count, the perihelion precession of Mercury about a spherical sun would differ from that predicted by general relativity, one can modify the present formalism to get agreement with observation. In determining the parameters β and γ, let us require that, in the case of a weak, static field with T_{kk} negligible, Eq. (13) be replaced by

$$\psi=-\lambda\phi, \tag{29}$$

with λ constant $(0<\lambda<1)$. This, together with Eq. (12), leads to the relations

$$\beta=\lambda^2\alpha+2, \quad \gamma=2\lambda\alpha. \tag{30}$$

The field equations are now somewhat more complicated and will not be given here. It is found that the static, spherically symmetric solution for empty space has the form

$$\begin{aligned}\Phi&=1-m/r+\tfrac{1}{4}(1+\lambda)m^2/r^2+\cdots,\\ \Psi&=1+\lambda m/r+\cdots,\end{aligned} \tag{31}$$

where Eq. (29) has been taken into account in first order.

Using this solution, one finds that the precession of the perihelion of a planet now depends on λ. If the angular velocity of precession is denoted by $\omega(\lambda)$, so that $\omega(1)$ is that given by general relativity, one obtains

$$\Delta\equiv[\omega(1)-\omega(\lambda)]/\omega(1)=(7/12)(1-\lambda). \tag{32}$$

For example, for $\lambda=6/7$, $\Delta=\frac{1}{12}=0.083$.

The value of λ also determines the deflection of light by the sun; one now gets a value which is $\frac{1}{2}(1+\lambda)$ times that given by general relativity. On the other hand, as one sees immediately from Eq. (31), the gravitational red shift, which depends only on g_{00}, is (in first order) independent of λ.

PHYSICAL REVIEW D VOLUME 3, NUMBER 10 15 MAY 1971

Scattering of Electromagnetic and Gravitational Waves by a Static Gravitational Field: Comparison Between the Classical (General-Relativistic) and Quantum Field-Theoretic Results*

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(Received 7 December 1970)

The classical general-relativistic cross sections for the scattering of either an electromagnetic wave or a gravitational wave by a scalar particle are calculated and found to agree with the results of the quantized linearized field theory.

I. INTRODUCTION

The quantum-mechanical cross section for the scattering of light by light has recently been shown to be equal to the classical cross section, in the limit of large impact parameter.[1] The quantum-mechanical cross section for the scattering of light by a boson was also shown to equal the classically calculated cross section,[1] in the limit of large impact parameter. In this paper the classical calculation of the light-by-boson scattering cross section is extended so as to apply for all scattering angles. The result agrees exactly with the quantum-mechanical result.[1] The classical calculation of the scattering of a graviton by a boson is also shown to agree with the quantum-mechanical result[2] for all scattering angles.

This latter result has a bearing on Weber's analysis of the mass quadrupole gravitational wave detector. Weber obtains a finite cross section of the order of λ^2 for an oscillator damped solely by radiation resistance.[3] Here λ is the wavelength of the incident gravitational radiation. The calculation presented in this paper shows the cross section of a single point mass to be infinite, a not too surprising result in view of the long-range nature of the gravitational interaction. Thus in order to obtain the total cross section, assuming Weber's calculation to be valid insofar as the quadrupole contribution is concerned, Weber's cross section should be augmented by the infinite one obtained herein. The validity of Weber's quadrupole cross section is a topic I will treat in a forthcoming paper.

In Sec. II the wave equation in flat space for the electromagnetic four-potential in the presence of a weak, but otherwise arbitrary, gravitational perturbation is derived. This equation is simplified for the case when the gravitational perturbation is static and Newtonian. An alternative derivation of this wave equation based

* Work performed under the auspices of the U. S. Atomic Energy Commission (Report No. NYO-2262-TA-222).

[1] P. J. Westervelt and L. F. Karr, Nuovo Cimento 66B, 129 (1970).

[2] B. S. DeWitt, Phys. Rev. 162, 1239 (1967).

[3] J. Weber, *General Relativity and Gravitational Waves* (Interscience, New York, 1961).

on a suggestion by Landau and Lifshitz[4] is given in Appendix B.

The photon-boson cross section is obtained in Sec. III. In Sec. IV the exact wave equation in flat space is obtained for the gravitational potential. This equation is simplified so as to apply to gravitational wave propagation in a static Newtonian background field. The graviton-boson cross section is obtained in Sec. V.

All notation and symbols with the exception of the gravitational constant, herein referred to as G, conform with Landau and Lifshitz's *The Classical Theory of Fields*.[4] In particular Latin indices refer to four-space; Greek indices refer to three-space with metric $\gamma^{\alpha\beta}$.

