# Theory of Gravitation

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Since there seems to exist a preferred frame of reference in the universe, determined by the large-scale distribution of matter, a theory of gravitation is considered which resembles the general theory of relativity in being based on the equivalence principle, but is without the covariance principle. This theory leads to the same results as general relativity in the three crucial tests. The formalism can be modified to take into account the solar oblateness observed by Dicke and Goldenberg.

## I. INTRODUCTION

THE Einstein general theory of relativity,[1] based on the covariance and equivalence principles, has been very successful in describing gravitational phenomena. It has also served as a basis for models of the universe. The homogeneous isotropic expanding model, based on general relativity, appears to provide a good approximation to the observed large-scale properties of the universe.

However, in the case of a homogeneous isotropic universe one has a preferred frame of reference, that in which the matter is at rest, which resembles the primary inertial system, or absolute space, of Newton; and one has the cosmic time given by a clock at rest in this reference frame, which resembles the absolute time of Newton. If the universe is expanding, one can show[2] that an observer in a closed, freely moving laboratory can, in principle, detect his motion with respect to this "absolute space" by means of mechanical or optical experiments. This appears to be in contradiction to the principle of covariance, which treats all coordinate systems on an equal footing and rejects the concepts of absolute space and time.

This situation raises doubts concerning the validity of the covariance principle of the general relativity theory within the framework of the universe. The question arises whether one can set up a theory of gravitation which, like general relativity, is based on the equivalence principle, but which however does not accept the principle of covariance.

## II. A POSSIBLE THEORY

One way of developing such a theory is to follow general relativity in satisfying the equivalence principle by assigning a Riemannian geometry to space-time, but to require from the beginning that there exists a preferred coordinate system, determined by the large-scale distribution of matter in the universe, in which the laws of physics have their simplest form. With respect to this coordinate system the following assumptions are made: (a) The time direction is orthogonal to the three-dimensional space; (b) the three-space is locally isotropic; (c) in a region of space-time in which the gravitational field is weak and slowly varying, the special theory of relativity is valid.

On the basis of the above assumptions the line element in the preferred coordinate system can be written in the form

$$ds^2=\Phi^2dt^2-\Psi^2(dx^2+dy^2+dz^2)\,, \qquad (1)$$

where $\Phi$ and $\Psi$ are functions of $(t,x,y,z)\equiv(x^0,x^1,x^2,x^3)$ (and can be regarded as scalars). They are to be determined by the energy-momentum density tensor $T^{\mu\nu}$ of the matter and other (nongravitational) fields by means of the field equations. It will be assumed that this tensor satisfies the equation

$$T_{\mu}{}^{\nu}{}_{;\nu}=0\,, \qquad (2)$$

representing the energy-momentum relations and, in the case of matter, the equations of motion. Since there are two field variables, two field equations are needed, and they are taken in the form

$$F=-8\pi T_{00}\,, \qquad (3)$$

$$G=-8\pi T_{kk}\,, \qquad (4)$$

where a Latin index takes on the values 1, 2, 3, and where the left-hand members are functions of $\Phi$, $\Psi$, and their first and second derivatives.

It will be assumed that the field equations are associated with a variational principle. Given an integral $I$ over a four-dimensional region,

$$I=\int \mathfrak{L}d\tau\,, \qquad (5)$$

where $\mathfrak{L}$ is a function of $\Phi$, $\Psi$, and their first derivatives, let us write, for arbitrary variations $\delta\Phi$ and $\delta\Psi$ vanishing on the boundary,

$$\delta I=\int[F\delta(\Phi^{-2})-G\delta(\Psi^{-2})](-g)^{1/2}d\tau\,. \qquad (6)$$

The metric tensor $g_{\mu\nu}$ and its determinant $g$ in our coordinate system are related to $\Phi$ and $\Psi$ in accordance with Eq. (1). One readily finds that

$$F=-(\Phi^2/2\Psi^3)\delta I/\delta\Phi\,,\quad G=(1/2\Phi)\delta I/\delta\Psi\,. \qquad (7)$$

It follows from (3), (4), and (7) that Eq. (2) can be written as a conservation law:

$$[(-g)^{1/2}(T_{\mu}{}^{\nu}+\theta_{\mu}{}^{\nu})]_{,\nu}=0\,, \qquad (8)$$

[1] A. Einstein, Ann. Physik **49**, 769 (1916).

