Compton Fokker-Planck Equation for Hot Plasmas*

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The Fokker-Planck equation for Compton scattering in a plasma is developed without recourse to a nonrelativistic expansion. Using calculated energy exchange rates for scattering in a relativistic Maxwellian plasma, an equation is obtained which is valid for electron temperatures up to 100 keV and photon energies up to 1 MeV. Numerical comparisons are presented which show excellent agreement between Fokker-Planck and Monte Carlo calculations. Results using the nonrelativistic Fokker-Planck, valid to only 20 keV, are also presented to indicate the improvement made.

I. INTRODUCTION

In various astronomical events, such as supernova explosions or the flare-up of thermal x-ray sources, the analysis of time-varying phenomena in highly ionized gases is of importance. In particular, for the problem of nonequilibrium radiation transport in such systems, Compton scattering with electrons may often be the dominant mechanism for energy transfer. Since the scattering integral for this process is extremely complicated, one often uses a Fokker-Planck approximation. This consists of expanding the collision integral in powers of the energy transfer through second order and is a valid approximation as long as the energy transfer per Compton scattering is small.

In the limit of low electron temperatures and photon energies, this is not only a valid approximation but is also very convenient, since the resulting equation may be expressed in a simple analytic form.[1–3] This will be called the nonrelativistic limit, though in reality it is correct through second order in v/c and $h\nu/mc^2$. On the basis of an analysis of the energy exchange rate between Maxwellian electrons and Planckian (blackbody) photons, the nonrelativistic approximation is seen to break down (give errors greater than 10%) for electron temperatures greater than 20 keV or photon temperatures greater than 2.5 keV.[4] The Fokker-Planck approximation, however, is still valid beyond these limits; the errors above result mainly from the nonrelativistic development.

The present paper remedies this by deriving the Fokker-Planck equation for an isotropic photon distribution without recourse to a nonrelativistic approximation.[5] It will be shown that only the average energy exchange per unit time need be known. In the case of arbitrary electron temperatures and photon energies, this quantity must be evaluated numerically since the integrals are far too complicated to be done analytically. However, using the numerical results of Stone and Nelson,[6] a fairly accurate fit to the energy exchange may be made for electron temperatures less than 100 keV and photon energies less than 1 MeV. The resulting Fokker-Planck equation using this fit assumes practically the same simple form as the nonrelativistic one.

To check the equation, a series of test problems for the Compton cooling of hot electrons were solved numerically and compared with Monte Carlo results. For the cases where the initial photon distribution had a temperature not too different from that of the electrons, excellent agreement was obtained. As the disparity in initial temperatures increased, the comparison became poorer, indicating a breakdown in the Fokker-Planck approximation itself.

A few comparison problems are presented to show the improvement made in calculating with the new Fokker-Planck equation as compared with the nonrelativistic form.

II. FOKKER-PLANCK EXPANSION

For this development, the precise form of the scattering integral need not be known. Instead, it is sufficient to consider the master equation for an isotropic system,

$$k^2\frac{\partial n(k,t)}{\partial t}=\int_0^\infty dk'\{S(k',k;\theta)[1+n(k,t)]n(k',t) -S(k,k';\theta)[1+n(k',t)]n(k,t)\}, \quad (1)$$

where $k=h\nu$ is the photon energy, $n(k,t)$ is the photon distribution function normalized to

$$\frac{8\pi}{(hc)^3}\int_0^\infty dk\, k^2 n(k,t)=N \quad (=\text{number/volume}), \quad (2)$$

and $S(k,k';\theta)/k^2$ is the transition rate for scattering from k to k', suitably averaged over a relativistic Maxwellian distribution of temperature θ. The factors $(1+n)$ represent the induced scattering enhancement due to the presence of photons in the final state.

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[1] A. R. Fraser, Atomic Weapons Research Establishment Report No. 0-82/65, 1965 (unpublished).

[2] G. C. Pomraning, Los Alamos Report No. LA-4006 MS, 1968 (unpublished).

[3] R. Weymann, Phys. Fluids **8**, 2112 (1965).