II. WAVE EQUATION FOR VECTOR POTENTIAL

In the absence of real sources the electromagnetic potential A_i and field tensor F^{ik} satisfy the equations

$$F_{lm}=A_{m,l}-A_{l,m}, \tag{1}$$

$$F^{ik}{}_{;k}=0. \tag{2}$$

These equations may be combined and rewritten with the help of the metric tensor g_{ik}:

$$[(\sqrt{-g})g^{il}g^{km}(A_{m,l}-A_{l,m})]_{,k}=0. \tag{3}$$

Linearization is now achieved in terms of the flat-space metric ${}^0g_{ik}=(1,\,1,\,1,\,-1)$ and the weak-field gravitational potentials ψ^{ik}. Thus Eq. (3) becomes

$$[(1+\tfrac{1}{2}\psi)\,{}^0g^{\alpha\beta 0}g^{km}(A_{m,\beta}-A_{\beta,m})]_{,k} =[({}^0g^{\alpha l}\psi^{km}+{}^0g^{km}\psi^{\alpha l})(A_{m,l}-A_{l,m})]_{,k}, \tag{4}$$

in which $g^{il}={}^0g^{il}(1+\frac{1}{2}\psi)-\psi^{il}$.

Next Eq. (4) is expanded in terms of the three dimensional flat-space curl, the vector potential $\mathbf{A}$, for which $(\mathbf{A})_\alpha=A_\alpha$, and A_0 the timelike component of A_i,

$$\begin{aligned}
&[\nabla\times(1+\tfrac{1}{2}\psi)\nabla\times\mathbf{A}]^\alpha+\frac{1}{c^2}\frac{\partial}{\partial t}\left[(1+\tfrac{1}{2}\psi)\frac{\partial\mathbf{A}}{\partial t}\right]^\alpha\\
&=[(1+\tfrac{1}{2}\psi)\,{}^0g^{\alpha\beta}A_{0,\beta}]_{,0}+[{}^0g^{\alpha\beta}\psi^{\lambda\nu}(A_{\nu,\beta}-A_{\beta,\nu})]_{,\lambda}\\
&\quad+[{}^0g^{\alpha\beta}\psi^{0\nu}(A_{\nu,\beta}-A_{\beta,\nu})]_{,0}\\
&\quad+[{}^0g^{\alpha\beta}\psi^{\nu 0}(A_{0,\beta}-A_{\beta,0})]_{,\nu}\\
&\quad+[{}^0g^{\alpha\beta}\psi^{00}(A_{0,\beta}-A_{\beta,0})]_{,0}\\
&\quad+[{}^0g^{km}\psi^{\alpha\beta}(A_{m,\beta}-A_{\beta,m})]_{,k}\\
&\quad+[{}^0g^{km}\psi^{\alpha 0}(A_{m,0}-A_{0,m})]_{,k}.
\end{aligned} \tag{5}$$

This general equation simplifies considerably in two cases of physical interest. First, in order to study the scattering of plane electromagnetic waves by plane gravitational waves, we may impose the following conditions on the potentials:

$$A_0=A^\alpha{}_{,\alpha}=\psi^{00}=\psi^{i0}=\psi=\psi^{ik}{}_{,k}=0.$$

In this case Eq. (5) becomes

$$\Box A^\alpha=[{}^0g^{\alpha\beta}\psi^{\lambda\nu}(A_{\nu,\beta}-A_{\beta,\nu})]_{,\lambda} +[{}^0g^{km}\psi^{\alpha\beta}(A_{m,\beta}-A_{\beta,m})]_{,k}, \tag{6}$$

in which $\Box$ stands for the flat-space d'Alembertian. No further reference to this result occurs in this paper.

In the scattering of plane electromagnetic waves by a static Newtonian gravitational field, we may impose the following conditions on the potentials: $A_0=0$; all ψ^{ik} except ψ^{00} are zero; and ψ^{00} is time independent. In this case Eq. (5) becomes

$$\Box\mathbf{A}=\frac{1}{2}\frac{\psi^{00}}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\tfrac{1}{2}\nabla\times(\psi^{00}\times\mathbf{A}). \tag{7}$$

Inasmuch as the right-hand side of this equation is of first order in ψ^{00}, the electric and magnetic fields may be approximated by

$$\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\quad\text{and}\quad\mathbf{H}=\nabla\times\mathbf{A},$$

respectively, and then introduced in place of the vector potential. Also ψ^{00} may be expressed in terms of the Newtonian potential ϕ:

$$\psi^{00}=-4\phi/c^2.$$

Upon making these substitutions, Eq. (7) may be written as follows:

$$\Box\mathbf{A}=-(4\pi/c)\mathbf{j}, \tag{8}$$

in which $\mathbf{j}$ represents a virtual source distribution corresponding to the following ficticious current density

$$\mathbf{j}=\nabla\times\left(-\frac{\phi}{2\pi c}\mathbf{H}\right)+\frac{\partial}{\partial t}\left(-\frac{\phi}{2\pi c^2}\mathbf{E}\right). \tag{9}$$