[2] N. Rosen, Proc. Israel Acad. Sci. Humanities, No. 12 (1968); Nuovo Cimento Letters **1**, 42 (1969).

with the gravitational energy-momentum density given by

$$16\pi\theta_\mu{}^\nu=(-g)^{-1/2}\left[\left(\frac{\partial\mathfrak{L}}{\partial\Phi_{,\nu}}\right)\Phi_{,\mu}+\left(\frac{\partial\mathfrak{L}}{\partial\Psi_{,\nu}}\right)\Psi_{,\mu}-\mathfrak{L}\delta_\mu{}^\nu\right]. \quad (9)$$

The field equations depend on the choice of $\mathfrak{L}$. Let us take a simple form for it:

$$\mathfrak{L}=(-g)^{1/2}(\alpha g^{\mu\nu}\Psi_{,\mu}\Psi_{,\nu}/\Psi^2+\beta g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}/\Phi^2+\gamma g^{\mu\nu}\Phi_{,\mu}\Psi_{,\nu}/\Phi\Psi), \quad (10)$$

where $\alpha$, $\beta$, and $\gamma$ are constants. To determine these constants, one writes out the field equations (3) and (4) and considers them for the case of a weak field characterized by

$$\Phi=1+\phi, \quad \Psi=1+\psi, \quad (11)$$

where $\phi$, $\psi$, and their derivatives are small quantities, and one takes the static case with $T_{kk}$ negligible compared to $T_{00}=\rho$. If one now requires that

$$\nabla^2\phi=4\pi\rho \quad (12)$$

as in Newtonian mechanics and

$$\psi=-\phi \quad (13)$$

as in general relativity,[3] one gets

$$\beta=\alpha+2, \quad \gamma=2\alpha. \quad (14)$$

The field equations (3) and (4) now take the form

$$(\alpha+2)[\Phi_{,00}/\Phi-\tfrac{3}{2}(\Phi_{,0})^2/\Phi^2+3\Phi_{,0}\Psi_{,0}/\Phi\Psi-\Phi\Phi_{,kk}/\Psi^2+\tfrac{1}{2}\Phi_{,k}\Phi_{,k}/\Psi^2-\Phi\Phi_{,k}\Psi_{,k}/\Psi^3]+\alpha[\Psi_{,00}/\Psi+\tfrac{5}{2}(\Psi_{,0})^2/\Psi^2-\Phi^2\Psi_{,kk}/\Psi^3+\tfrac{1}{2}\Phi^2\Psi_{,k}\Psi_{,k}/\Psi^4]=-8\pi T_{00}, \quad (15)$$

$$\alpha[-\Psi^2\Phi_{,00}/\Phi^3-\Psi\Psi_{,00}/\Phi^2+\Psi\Phi_{,0}\Psi_{,0}/\Phi^3-\tfrac{1}{2}(\Psi_{,0})^2/\Phi^2+\Phi_{,kk}/\Phi+\Psi_{kk}/\Psi+\Phi_{,k}\Psi_{,k}/\Phi\Psi-\tfrac{1}{2}\Psi_{,k}\Psi_{,k}/\Psi^2]+(3+\tfrac{7}{2}\alpha)\Psi^2(\Phi_{,0})^2/\Phi^4-(1+\tfrac{1}{2}\alpha)\Phi_{,k}\Phi_{,k}/\Phi^2=-8\pi T_{kk}. \quad (16)$$

The static, spherically symmetric solution for empty space is found to be given in polar coordinates by the expansions

$$\begin{aligned}\Phi&=1-m/r+\tfrac{1}{2}m^2/r^2+\cdots,\\ \Psi&=1+m/r+\tfrac{1}{2}(1+1/\alpha)m^2/r^2+\cdots,\end{aligned} \quad (17)$$

where use has been made of Eq. (13) for the first-order terms.

