$.) The time $\tau_0=e^2/mc^3$ is the characteristic time required for light to traverse the classical electron radius. As mentioned before, it is a natural characteristic time to associate with electron self-interaction (this may be seen particularly from the analogy with classical electrodynamics).

The contribution from (4.4) is $E_{2,2S}=\frac{1}{2}KZ^4\alpha^3$ Ry, so that the total shift is given by

$$E_{2S\text{-}2P}=E_{1,2S}+E_{2,2S}-E_{1,2P}=1.05KZ^4\alpha^3\ \text{Ry}$$
$$=1347KZ^4\ \text{MHz}. \quad (4.7)$$

If one uses the vacuum-polarization result of Uehling,[9] the above result is corrected to $E_{2S\text{-}2P}=1321\ Z^4K$ MHz. If one adjusts K to the hydrogen experimental value corresponding to $E_{2S\text{-}2P}=1058$ MHz, then $K=0.80$ (with vacuum polarization) or $K=0.78$ (without vacuum polarization). We will use these values (note the closeness to $K=\frac{2}{3}$) in subsequent determinations of the $1S$-$2P$ and $3S$-$3P$ shifts.

In Appendix B, we show that $E_{1,1S}=4KZ^4\alpha^3$ Ry, while from (4.4), $E_{2,1S}=4KZ^4\alpha^3$ Ry. From the previous result the $2P$ contribution is small and equal to $-0.0278KZ^4\alpha^3$ Ry. The vacuum polarization corresponding to the $2S$ state is $-0.170Z^2\alpha^3$ Ry. Thus for

[9] E. A. Uehling, Phys. Rev. **48**, 55 (1935). We include this here only for comparison, as is done in Ref. 4, although a derivation is not given in the present theory. A screening of the potential interaction might give this result within the present treatment. In any event, the effect is seen to be negligible.

TABLE I. Values of Lamb shift in hydrogen, with and without adjustable constant, and with and without vacuum polarization. The value of K has been adjusted to 0.80 or 0.78 to give the correct $2S$-$2P$ shift. The results are compared with experiment (references given in Ref. 4) and with quantum electrodynamic (QED) theory. If the $2S$-$2P$ shift is taken as 1011.4 MHz (no vacuum polarization, anomalous magnetic moment, or higher-order effects), then adjustment gives $K=0.765$ and the $1S$-$2P$ shift is 0.270 cm^{-1} while the $3S$-$3P$ shift is 0.0075 cm^{-1}. Also, if $K=\frac{2}{3}$ (classical coefficient), the respective values are 0.227 cm^{-1}, 898 MHz, and 0.0065 cm^{-1}. See Eq. (4.7) for conversion from Ry to MHz; 1 Ry = 109 678 cm^{-1}.

nS-nP	$K=1$ (no vac. pol.)	$K=1$ (vac. pol.)	$K=0.78$ (vac. pol.)	$K=0.80$ (no vac. pol.)	Expt	QED
$1S$-$2P$	0.340 cm^{-1}	0.334 cm^{-1}	0.276 cm^{-1}	0.277 cm^{-1}	0.262 ± 0.038 cm^{-1}	0.2726 cm^{-1}
$2S$-$2P$	1347 MHz	1321 MHz	1058 MHz	1058 MHz	1057.77 ± 0.10 MHz	1057.19 MHz
$3S$-$3P$	0.0098 cm^{-1}	0.0095 cm^{-1}	0.0076 cm^{-1}	0.0078 cm^{-1}	$0.0083_{-0.003}{}^{+0.002}$ cm^{-1}	0.0105 cm^{-1}

hydrogen

$$E_{1S\text{-}2P}=7.86KZ^4\alpha^3\ \mathrm{Ry}=0.334K\ \mathrm{cm}^{-1}. \quad (4.8)$$