[4] P. Woodward, Phys. Rev. D **1**, 2731 (1970).

[5] For an earlier development for low-temperature electrons see B. E. Freeman, Systems, Science, and Software Report No. 3SIR-43, 1969 (unpublished).

[6] S. Stone and R. G. Nelson, LRL Report No. UCRL-14918-T, 1966 (unpublished).

To make the Fokker-Planck expansion, assume that $n(k,t)$ is analytic and that $S(k,k';\theta)$ is peaked about k within $|k'-k|/k\ll 1$. Moreover, note that detailed balance gives $e^{k/\theta}S(k',k;\theta)=e^{k'/\theta}S(k,k';\theta)$. Then expand both the exponential and $n(k',t)$ about $k'=k$ to convert Eq. (1) into

$$k^2\frac{\partial n(k,t)}{\partial t}\simeq\langle(k'-k)\rangle\left\{\frac{n(k,t)}{\theta}[1+n(k,t)]+\frac{\partial n(k,t)}{\partial k}\right\}$$
$$+\langle(k'-k)^2\rangle\left\{[n(k,t)+1]\left(\frac{n(k,t)}{2\theta^2}\right.\right.$$
$$\left.\left.+\frac{1}{\theta}\frac{\partial n(k,t)}{\partial k}\right)+\frac{1}{2}\frac{\partial^2 n(k,t)}{\partial k^2}\right\}, \quad (3)$$

where

$$\langle(k'-k)^n\rangle\equiv\int_0^\infty dk'(k'-k)^n S(k,k';\theta). \quad (4)$$

As shown in the Appendix, consistent with the order of the expansion of Eq. (1), the quantities $\langle(k'-k)\rangle$ and $\langle(k'-k)^2\rangle$ are related by

$$\frac{\partial\langle(k'-k)^2\rangle}{\partial k}\simeq\frac{\langle(k'-k)^2\rangle}{\theta}+2\langle(k'-k)\rangle. \quad (5)$$

When inserted into Eq. (3), this gives

$$k^2\frac{\partial n(k,t)}{\partial t}$$
$$=\frac{\partial}{\partial k}\left[\alpha(k,\theta)\left(n(k,t)[1+n(k,t)]+\theta\frac{\partial n(k,t)}{\partial k}\right)\right], \quad (6)$$

where

$$\alpha(k,\theta)=\frac{1}{2}\frac{\langle(k'-k)^2\rangle}{\theta}=\frac{1}{2}\frac{\partial\langle(k'-k)^2\rangle}{\partial k}-\langle(k'-k)\rangle. \quad (7)$$

It should be noted that Eq. (6) guarantees the two physically important intrinsic properties of Eq. (1), namely, conservation of photons and a Bose-Einstein equilibrium.

The complete determination of the Fokker-Planck equation rests now in calculating the coefficient $\alpha(k,\theta)$. In the nonrelativistic limit, the integrations may be carried out to give[1-3]

$$\alpha_{\rm NR}{}^{(k)}=\sigma_T\rho_e k^4/mc \quad (8)$$

with σ_T, ρ_e and m the Thomson cross section and electron density and mass, respectively. For regions where this limit does not apply, one must perform a very complicated multiple integration involving a relativistic Maxwellian electron distribution, the Klein-Nishina cross section, and various Lorentz transformations. This is clearly a numerical problem and has been done in various forms by several authors.[6-9] In Sec. III, some of these results will be used to evaluate $\alpha(k,\theta)$ for a region of temperature and energies beyond which the nonrelativistic answer fails.

III. COEFFICIENT EVALUATION

To evaluate $\alpha(k,\theta)$, the work of Stone and Nelson[6] will be used. These authors present a calculation for $\langle(k-k')\rangle/k^3$ for a series of electron temperatures up to 120 keV and photon energies up to 1 MeV. Using Eq. (5) and fitting their results with the form

$$\langle(k-k')\rangle=\tilde{\alpha}-\theta\partial\tilde{\alpha}/\partial k, \quad (9)$$

with $\tilde{\alpha}\to 0$ as $k\to 0$, one may make the unique identification $\tilde{\alpha}=\alpha(k,\theta)$. The $k\to 0$ condition is imposed because $\alpha(k,\theta)$ vanishes at $k=0$ and $\tilde{\alpha}$ is determined only up to a constant times $e^{k/\theta}$.