In complete analogy with the theory of electrodynamics of continuous media, we may consider this ficticious current density as constituting a vacuum polarization for which $\mathbf{p}$ and $\mathbf{m}$, the equivalent induced electric and magnetic dipole moments per unit volume, respectively, are given by

$$\mathbf{p}=-(\phi/2\pi c^2)\mathbf{E}\quad\text{and}\quad\mathbf{m}=-(\phi/2\pi c^2)\mathbf{H}. \tag{10}$$

This result is compatible with an electric and magnetic "permeability" $\epsilon=\mu=(1-2\phi/c^2)$, which ought not be confused with the permeability defined by Landau and Lifshitz[5] in a related problem discussed in Appendix B.

Finally, combining Eqs. (8)–(10) allows the wave equation to be written:

$$\Box\mathbf{A}=-\frac{4\pi}{c}\left(\nabla\times\mathbf{m}+\frac{1}{c}\frac{\partial\mathbf{p}}{\partial t}\right). \tag{11}$$

[4] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 2nd ed. (Addison-Wesley, Reading, Mass., 1962).

[5] See problem 3, p. 296 of Ref. 4.

III. SCATTERING OF A PLANE ELECTROMAGNETIC WAVE BY A STATIC GRAVITATIONAL FIELD

In applying Eq. (11) of the preceding section, $\mathbf{p}$ and $\mathbf{m}$ are obtained from the known incident electric and magnetic fields. The vector potential for the scattered wave in the wave zone is given by the well known solution to Eq. (11),

$$\mathbf{A}=\frac{1}{Rc}\int[\dot{\mathbf{p}}+\times\dot{\mathbf{m}}\times\mathbf{n}]_{t'}dv, \tag{12}$$

in which dots correspond to time differentiation, $\mathbf{n}=\mathbf{R}/R$, and $\mathbf{R}$ is the vector originating in the scattering region and terminating at the field point. The electric and magnetic fields of the scattered wave in the radiation zone are

$$\mathbf{E}=(1/c)(\dot{\mathbf{A}}\times\mathbf{n})\times\mathbf{n} \tag{13}$$

and

$$\mathbf{H}=(1/c)\dot{\mathbf{A}}\times\mathbf{n}, \tag{14}$$

respectively. If the incident plane wave propagates in the z direction, Eq. (10) ensures the relation

$$\mathbf{m}=\mathbf{z}\times\mathbf{p}, \tag{15}$$

where $\mathbf{z}$ is a unit vector in the z direction.

Combining Eqs. (12), (13), and (15) permits the scattered electric field to be expressed entirely in terms of the retarded second time derivative of the induced electric dipole moment density:

$$\begin{aligned}\mathbf{E}&=\frac{1}{Rc^2}\int[\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{p}})+\mathbf{n}\times(\mathbf{z}\times\ddot{\mathbf{p}})]_{t'}dv\\&=\frac{1}{Rc^2}\int[(\mathbf{n}\cdot\ddot{\mathbf{p}})(\mathbf{n}+\mathbf{z})-(1+\mathbf{n}\cdot\mathbf{z})\ddot{\mathbf{p}}]_{t'}dv.\end{aligned} \tag{16}$$

Notice that there is no backscattering; i.e., Eq. (16) shows that $\mathbf{E}=0$ when $\mathbf{n}=-\mathbf{z}$. This means that the characteristic wave impedance of the vacuum is unchanged by the static gravitational field.[6]

The scattered time-averaged intensity $\langle dI\rangle$ per unit solid angle $d\Omega$ is

$$\left\langle\frac{dI}{d\Omega}\right\rangle=\frac{cR^2}{4\pi}\langle E^2\rangle. \tag{17}$$

Combining Eqs. (16) and (17) along with the fact that $\mathbf{z}\cdot\mathbf{p}=0$ and the definition $\cos\theta=\mathbf{z}\cdot\mathbf{n}$, it is easy to show that

$$\left\langle\frac{dI}{d\Omega}\right\rangle=\frac{(1+\cos\theta)^2}{4\pi c^3}\left\langle\left[\int\ddot{\mathbf{p}}_{t'}dv\right]^2\right\rangle. \tag{18}$$