When $K=0.8$, this is close to the experimental value of 0.262 ± 0.038 cm^{-1}. The corresponding calculation for the $3S$-$3P$ transition in Appendix C gives the following results:

$$E_{2,3S}=4Z^4K\alpha^3/27\ \mathrm{Ry}=0.00630K\ \mathrm{cm}^{-1}, \quad (4.9)$$

$$E_{1,3P}=0.0584Z^4K\alpha^3\ \mathrm{Ry}=0.00248K\ \mathrm{cm}^{-1}, \quad (4.10)$$

and

$$E_{1,3S}=0.138KZ^4\alpha^3\ \mathrm{Ry}=0.00586K\ \mathrm{cm}^{-1}. \quad (4.11)$$

Thus we have

$$E_{3S\text{-}3P}=0.00977KZ^4\ \mathrm{cm}^{-1}. \quad (4.12)$$

For comparison with experimental and quantum electrodynamic theory calculations for hydrogen, see Table I. From this table we see that even without the use of the constant K, the values are in reasonably good agreement (although consistently higher). Adjusting to the $2S$-$2P$ transition, however, brings the $1S$-$2P$ and $3S$-$3P$ values in rather striking agreement with experimental values. In the $1S$-$2P$ the value is within experimental error and is very close to the theoretical quantum-electrodynamic result. In the $3S$-$3P$ transition, the result matches experiment more closely than the quantum-electrodynamic result (this is true even if K is unity).

It is quickly seen that the vacuum-polarization correction is not significant; thus for all purposes one can avoid here the question of introducing it altogether.

V. CONCLUSIONS

The theory presented is potentially useful in describing effects attributed to self-interaction of a single charged particle. In a certain sense the description as "self-interaction" is a misnomer; this is so since the model requires the vanishing of effect when the external force is absent. On the other hand, one can view the effect as a distortion of the electron charge distribution which is due to a finite-size effect (the interpretation of the classical electron radius in the parameter ϵ suggests this). Thus if (in the traditional classical sense) the charge is thought to possess a self-interaction due to finite size (recall the Lorentz classical electron) even in the absence of forces, the effect here can be interpreted as due to the *change* of that distribution due to an external force.[10] It is also assumed, by analogy with the classical result, that a limiting force could follow as the radius of the charge is made vanishingly small.

One of the difficulties of the usual classical theory is that self-interaction is not assumed to vanish in the force-free case. This leads to the appearance of difficulties characterized as "self-accelerated" solutions.[11] Here, we construct the self-interaction problem in such a way from the beginning that these effects do not occur; thus we constructed Eq. (2.10) such that the self-interaction term vanished when $\mathbf{f}=0$.

The above interpretation, of course, deviates from the detailed description of virtual photon emission and absorption of an electron (whether isolated or not). The divergences associated with the quantum electrodynamic treatment of this problem are well known. What we argue for is an approach which eliminates these in a formalism which does not introduce independent field quantities.

Recently[3] it was argued that a time-symmetric basis could be used to construct a quantum electrodynamics in which the particle variables are emphasized. It was indicated there that it followed that many quantum electrodynamic results could be obtained in a formalism which is free even of mass renormalization. The essence of classical electrodynamic ideas was exploited (as is true to some extent in the present paper) to argue that the usual classical limit with radiation damping can be seen to emerge. However, in a typical Lamb-shift calculation, as would follow even from this last approach, one would still be forced to contend with a cutoff in order to get usual results.

Here there is no cutoff. Therein lies the incentive for exploring these effects in the present formalism. The agreement is reasonable, which may be surprising when we consider that the only wave functions used in calculating effects are those associated with the two levels concerned. Contrast this with the classic perturbation theoretic result of Bethe[12] which introduces all

[10] This is offered as a plausible argument which follows from classical analogy. However, we are open to other interpretations.

[11] See, e.g., F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965).

[12] H. A. Bethe, Phys. Rev. **72**, 339 (1947).

states in virtual transitions. On the other hand, recent two-level calculations have been given elsewhere[4]; moreover it is really not apparent (at least to this author) that the complete set of states is needed to calculate effects dealing with two given levels.

Note that for all three calculations made the agreement compares favorably with the usual results. Thus the $3S$-$3P$ theoretical quantum-electrodynamic result and the result here compare favorably; the same is true for the $1S$-$2P$ result. However, we would point out that refinements of the model (including relativistic modification) should be made. Further refinement of the calculations, as well as extension to other effects, would follow from the time-dependent equations, (2.10) and (2.11).