From Eq. (2) one can show that a test particle in a gravitational field will move along a geodesic. From the solution (17) one then obtains, to the usual accuracy, the same results as in general relativity for the precession of the perihelion of Mercury, the deflection of light by the sun, and the gravitational red shift, for arbitrary values of $\alpha$ ($\neq 0$).

In the case of a weak field, Eqs. (15) and (16) can be combined to give

$$\phi_{,00}-\phi_{,kk}=-4\pi(\rho+3p), \quad (18)$$

$$\psi_{,00}-\psi_{,kk}=4\pi\rho+12\pi(1+2/\alpha)p, \quad (19)$$

where, in the present approximation,

$$p=-\tfrac{1}{3}T_k{}^k=\tfrac{1}{3}T^{kk}, \quad \rho=T_0{}^0=T^{00}. \quad (20)$$

The form of Eqs. (18) and (19) in empty space is in agreement with our assumption (c). From their form in the general case one sees that it is possible for a material system to emit gravitational waves even if it is spherically symmetric.

If one calculates the Riemann-Christoffel tensor in the linear approximation, one obtains (among other components)

$$R^k{}_{0m0}=\phi_{,km}-\delta_k{}^m\psi_{,00}. \quad (21)$$

If $\phi$ and $\psi$ correspond to a plane wave traveling, say, in the $x^1$ direction, then in general both $R^1{}_{010}$ and $R^2{}_{020}$ will be different from zero. This means that gravitational waves have both longitudinal and transverse components; i.e., they can produce both longitudinal and transverse motion in a suitable physical system.[4]

Let us go back to the gravitational energy-momentum density tensor given by Eq. (9). One finds that, with $\mathfrak{L}$ given by Eq. (10), one can write

$$16\pi\theta_{\mu\nu}=(2\alpha/\Psi^2)\Psi_{,\mu}\Psi_{,\nu}+(2\beta/\Phi^2)\Phi_{,\mu}\Phi_{,\nu}+(\gamma/\Phi\Psi)(\Phi_{,\mu}\Psi_{,\nu}+\Phi_{,\nu}\Psi_{,\mu})-(-g)^{-1/2}\mathfrak{L}g_{\mu\nu}, \quad (22)$$

which is symmetric. If $T^{\mu\nu}$ is also symmetric, and if one defines

$$\Theta^{\mu\nu}=T^{\mu\nu}+\theta^{\mu\nu}, \quad (23)$$

then Eq. (8) gives conservation laws for energy and momentum,

$$(\Phi^3\Psi^3\Theta^{0\nu})_{,\nu}=0, \quad (24)$$

$$(\Phi\Psi^5\Theta^{k\nu})_{,\nu}=0, \quad (25)$$

and also leads to conservation of angular momentum,

$$(\Phi\Psi^5M^{jk\nu})_{,\nu}=0, \quad (26)$$

where

$$M^{\sigma\mu\nu}=x^\sigma\Theta^{\mu\nu}-x^\mu\Theta^{\sigma\nu}. \quad (27)$$

With the constants $\beta$ and $\gamma$ fixed by (14), one finds that the gravitational energy density is given by

$$16\pi\theta_{00}=2[(\Phi_{,0})^2/\Phi^2+\Phi_{,k}\Phi_{,k}/\Psi^2]+\alpha(\Phi_{,0}/\Phi+\Psi_{,0}/\Psi)^2+\alpha(\Phi^2/\Psi^2)(\Phi_{,k}/\Phi+\Psi_{,k}/\Psi)(\Phi_{,k}/\Phi+\Psi_{,k}/\Psi). \quad (28)$$

For this to be positive definite, one must have $\alpha\geqslant 0$. From a theoretical standpoint one can choose $\alpha$ arbitrarily, and it is aesthetically attractive to take a simple value as, for example, $\alpha=2$ ($\beta=4$, $\gamma=4$). However, we see from Eq. (19) that it should be possible, in principle, to arrive empirically at the value of $\alpha$ by a study of gravitational radiation.

## III. SOLAR OBLATENESS

If the conclusions of Dicke and Goldenberg[5] are valid, that the sun is oblate and that, if this is taken into ac-

[3] A. S. Eddington, *The Mathematical Theory of Relativity*, 2nd ed. (Cambridge U. P., London, 1924), p. 101.