The fit to the calculation was aided by noting that three limiting forms to $\langle(k-k')\rangle$ are known. First, α must reduce to the proper nonrelativistic limit. Second, when $\theta=0$ the problem reduces to scattering photons off of electrons at rest and $\langle(k-k')\rangle$ may be evaluated exactly, though the expression is somewhat lengthy.[10] For photon energies up to 1 MeV, this result is represented very accurately by (k in keV)

$$\alpha(k,0)=\alpha_{\rm NR}/(1+9\times10^{-3}k+4.2\times10^{-6}k^2), \quad (10)$$

an expression which is also convenient for numerical work. As $k\to 0$, a recent expansion of the energy exchange by Woodward[4] indicates that through third order in θ/mc^2,

$$\alpha(k,\theta)\underset{k\to 0}{\longrightarrow}=\alpha_{\rm NR}[1+f(\theta)], \quad (11)$$
$$f(\theta)=\frac{5}{2}\frac{\theta}{mc^2}+\frac{15}{8}\left(\frac{\theta}{mc^2}\right)^2\left(1-\frac{\theta}{mc^2}\right). \quad (12)$$

This is the third limiting form.

Empirically it was found that Eq. (10) multiplied by $1+f(\theta)/(1+0.02k)$ gave an excellent fit to the Stone-Nelson results. This satisfies the three limiting forms since the fit fator $1/(1+0.02k)$ does not change Eq. (11). The calculated results and fit are presented in Fig. 1. Using this fit, the Fokker-Planck approximation given by Eq. (6) with

$$\alpha(k,\theta)=\alpha_{\rm NR}[1+f(\theta)/(1+0.02k)]$$
$$\times[(1+9\times10^{-3}k+4.2\times10^{-6}k^2)]^{-1} \quad (13)$$

should be applicable for electron temperatures up to 100 keV and photon energies to 1 MeV. Since numerical calculations usually require photon energies up to

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[8] L. Matteson, G. C. Pomraning, and H. L. Wilson, Gulf General Atomic Report No. GA-9694, 1969 (unpublished).
[9] E. G. Corman, Phys. Rev. D 1, 2734 (1970).
[10] R. D. Evans, *The Atomic Nucleus* (McGraw-Hill, New York, 1955), p. 688.

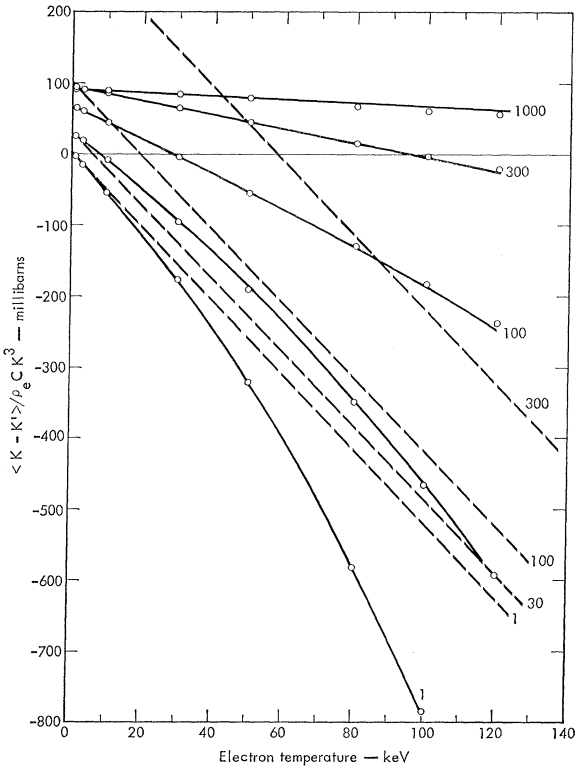

FIG. 1. Exact temperature-averaged energy-exchange coefficient of Stone and Nelson (circles) compared with the analytical fit (solid lines) and nonrelativistic answer (dashed lines) for various photon energies in keV.