It has been pointed out that in view of the fact that the calculation is nonrelativistic, the inclusion of vacuum polarization (as in Table I) is not legitimate. We can not really argue with this; it was included mainly for comparison with the procedure in Ref. 4. Also, the anomalous-magnetic-moment contribution to the Lamb shift would presumably follow in a more satisfactory treatment including spin and higher-order effects. The latter (of order α^4Z^4 Ry) may be obtained by an iterative procedure on (3.1). If K is adjusted to the $2S$-$2P$ value calculated from quantum electrodynamics without inclusion of spin, vacuum polarization or higher order effects, then one obtains values for the $1S$-$2P$ shift of 0.270 cm^{-1} and 0.0075 cm^{-1} for the $3S$-$3P$ shift. Again we would repeat that a relativistic modification is needed to deal most effectively with the above corrections.

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APPENDIX A

The integrals following from (4.5) are of the form ($x\equiv Zr/2a_0$):

$$\int_1^\infty (1-x)^2[\ln(x-1)-\ln x+x^{-2}+x^{-1}]x^2e^{-2x}dx \quad \text{(A1)}$$

and

$$\int_0^1 (1-x)^2[\ln(1-x)-\ln x+x^{-2}+x^{-1}]x^2e^{-2x}dx. \quad \text{(A2)}$$

Defining $z\equiv x-1$ in (A1) and $z\equiv 1-x$ in (A2), we obtain

$$\int_0^\infty z^2(z+1)^2(\ln z)e^{-2z-2}dz \quad \text{(A3)}$$

and

$$\int_0^1 z^2(1-z)^2(\ln z)e^{2z-2}dz, \quad \text{(A4)}$$

respectively. Using the form

$$\int_0^1 e^{2x}(2x+n+1)x^n\ln x\,dx = e^2\sum_{k=0}^{k=n}(-)^{k-1}\frac{n!}{(n-k)!2^{k+1}}+(-)^n\frac{n!}{2^{n+1}},$$

Eq. (A4) is evaluated as

$$21/16-157e^{-2}/16. \quad \text{(A5)}$$

The other integration (A3) gives

$$53e^{-2}/16-(7e^{-2}/4)(C+\ln 2), \quad \text{(A6)}$$

when

$$\int_0^\infty x^ne^{-2x}\ln x\,dx=\frac{n!}{2^{n+1}}\left[1+\tfrac{1}{2}+\cdots\frac{1}{n}-C-\ln 2\right] \quad \text{(A7)}$$

is used. Above, $C=0.58$. To (A5) and (A6) we must add

$$-\int_0^\infty x^2(1-x)^2e^{-2x}\ln x\,dx. \quad \text{(A8)}$$

This is evaluated in similar fashion to (A3) to give

$$-\tfrac{9}{16}+\tfrac{1}{4}(C+\ln 2). \quad \text{(A9)}$$

To this we must add, from (4.5),

$$\int_0^\infty (1-x)^2e^{-2x}dx=\tfrac{1}{4} \quad \text{(A10)}$$

and

$$\int_0^\infty (1-x)^2xe^{-2x}dx=\tfrac{1}{8}. \quad \text{(A11)}$$

Adding up (A5), (A6), (A9), (A10), and (A11) gives a value of 0.261. The contribution from $E_{2,2S}$ is $2\epsilon\hbar e^2Z^4/(2a_0)^3m=\frac{1}{2}K\alpha^3Z^4$ Ry, where $\epsilon=Ke^2/mc^3$ has been used, in the result which follows from (4.4). Combining this with the above result, whose coefficient is $8\epsilon\hbar Z^4e^2/(2a_0)^3m=2KZ^4\alpha^3$ Ry, we get

$$E_{1,2S}+E_{2,2S}=1.022Z^4\alpha^3K \text{ Ry}. \quad \text{(A12)}$$

The $E_{2,2P}$ calculation follows, and from (4.6) we take the expectation to get

$$\frac{2\epsilon\hbar Z^4e^2}{m(2a_0)^3}\left[\left(\frac{(2a_0)^3}{3Z^3}\right)\left\langle\frac{1}{r^3}\right\rangle_{2P}-\frac{(2a_0)^2}{2Z^2}\left\langle\frac{1}{r^2}\right\rangle_{2P}\right]$$

$$=\frac{2\epsilon\hbar Z^4e^2}{m(2a_0)^3}(\tfrac{1}{9}-\tfrac{1}{6})$$

$$=(-1/36)KZ^4\alpha^3 \text{ Ry}=-0.0278Z^4K\alpha^3 \text{ Ry}. \quad \text{(A13)}$$