[4] Cf. J. Weber, Phys. Rev. **117**, 306 (1960).

[5] R. H. Dicke and H. M. Goldenberg, Phys. Rev. Letters **18**, 313 (1967).

count, the perihelion precession of Mercury about a spherical sun would differ from that predicted by general relativity, one can modify the present formalism to get agreement with observation. In determining the parameters $\beta$ and $\gamma$, let us require that, in the case of a weak, static field with $T_{kk}$ negligible, Eq. (13) be replaced by

$$\psi=-\lambda\phi, \tag{29}$$

with $\lambda$ constant ($0<\lambda<1$). This, together with Eq. (12), leads to the relations

$$\beta=\lambda^2\alpha+2, \quad \gamma=2\lambda\alpha. \tag{30}$$

The field equations are now somewhat more complicated and will not be given here. It is found that the static, spherically symmetric solution for empty space has the form

$$\begin{aligned}\Phi&=1-m/r+\tfrac{1}{4}(1+\lambda)m^2/r^2+\cdots,\\ \Psi&=1+\lambda m/r+\cdots,\end{aligned} \tag{31}$$

where Eq. (29) has been taken into account in first order.

Using this solution, one finds that the precession of the perihelion of a planet now depends on $\lambda$. If the angular velocity of precession is denoted by $\omega(\lambda)$, so that $\omega(1)$ is that given by general relativity, one obtains

$$\Delta\equiv[\omega(1)-\omega(\lambda)]/\omega(1)=(7/12)(1-\lambda). \tag{32}$$

For example, for $\lambda=6/7$, $\Delta=\frac{1}{12}=0.083$.

The value of $\lambda$ also determines the deflection of light by the sun; one now gets a value which is $\frac{1}{2}(1+\lambda)$ times that given by general relativity. On the other hand, as one sees immediately from Eq. (31), the gravitational red shift, which depends only on $g_{00}$, is (in first order) independent of $\lambda$.

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# Scattering of Electromagnetic and Gravitational Waves by a Static Gravitational Field: Comparison Between the Classical (General-Relativistic) and Quantum Field-Theoretic Results*

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The classical general-relativistic cross sections for the scattering of either an electromagnetic wave or a gravitational wave by a scalar particle are calculated and found to agree with the results of the quantized linearized field theory.

## I. INTRODUCTION

The quantum-mechanical cross section for the scattering of light by light has recently been shown to be equal to the classical cross section, in the limit of large impact parameter.[1] The quantum-mechanical cross section for the scattering of light by a boson was also shown to equal the classically calculated cross section,[1] in the limit of large impact parameter. In this paper the classical calculation of the light-by-boson scattering cross section is extended so as to apply for all scattering angles. The result agrees exactly with the quantum-mechanical result.[1] The classical calculation of the scattering of a graviton by a boson is also shown to agree with the quantum-mechanical result[2] for all scattering angles.

This latter result has a bearing on Weber's analysis of the mass quadrupole gravitational wave detector. Weber obtains a finite cross section of the order of $\lambda^2$ for an oscillator damped solely by radiation resistance.[3] Here $\lambda$ is the wavelength of the incident gravitational radiation. The calculation presented in this paper shows the cross section of a single point mass to be infinite, a not too surprising result in view of the long-range nature of the gravitational interaction. Thus in order to obtain the total cross section, assuming Weber's calculation to be valid insofar as the quadrupole contribution is concerned, Weber's cross section should be augmented by the infinite one obtained herein. The validity of Weber's quadrupole cross section is a topic I will treat in a forthcoming paper.

In Sec. II the wave equation in flat space for the electromagnetic four-potential in the presence of a weak, but otherwise arbitrary, gravitational perturbation is derived. This equation is simplified for the case when the gravitational perturbation is static and Newtonian. An alternative derivation of this wave equation based

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[1] P. J. Westervelt and L. F. Karr, Nuovo Cimento 66B, 129 (1970).

[2] B. S. DeWitt, Phys. Rev. 162, 1239 (1967).

[3] J. Weber, *General Relativity and Gravitational Waves* (Interscience, New York, 1961).