about ten times the final equilibrium temperature of the photon distribution, the given equation can handle photon temperatures up to 100 keV. Of course, the range of problems which may be solved with the given Fokker-Planck equation is still limited by the fact that in converting the scattering integral into this differential form, it was assumed that contributions from terms in third- and higher-order derivatives of the photon distribution were negligible. This means roughly that the photon distribution cannot vary greatly over energies of the order of the electron temperature.

IV. NUMERICAL COMPARISONS

The Fokker-Planck equations using the α's given by Eqs. (8) and (13) were solved as initial value problems using a finite difference scheme described in Ref. 11. To ensure very accurate solutions, meshes of from 60 to 100 points were employed; in practical problems where errors up to 1% may be tolerated, 20 to 30 points are usually sufficient. The time steps were chosen fairly small between 1/200 and 1/500 of the equilibration time, in order to obtain precise temporal development. All of the cases studied used initial photon distributions of the form

$$n(k) = (Ce^{k/\theta_r} - 1)^{-1}, \tag{14}$$

corresponding to a Bose-Einstein distribution with temperature θ_r.

Two types of problems were considered. First, for initial electron temperatures varying from 10 to 100 keV, a Planckian distribution ($C=1$) with $\theta_r=1$ keV was used. Second, for electrons from 50 to 100 keV, values of θ_r equal to one-half the electron temperature were used. The latter cases used $C=\theta_r^3$, (θ_r in keV), corresponding to approximately the same number of photons as a 1-keV Planckian distribution. In all of the problems, the electron density was taken as 6.025×10^{22} per cm^3 so that the free-electron specific heat ($\frac{3}{2}\rho_e$) was about equal to the photon specific heat ($\sim 3N$). This choice was made so that the system was dominated by neither radiation nor material.

As a comparison, the same problems were calculated using a Monte Carlo code developed by E. G. Corman at Lawrence Radiation Laboratory, Livermore. The results from this sampling scheme agree with the exact answers from the scattering integral, Eq. (1), except for statistical fluctuations which are estimated at about 1%.

The calculations were performed on CDC 6600 and 7600 computers. Running times for the Monte Carlo code on the 7600 were from 3 to 5 min. This is to be

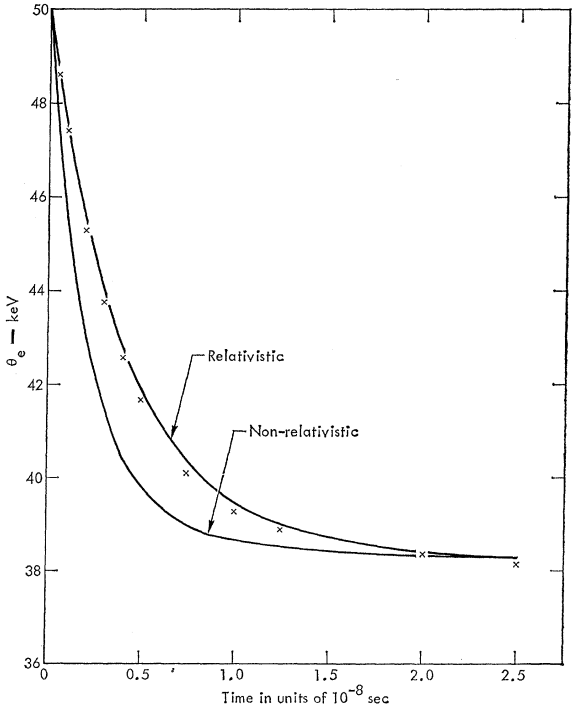

FIG. 2. Time evolution of the electron temperature, initially 50 keV, for the case of photons initially in a 25-keV Bose-Einstein distribution with $C=(25)^3$. Monte Carlo results (×) are presented for the comparison.