Thus the total shift is given by (4.7) when we subtract (A13) from (A12). Note that the $E_{2,2P}$ correction is small compared to the $2S$ terms, as is to be expected.

The vacuum polarization given by Uehling[9] is

$$E_{2S}=-Z^2\alpha^3/15\pi \text{ Ry}=-Z^2\alpha^3(0.021) \text{ Ry}.$$

If we include the vacuum polarization, then

$$E_{2S\text{-}2P}=(1.05Z^4K\alpha^3-0.021Z^2K\alpha^3) \text{ Ry}.$$

APPENDIX B

The $1S$-$2P$ transition is obtained immediately from the result (A13) and the calculation of $E_{1,1S}$. $E_{2,1S}$ follows directly from (4.4) as $E_{2,1S}=4KZ^4\alpha^3$ Ry:

$$E_{1,1S}=-\left\langle\frac{\epsilon\hbar Z^2e^2}{(ma_0)}\int\frac{d}{dr}\left(\frac{1}{r^2}\right)dr\right\rangle_{1S}$$

$$=(\epsilon he^2Z^2/ma_0)\langle 1/r^2\rangle_{1S}$$

$$=4KZ^4\alpha^3 \text{ Ry}.$$

Thus

$$E_{1S\text{-}2P}=(8.00+0.028)KZ^4\alpha^3 \text{ Ry}.$$

The vacuum polarization estimate[9] is $-8Z^2\alpha^3/15\pi$ Ry $=-0.170Z^2\alpha^3$ Ry. Thus *with* this term we get (4.8) for $E_{1S\text{-}2P}$, for hydrogen. In general,

$$E_{1S\text{-}2P}=8.03Z^4\alpha^3-0.170Z^2K\alpha^3 \text{ Ry}. \quad \text{(B1)}$$

APPENDIX C

The wave functions used in the $3S$-$3P$ calculation are

$$\psi_{3Pz}=(\sqrt{2}/81\sqrt{\pi})(Z/a_0)^{3/2}(6-Zr/a_0)\times(Zr/a_0)e^{-Zr/3a_0}\cos\theta \quad \text{(C1)}$$

and

$$\psi_{3S}=[1/81\sqrt{(3\pi)}](Z/a_0)^{3/2}[27-18(Zr/a_0)+2(Zr/a_0)^2]e^{-Zr/3a_0}. \quad \text{(C2)}$$

Again $E_{2,3S}=0.148Z^4K\alpha^3$ Ry, directly from (4.4), using (C2).

The integral corresponding to (4.2) for the $3S$ state is

$$\frac{2\epsilon\hbar Z^4e^2}{m(a_0)^3}\int\left(\frac{-18+4y}{27-18y+2y^2}-\frac{1}{3}\right)\frac{dy}{y^3},$$

where $y=Zr/a_0$. The integral above leads to

$$-(6/81)\ln[y^2/(27-18y+2y^2)]-(10/81\sqrt{3})\ln[(-9+2y-3\sqrt{3})/(-9+2y+3\sqrt{3})]+8/27y+1/2y^2. \quad \text{(C3)}$$

The last two terms in (C3) give

$$(8/27)\langle 1/y\rangle_{3S}+\tfrac{1}{2}\langle 1/y^2\rangle_{2S}=17/9(27). \quad \text{(C4)}$$