With no loss of generality the incident wave may be assumed plane polarized and characterized by the electric field $\mathbf{E}_{\text{inc}}$:

$$\mathbf{E}_{\text{inc}}=\mathbf{E}_0e^{ikz-i\omega t}, \tag{19}$$

in which $\mathbf{E}_0$ is constant and real. Furthermore

$$\mathbf{p}=-(\phi/2\pi c^2)\mathbf{E}_{\text{inc}}. \tag{20}$$

With the help of Eqs. (19) and (20), Eq. (18) becomes

$$\left\langle\frac{dI}{d\Omega}\right\rangle=\frac{\omega^4(1+\cos\theta)^2}{16\pi^3c^3}E_0^2\left\langle\left[\mathrm{Re}\int\left(\frac{\phi}{c^2}e^{ikz-i\omega t}\right)_{t'}dv\right]^2\right\rangle. \tag{21}$$

The average flux of energy $\langle S_z\rangle$ of the incident wave is

$$\langle S_z\rangle=(c/4\pi)\langle E^2\rangle=(c/8\pi)E_0^2. \tag{22}$$

The differential scattering cross section $d\sigma$ is given by the ratio of the two preceding equations

$$d\sigma=\frac{1}{\langle S_z\rangle}\left\langle\frac{dI}{d\Omega}\right\rangle=\frac{\omega^4(1+\cos\theta)^2}{2\pi^2c^4}\langle F^2\rangle, \tag{23}$$

in which

$$F=\mathrm{Re}\int\left(\frac{\phi}{c^2}e^{ikz-i\omega t}\right)_{t'}dv. \tag{24}$$

This result is valid for arbitrary Newtonian potential ϕ. In case ϕ arises from a single point mass m, the expression

$$\phi=-mG/r$$

may be introduced into Eq. (24) to give

$$F=\mathrm{Re}\left(\frac{-mGe^{-i\omega t}}{c^2}\int\frac{e^{ik(z+|\mathbf{r}-\mathbf{r}'|)}}{r'}dv'\right). \tag{25}$$

The integral in the above expression is the Born approximation to Rutherford scattering, from which fact it is easy to show[7] that

$$F=\mathrm{Re}\left[\frac{-mG\pi e^{-i\omega t}}{\omega^2\sin^2(\frac{1}{2}\theta)}\right] \tag{26}$$

and

$$\langle F^2\rangle=\frac{2m^2G^2\pi^2}{\omega^4(1-\cos\theta)^2}. \tag{27}$$

Combining this result with Eq. (23), the scattering cross section for electromagnetic waves scattered by the gravitational field of a point mass is found to be

$$d\sigma=\frac{m^2G^2}{c^4}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^2, \tag{28}$$

a value in exact agreement with the quantum-mechanical result first correctly displayed by Westervelt and Karr.[1]

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[6] This result is at variance with the claim in a recent paper [K. Nordtvedt, Jr., Phys. Rev. **186**, 1352 (1969)] that back reflection does occur for vector waves in a static gravitational field.

[7] M. Born, *Atomic Physics*, 5th ed. (Blackie, London, 1951).

IV. WAVE EQUATION FOR GRAVITATIONAL POTENTIAL

The Landau-Lifshitz energy-momentum complex $\Theta_L{}^{ik}$ is well known to be given by

$$\Theta_L{}^{ik}=(-g)(T^{ik}+t_L{}^{ik}) \\ =(c^4/16\pi G)[(-g)(g^{ik}g^{lm}-g^{il}g^{km})]_{,ml}. \quad (29)$$

What is not so well appreciated is the fact that Eq. (29) is entirely equivalent to the exact field equations of general relativity. This feature may be clarified by introducing the variables

$$\Psi^{ik}={}^0g^{ik}-\phi^{ik}, \quad \phi^{ik}=(\sqrt{-g})g^{ik},$$

with the requirement that Ψ^{ik} satisfies the de Donder condition $\Psi^{ik}{}_{,k}=0$. With these substitutions the exact field equation now becomes[8]

$$\Box\Psi^{ik}=-(16\pi G/c^4)\Theta_L{}^{ik}-\Psi^{il}{}_{,m}\Psi^{km}{}_{,l}+\Psi^{lm}\Psi^{ik}{}_{,lm}, \quad (30)$$

in which $\Box$ is the flat-space d'Alembertian. This elegant result is exact and valid without limits on the "strength" of Ψ^{ik}. For our purpose it suffices to consider weak fields ψ^{ik} in the absence of real sources, thus $T^{ik}=0$. In this case the wave equation becomes

$$\Box\psi^{ik}=-(16\pi G/c^4)t_L{}^{ik}-\psi^{il}{}_{,m}\psi^{km}{}_{,l}+\psi^{lm}\psi^{ik}{}_{,lm}, \quad (31)$$

in which $t_L{}^{ik}$ is the weak-field approximation of the Landau-Lifshitz pseudotensor density

$$t_L{}^{ik}=(c^4/16\pi G)[\tfrac{1}{2}\psi_{pq}{}^{,i}\psi^{pq,k}-\tfrac{1}{4}\psi^{,i}\psi^{,k} \\ -\psi^i{}_{m,n}\psi^{mn,k}-\psi^k{}_{m,p}\psi^{mp,i}+\psi^i{}_{m,n}\psi^{km,n} \\ +{}^0g^{ik}(\tfrac{1}{8}\psi_{,m}\psi^{,m}-\tfrac{1}{4}\psi_{pq,m}\psi^{pq,m}+\tfrac{1}{2}\psi^n{}_{m,p}\psi^{mp}{}_{,n})]. \quad (32)$$