[11] J. S. Chang and G. E. Cooper, J. Comp. Phys. 6, 1 (1970).

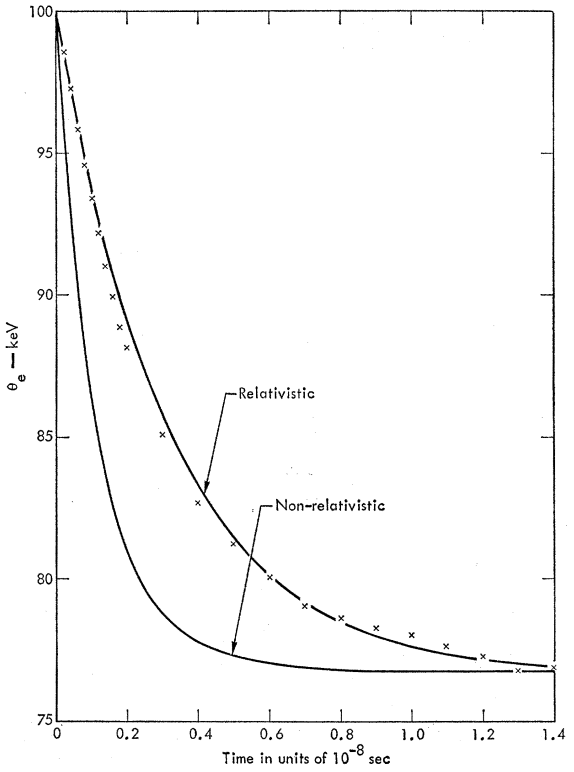

FIG. 3. Time evolution of the electron temperature, initially 100 keV, for the case of photon initially in a 50-keV Bose-Einstein distribution with $C=(50)^3$. Monte Carlo results (×) are presented for the comparison.

compared with a few hundredths of a minute for the Fokker-Planck.

For the case of an initial Planckian distribution at 1 keV, and electron temperatures below 10 keV all three codes gave essentially the same result for the average photon energy versus time. This is to be expected since these are energies in the nonrelativistic region and is taken as a "calibration" for the three codes.

The problems which initially had the photons in a Bose-Einstein distribution at one-half the initial electron temperature showed excellent agreement between Monte-Carlo calculations and the Fokker-Planck results presented here. However, the nonrelativistic Fokker-Planck results deviated significantly since the electron temperatures of 50 to 100 keV were quite outside its domain of validity. Figures 2 and 3 show the electron temperatures as a function of time, calculated from the energy gain of the photons using the free-electron specific heat $\frac{3}{2}\rho_e$ for initial values of 50 to 100 keV.

As an example of where all three calculations disagree, Fig. 4 presents the results for a 1-keV Planckian distribution and a 50-keV electron distribution. This shows the effects of truncating the Fokker-Planck series at the second derivative since, during the early time development, the variation of the photon distribution in k is quite severe when compared to the electron temperature. Obviously, one must carefully examine what source terms are present if the Fokker-Planck is to be used; special attention must be applied to cases which have narrow lines in the radiation spectrum resulting from discrete line transitions in partially ionized constituents. However, for temperatures of the order presented here, low-Z gases may usually be taken as fully ionized, in which case the only internal source is bremsstrahlung which has a width of the same order as the electron temperature.

Besides the energy transfer with the electrons, these calculations also provide a time-dependent nonequilibrium distribution function for the photons. This aspect of the problem is relevant to some experimental observations, especially in astrophysics. Comparisons of the photon spectra from the three codes indicates that the new Fokker-Planck and Monte Carlo results are in essential agreement for the cases where the initial photon temperature is one-half the electron temperature. The nonrelativistic Fokker-Planck spectra differ significantly in the high-energy tail; this is because the transition rate for high-energy photons is reduced in the relativistic case because the Klein-Nishina cross section is lower than the Thomson cross section. An illustration of this effect is presented in Fig. 5, where it is seen that above about 200 keV the nonrelativistic