If we set $x\equiv 2y$, and observe that the logarithm in the first term in (C3) may be written

$$2\ln x-\ln 2-\ln(-9+3\sqrt{3}+x)-\ln(-9-3\sqrt{3}+x), \quad \text{(C5)}$$

we have integrals of the type given previously in (A3) and (A4) [for the latter two logarithms in (C5)] and the first in (C5) is of the type given in (A7). The total contribution from the first two terms of (C5) is

$$-96/243-(6/81)[(2\ln 3-\ln 2)-2C]. \quad \text{(C6)}$$

To evaluate the other ln terms in the manner of (A3) and (A4), we make a transformation $\zeta=\frac{1}{3}x$. Under this transformation $\ln(-9\pm 3\sqrt{3}+x)=\ln 3+\ln(-\sqrt{3}\pm\sqrt{3}+\zeta)$. The ln3 term integrates to

$$(6/81)2\ln 3. \quad \text{(C7)}$$

The remaining logarithm integrands in (C3) become, after using (C5),

$$[(6\mp 10/\sqrt{3})/81]\exp(-3\mp\sqrt{3})\int_0^\infty \eta^2(\eta\pm 2\sqrt{3})^2\times(\eta+3\pm\sqrt{3})^2e^{-\eta}\ln\eta\,d\eta, \quad \text{(C8)}$$

where $\eta\equiv\zeta-3\mp\sqrt{3}$, and

$$[(6\mp 10/\sqrt{3})/81]\exp(-3\mp\sqrt{3})\int_0^1 \xi^2(\xi-1)^2\times\left(\xi\mp\frac{2\sqrt{3}}{3\pm\sqrt{3}}\right)^2e^{(3\pm\sqrt{3})\xi}\ln\xi\,d\xi, \quad \text{(C9)}$$

where $\xi\equiv 1-\zeta/(3\pm\sqrt{3})$. It is seen that (C8) is of the form (A3) while (C9) is of the form (A4), except that the order of the polynomial is higher in the integrand.

(C8) becomes, when $3S$-function normalization is included,

$$21.5/72=0.299. \quad \text{(C10)}$$

The other form, (C9), leads (after lengthy, but straightforward calculation) to

$$-0.0768. \quad \text{(C11)}$$

Adding up (C6), (C7), (C10), and (C11) gives

$$-96/243+(6/81)\ln 2+(12/81)C+0.299-0.0768=-0.395+0.0514+0.0859+0.299-0.0768=-0.0355. \quad \text{(C12)}$$

When (C12) is added to (C4) the result leads to $+0.0345$, which gives finally

$$E_{1,3S}=[2\epsilon he^2Z^4/m(2a_0)^3](8\times 0.0345)=0.138KZ^4\alpha^3 \text{ Ry}. \quad (4.11)$$

We finally compute the contribution of $E_{1,3P}$ using (C1). The integral corresponding to (4.2) for the $3P$ state leads to

$$[2\epsilon\hbar Z^4e^2/m(6a_0)^3]\{-\tfrac{1}{3}(6a_0/rZ)^3+\tfrac{3}{2}(6a_0/Zr)^2+(6a_0/Zr)+\ln[(6a_0/Zr)-1]\}, \quad r<6a_0Z^{-1} \quad \text{(C13)}$$

and

$$[2\epsilon\hbar Z^4e^2/m(6a_0)^3]\{-\tfrac{1}{3}(6a_0/Zr)^2+\tfrac{3}{2}(6a_0/Zr)^2+(6a_0/Zr)+\ln[1-(6a_0/Zr)]\}, \quad r>6a_0Z^{-1}.$$

The first three terms in (C13) give directly, upon taking

the expectation,

$$[2\epsilon\hbar Z^4e^2/m(2a_0)^3](1/27)[(-8/9)+(4/3)+(6/9)]$$
$$=[5/9(27)]Z^4K\alpha^3\ \mathrm{Ry}=0.0206Z^4K\alpha^3\ \mathrm{Ry}. \quad \text{(C14)}$$