The gravitational wave is characterized by having $\psi^{0\alpha}=\psi^{00}=\psi_\alpha{}^\alpha=0$. This wave is scattered by a static field characterized by $\psi^{00}=-4\phi/c^2$. We thus will retain on the right-hand side of the field equation only terms bilinear in $\psi^{00}{}_{(\mathrm{static})}$ and $\psi^{\alpha\beta}{}_{(\mathrm{wave})}$. This enables us to further simplify the field equation

$$\Box\psi^{\alpha\beta}=(1/c^2)\psi^{00}\ddot{\psi}^{\alpha\beta}. \quad (33)$$

This wave equation may also be written

$$\Box\psi^{\alpha\beta}=-(16\pi G/c^4)\tau^{\alpha\beta}, \quad (34)$$

in which $\tau^{\alpha\beta}$ represents a virtual source distribution corresponding to the following stress:

$$\tau^{\alpha\beta}=(1/4\pi G)\phi\ddot{\psi}^{\alpha\beta}. \quad (35)$$

This stress distribution can be thought of as a vacuum-polarization equivalent to the quadrupole moment density $q^{\alpha\beta}$ given by

$$q^{\alpha\beta}=(3/2\pi G)\phi\psi^{\alpha\beta}. \quad (36)$$

V. SCATTERING OF A PLANE GRAVITATIONAL WAVE BY A STATIC GRAVITATIONAL FIELD

In applying Eq. (34) of the preceding section, $\tau^{\alpha\beta}$ is obtained through Eq. (35) from the known incident field components. The field components for the scattered wave in the wave zone is given by the well-known solution to Eq. (34):

$$\psi^{\alpha\beta}=\frac{4G}{Rc^4}\int[\tau^{\alpha\beta}]_{t'}dv. \quad (37)$$

Assuming a time dependence of $e^{-i\omega t}$ for the incident wave $\psi^{\alpha\beta}{}_{\mathrm{inc}}$ and referring to Eqs. (35) and (36) permits the scattered field to be expressed in terms of the incident field

$$\psi^{\alpha\beta}=-\frac{\omega^2}{\pi Rc^4}\int[\phi\psi^{\alpha\beta}{}_{\mathrm{inc}}]_{t'}dv. \quad (38)$$

We assume the incident wave is circularly polarized and of the form

$$\psi^{\alpha\beta}{}_{\mathrm{inc}}={}_0\psi^{\alpha\beta}{}_{\mathrm{inc}}e^{i(kz-\omega t)},$$

with (39)

$${}_0\psi^{\alpha\beta}{}_{\mathrm{inc}}=(\delta^{\alpha1}\delta^{\beta1}-\delta^{\alpha2}\delta^{\beta2})\pm i(\delta^{\alpha1}\delta^{\beta2}+\delta^{\alpha2}\delta^{\beta1}).$$

The incident wave traveling in the $x^3=z$ direction has the components

$$\psi^{11}{}_{\mathrm{inc}}=-\psi^{22}{}_{\mathrm{inc}}=\mp i\psi^{12}{}_{\mathrm{inc}}=\mp i\psi^{21}{}_{\mathrm{inc}}.$$

The scattered intensity per unit solid angle is given by the following expression, valid when $\psi_\alpha{}^\alpha=0$:

$$\frac{dI}{d\Omega}=\frac{c^3R^2}{64\pi G}[(\dot{\psi}_{\alpha\beta}n^\alpha n^\beta)^2+2\dot{\psi}_{\alpha\beta}\dot{\psi}^{\alpha\beta}-4\dot{\psi}_{\alpha\beta}\dot{\psi}^\alpha{}_\nu n^\beta n^\nu]. \quad (40)$$

Calculating the time averages gives

$$\langle(\dot{\psi}_{\alpha\beta}n^\alpha n^\beta)^2\rangle=[\tfrac{1}{2}(n_1{}^4+n_2{}^4)+n_1{}^2n_2{}^2]B\langle F^2\rangle, \\ \langle 2\dot{\psi}_{\alpha\beta}\dot{\psi}^{\alpha\beta}\rangle=4B\langle F^2\rangle, \\ \langle -4\dot{\psi}_{\alpha\beta}\dot{\psi}_\nu{}^\alpha n^\beta n^\nu\rangle=-4(1-n_3{}^2)B\langle F^2\rangle,$$

where $\mathbf{n}$ is the unit vector from the scatterer to the field point, $B=\omega^6/\pi^2R^2c^4$, and

$$F=\mathrm{Re}\left\{\int\left[\frac{\phi}{c^2}e^{i(kz-\omega t)}\right]_{t'}\right\}dv.$$

Thus

$$\langle(\dot{\psi}_{\alpha\beta}n^\alpha n^\beta)^2+2\dot{\psi}_{\alpha\beta}\dot{\psi}^{\alpha\beta}-4\dot{\psi}_{\alpha\beta}\dot{\psi}^\alpha{}_\nu n^\beta n^\nu\rangle \\ =B\langle F^2\rangle[(n_1{}^2+n_2{}^2)^2+8n_3{}^2] \\ =B\langle F^2\rangle(\sin^4\theta+8\cos^2\theta) \\ =8B\langle F^2\rangle[\sin^8(\tfrac{1}{2}\theta)+\cos^8(\tfrac{1}{2}\theta)], \quad (41)$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{z}$.

[8] This result appears in P. J. Westervelt, Brown University Report No. NYO-2262-TA-218 (unpublished). In the event the de Donder condition is not satisfied, a more general expression results [H. Hurwitz, Brown University Lecture Notes for Physics, 1969, p. 210 (unpublished)], which is

$$\Box\Psi^{ik}=-(16\pi Gc^{-4})\theta^{ik}-\Psi^{il}{}_{,m}\Psi^{km}{}_{,l} \\ +\Psi^{lm}\Psi^{ik}{}_{,lm}-\Psi^{lm}{}_{,lm}({}^0g^{ik}-\Psi^{ik}) \\ +2\Psi^{lm}{}_{,l}\Psi^{ik}{}_{,m}+\Psi^{il}{}_{,lm}({}^0g^{km}-\Psi^{km}) \\ -\Psi^{il}{}_{,l}\Psi^{km}{}_{,m}+({}^0g^{il}-\Psi^{il})\Psi^{km}{}_{,lm}.$$

The average scattered intensity obtained by combining Eqs. (40) and (41) is

$$\left\langle\frac{dI}{d\Omega}\right\rangle=\frac{\omega^6}{8\pi^3Gc}[\sin^8(\tfrac{1}{2}\theta)+\cos^8(\tfrac{1}{2}\theta)]\langle F^2\rangle. \quad (42)$$

The average flux of energy $\langle ct^{03}\rangle$ in the incident wave is

$$\langle ct^{03}\rangle=(c^3/16\pi G)\langle(\dot\psi_{\mathrm{inc}}{}^{12})^2+\tfrac{1}{4}(\dot\psi_{\mathrm{inc}}{}^{11}-\dot\psi_{\mathrm{inc}}{}^{22})^2\rangle$$
$$=(\omega^2c^3/16\pi G). \quad (43)$$

The differential scattering cross section $d\sigma$ is given by the ratio of the two preceding equations:

$$d\sigma=\frac{1}{\langle ct^{03}\rangle}\left\langle\frac{dI}{d\Omega}\right\rangle=\frac{2\omega^4}{\pi^2c^4}[\sin^8(\tfrac{1}{2}\theta)+\cos^8(\tfrac{1}{2}\theta)]\langle F^2\rangle, \quad (44)$$

in which F is given by Eq. (24). This result is valid for arbitrary Newtonian potential ϕ. In case ϕ arises from a single point mass m we may determine from Eq. (27) that

$$\langle F^2\rangle=m^2G^2\pi^2/2\omega^4\sin^4(\tfrac{1}{2}\theta). \quad (45)$$

Combining this result with Eq. (44), the scattering cross section for gravitational waves scattered by the gravitational field of a point mass is found to be

$$d\sigma=\frac{m^2G^2[\sin^8(\tfrac{1}{2}\theta)+\cos^8(\tfrac{1}{2}\theta)]}{c^4\sin^4(\tfrac{1}{2}\theta)}, \quad (46)$$

a value in exact agreement with the quantum-mechanical result.[2]

It is clear from this result that the total cross section is infinite, which comes as no surprise in view of the long-range nature of the scattering potential.

In concluding this section let us speculate upon an astrophysical implication of Eq. (46). Suppose the scattering potential is the field of a neutron star. Our result should be valid for gravitational wavelength one order of magnitude larger than the star's radius, that is for $\lambda_G\geq 100$ km which corresponds to frequencies below about 3 kHz. Thus it appears that a neutron star is capable of focusing low-frequency gravitational waves in both the forward direction as well as, although rather weakly, in the backward direction. From the results of Sec. III we can now also say that a neutron star is capable of focusing low-frequency (less than 3 kHz) electromagnetic waves in the forward direction only.