Defining $x\equiv(Zr/6a_0)$, the angular integration gives $\frac{4}{3}\pi$ while the r integrals are of the form encountered previously in the $E_{1,3S}$ calculation:

$$e^{-4}\int_0^\infty (\ln x)(x+1)^4x^2\,e^{-4x}dx=0.00196, \quad \text{(C15)}$$

$$-\int_0^\infty (\ln x)x^4(1-x)^2e^{-4x}dx=-0.015, \quad \text{(C16)}$$

and

$$e^{-4}\int_0^1 (\ln x)(x-1)^4x^2e^{4x}dx=0.035. \quad \text{(C17)}$$

Adding (C14)–(C17) gives, with the coefficient,

$$E_{1,3P}=[0.0378+0.0206]KZ^4\alpha^3\ \mathrm{Ry}$$
$$=0.0584KZ^4\alpha^3\ \mathrm{Ry}=0.00248K\ \mathrm{cm}^{-1}. \quad (4.10)$$

Taking into account (4.9)–(4.11),

$$E_{3S\text{-}3P}=[0.148+0.138-0.0584]KZ^4\alpha^3\ \mathrm{Ry}$$
$$=0.228KZ^4\alpha^3\ \mathrm{Ry}=0.00977KZ^4\ \mathrm{cm}^{-1}. \quad (4.12)$$

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Nonrelativistic Quantum Mechanics for Particles with Arbitrary Spin*†

WILLIAM J. HURLEY
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
and
Department of Physics, Syracuse University, Syracuse, New York 13210‡

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Motivated by the difficulties which persist in the construction of a consistent description of interacting relativistic particles with spin greater than 1, we examine the analogous problem within the framework of nonrelativistic quantum mechanics and find that all of the difficulties vanish in the (Galilei-invariant) limit. It is found that a unique, first-order, Galilei-covariant wave equation describing massive, spin-s particles follows from general invariance assumptions and a minimality condition on the number of components of the wave function. The minimal theory has $6s+1$ components ($2s+1$ of which are independent) and admits a consistent quantum-mechanical interpretation. An external electromagnetic field interaction is introduced via the minimal-coupling replacement, and, in contrast with the relativistic case, a consistent theory emerges for arbitrary spin. The Galilean spin-s particles so described are found to possess only an electric charge and a magnetic dipole with a g factor of $1/s$. The Galilei-invariant addition of arbitrary moment terms is also described. The extension of the formalism to second-quantized spin-s fields is discussed, and it is found that in that case, too, the difficulties are peculiar to the relativistic case. Reasons for the simplicity of the Galilei case are presented. Finally, for the sake of completeness, the first-order form of the Schrödinger theory is presented, as well as examples of theories which violate the minimality condition.

I. INTRODUCTION

DESPITE more than thirty years of attention, there is still no satisfactory description of interacting relativistic particles with spin greater than 1.[1] Motivated by this classic problem, we ask whether the difficulties persist in the nonrelativistic limit, i.e., we seek wave equations which describe particles with arbitrary spin, which are Galilei-covariant, and which consistently admit the introduction of interactions. Thus we seek the higher-spin wave equations of nonrelativistic quantum mechanics.

The symmetry group of nonrelativistic quantum mechanics is the Galilei group defined by the transformations of space and time, $(b,\mathbf{a},\mathbf{v},R)$, such that

$$\mathbf{x}'=R\mathbf{x}+\mathbf{v}t+\mathbf{a}, \quad t'=t+b,$$

where $\mathbf{a}$ (b) is a space (time) translation, R is a space rotation matrix, and $\mathbf{v}$ is the relative velocity of the primed and unprimed frames. The specific manner in which quantum mechanics realizes this group became understood only long after the wave equation was originally introduced.[2] Likewise, the classification of the appropriate representations and their specific realization in

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‡ Present address.

[1] The following papers may serve as a brief introduction to the considerable literature on this subject: G. Velo and D. Zwanziger, Phys. Rev. **186**, 1337 (1969); **188**, 2218 (1969); A. S. Wightman, *Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energy*, edited by A. Perlmutter, C. A. Hurst, and B. Kurşunoğlu (Benjamin, New York, 1968); B. Schroer, R. Seiler, and J. A. Swieca, Phys. Rev. D **2**, 2927 (1970); M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939); and P. A. M. Dirac, *ibid.* **A155**, 447 (1936).

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