Finally we list some corrollaries of this paper and of a previous study. (a) In the short-wavelength limit when $\lambda\ll\phi|\nabla\phi|^{-1}$, Eq. (7) becomes

$$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\mathbf{A}=0, \quad (47)$$

in which $v=c(1+2\phi/c^2)$. This value of v for the velocity of light in a static gravitational field, leads to the correct result for both the Shapiro radar time delay and the bending of light. (b) Since Eq. (33) can be expressed as

$$\nabla^2\psi^{\alpha\beta}-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi^{\alpha\beta}=0, \quad (48)$$

with the velocity v being identical with that in Eq. (47), it follows that a gravitational wave will suffer the same deflection and time delay as does an electromagnetic wave. (c) Since the pseudotensor for a graviton t^{ik} has the same form as the energy-momentum tensor for a photon T^{ik}, we can use Eq. (31), the equation of motion for a graviton

$$[(\sqrt{-g})t^{ik}]_{,k}=0, \quad (49)$$

and the techniques of Ref. 1 to demonstrate that the graviton-photon and the graviton-graviton scattering cross sections for large impact parameter will be the same as the photon-photon cross section. These results are consonant with the fact that surfaces of constant phase for both electromagnetic and gravitational waves are perpendicular to null geodesics.

APPENDIX A: GRAVITATIONAL ENERGY FLUX

We give here a simple derivation of Eq. (40). In the weak-field limit the energy flux in a plane gravitational wave propagating along the x^1 axis can be expressed as follows[9]:

$$ct^{01}=(c^3/64\pi G)[\dot\psi^2+4(\dot\psi_{23}{}^2-\dot\psi_{22}\dot\psi_{33})]. \quad (A1)$$

This result can be generalized so as to determine the flux in a wave propagating in an arbitrary direction characterized by the unit vector n_α. To do this it suffices to construct from the components of the tensor $\dot\psi^{ik}$ and the vector n_α, a scalar quadratic in the $\dot\psi^{ik}$, which for $n_1=1$, $n_2=n_3=0$ reduces to the expression in the square bracket in Eq. (A1). This scalar times n^α yields the desired result[10]

$$ct^{0\alpha}=(c^3/64\pi G)(\dot\psi^2-2n_\delta n_\lambda e^{\delta\beta\rho}e^{\lambda\mu\nu}\dot\psi_{\beta\mu}\dot\psi_{\rho\nu})n^\alpha. \quad (A2)$$

Note in passing that a great simplification occurs in the event that $\psi_{\alpha\beta}$ is proportional to $v_\alpha v_\beta$, where v_α is an

[9] P. J. Westervelt, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu **4**, 333 (1966) [Soviet Phys. JETP Letters **4**, 225 (1966)].

[10] It is illuminating to think of gravitational waves as being strains, in an otherwise flat space, giving rise to a three-space with metric $\gamma_{\alpha\beta}$. The strains, of course, are generated by the retarded stress $\tau_{\alpha\beta}$ of the source. In this context, the energy flux of a wave traveling in, say, the x^1 direction, may be expressed in terms of the fractional change in cross-sectional area of an element of surface normal to x_1, i.e.,

$$(ds_1-ds_1{}^*)/ds_1=\delta s_1/ds_1$$

Here $ds_1{}^*$ refers to strained space, ds_1 to flat space; thus

$$ds_1{}^*/ds_1=(\gamma_{22}\gamma_{33}-\gamma_{23}\gamma_{23})^{1/2}.$$

In the limit of small strain,

$$\delta s_1/ds_1=-\tfrac{1}{8}\psi^2-\tfrac{1}{2}(\psi_{23}\psi_{23}-\psi_{22}\psi_{33}).$$

Combining this result with Eq. (A1), considering a simple harmonic time dependence, and performing a time average, the flux of energy becomes

$$ct^{01}=(-\tfrac{1}{4}\pi c^5/G\lambda^2)(\langle\delta s_1\rangle/ds_1).$$

arbitrary vector. In this case

$$ct^{0\alpha}=(c^3/64\pi G)\dot{\psi}^2 n^\alpha. \tag{A3}$$

By expressing the $e^{\alpha\beta\alpha}$ in terms of $\delta^{\alpha\beta}$, an expression equivalent to Eq. (A2) results:

$$ct^{0\alpha}=(c^3/64\pi G)[(\dot{\psi}^{\lambda\beta}n_\lambda n_\beta)^2+2\dot{\psi}_{\lambda\beta}\dot{\psi}^{\lambda\beta} -4\dot{\psi}_\lambda{}^\beta\dot{\psi}^{\lambda\delta}n_\beta n_\delta+\dot{\psi}^\mu{}_\mu(2\dot{\psi}^{\lambda\beta}n_\lambda n_\beta-\dot{\psi}^\delta{}_\delta)]n^\alpha. \tag{A4}$$

We now obtain the scattered intensity per unit solid angle $ct^{0\alpha}n_\alpha R^2$ for the case in which $\psi^\beta{}_\beta=0$:

$$\frac{dI}{d\Omega}=ct^{0\alpha}n_\alpha R^2=\frac{c^3R^2}{64\pi G}[(\dot{\psi}^{\lambda\beta}n_\lambda n_\beta)^2+2\dot{\psi}_{\lambda\beta}\dot{\psi}^{\lambda\beta}-4\dot{\psi}^\beta{}_\lambda\dot{\psi}^{\lambda\delta}n_\beta n_\delta],$$

in agreement with Eq. (40).

APPENDIX B: MAXWELL EQUATIONS IN THREE-DIMENSIONAL FORM

In this appendix the solution is given to a problem posed by Landau and Lifshitz, namely, writing the Maxwell equations, in a constant gravitational field, in three-dimensional form.[11] Their solution suffers from an incorrect identification of the three-dimensional vectors with the four-dimensional tensors as well as containing minor errors in the "constitutive" equations. Here we shall simply state the results, since once they are known their verification[12] is straightforward though slightly tedious.

The three-dimensional fields **E**, **H**, **D**, and **B** are defined in terms of F_{ik}, the electromagnetic field tensor, as follows:

$$E_\alpha=F_{\alpha0},\quad H_\alpha={}^*F_{0\alpha},\quad D^\alpha=(\surd-g_{00})F^{0\alpha},$$
and
$$B^\alpha=(\surd-g_{00})\,{}^*F^{\alpha0}, \tag{B1}$$

in which $^*F_{ik}$ denotes the dual to the tensor F_{ik}. We define a vector $\mathbf{g}$ such that $g^\alpha=g^{0\alpha}$.

In this appendix all vector operations are carried out in a three-dimensional space with metric $\gamma_{\alpha\beta}$.

The relations $F^{ik}=g^{ik}g^{kl}F_{ml}$ and $^*F^{ik}=g^{im}g^{kl}\,{}^*F_{ml}$ can be represented in the form of a pair of vector equations

$$\begin{aligned}\mathbf{D}&=(\surd-g_{00})[(-g^{00})\mathbf{E}+\mathbf{B}\times\mathbf{g}+(\mathbf{g}\cdot\mathbf{E})\mathbf{g}],\\ \mathbf{B}&=(\surd-g_{00})[(-g^{00})\mathbf{H}+\mathbf{g}\times\mathbf{D}+(\mathbf{g}\cdot\mathbf{H})\mathbf{g}].\end{aligned} \tag{B2}$$

Only in the weak-field limit when these equations become

$$\mathbf{D}=\mathbf{E}/(\surd-g_{00})+\mathbf{H}\times\mathbf{g}$$
and
$$\mathbf{B}=\mathbf{H}/(\surd-g_{00})+\mathbf{g}\times\mathbf{E} \tag{B3}$$

do we agree with Landau and Lifshitz.

The potential A_i satisfies the usual equation

$$F_{ik}=A_{k,i}-A_{i,k}, \tag{B4}$$

which may be represented by the two vector equations

$$\begin{aligned}\mathbf{E}&=-\nabla A_0-(1/c)\dot{\mathbf{A}},\\ \mathbf{B}&=\nabla\times\mathbf{A},\end{aligned} \tag{B5}$$

in which $\mathbf{A}\equiv A_\alpha$.

The Maxwell equations

$$(1/\surd-g)[(\surd-g)F^{ik}]_{,k}=0$$
and
$$F_{ik,l}+F_{kl,i}+F_{li,k}=0 \tag{B6}$$

can be represented by the vector equations

$$\nabla\cdot\mathbf{D}=0,\qquad \nabla\cdot\mathbf{B}=0,$$
$$\nabla\times\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t},\quad \nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}. \tag{B7}$$

These results are in exact agreement with Landau and Lifshitz.

It is possible to combine Eqs. (B5) and (B7) to obtain for **A** a wave equation in curved space with metric $\gamma^{\alpha\beta}$. By transforming all curved-space vector operators to flat-space operators, and letting $A_0=0$, we obtain an alternate derivation of Eq. (7) in Sec. II.

Finally, a comment on the observation that the difference from unity of the equivalent electric and magnetic permeability in a static gravitational field found by Landau and Lifshitz is one-half the value obtained by us in Sec. II. This apparent discrepancy is purely artificial and stems from the fact that their definition is based on an arbitrary constitutive equation.

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[11] This is problem 3 appearing on p. 296 in Ref. 4.
[12] P. J. Westervelt, Brown University Report No. NYO-2262-TA-218 (unpublished).