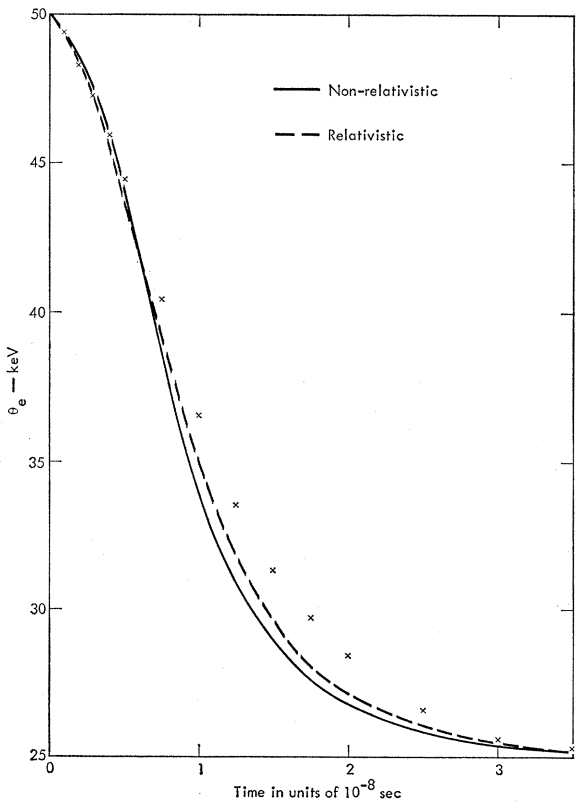

FIG. 4. Time evolution of the electron temperature, initially 50 keV, for the case of photons initially in a 1-keV Planckian distribution ($C=1$). Monte Carlo results (×) are presented for comparison.

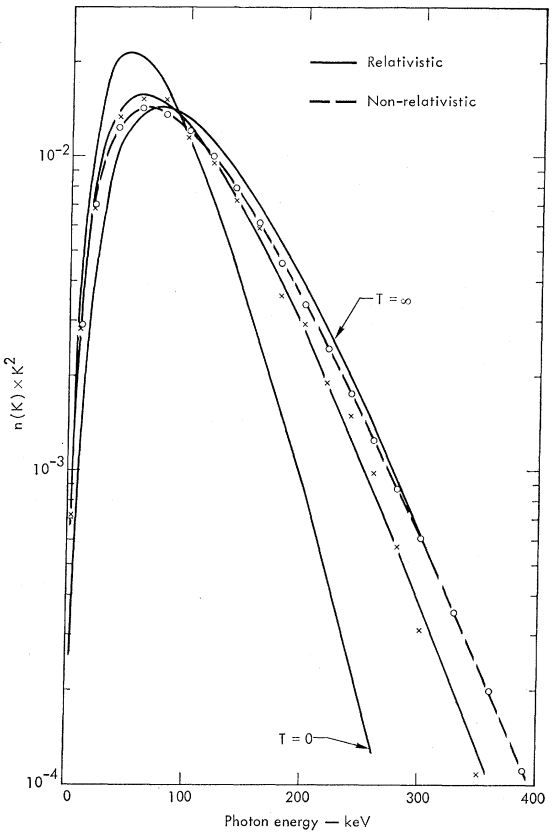

FIG. 5. Photon number spectra, normalized according to Eq. (2), at time 4×10^{-9} sec for the case of electrons initially at 50 keV and photons initially in a 25-keV Bose-Einstein distribution with $C=(25)^3$. Monte Carlo results (×) and initial ($T=0$) and equilibrium ($T=\infty$) spectra are presented for comparison.

spectrum attains its equilibrium value while the relativistic spectrum does not.

Spectra from the cases where the initial photon distribution is a 1-keV Planckian while the electrons are at temperatures in excess of 50 keV show disagreement between all three calculations; this is, of course, the set of problems where the Fokker-Planck approximation fails.

ACKNOWLEDGMENTS

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APPENDIX

To prove the relation between the first and second moments of the energy transfer, Eq. (5), it will be convenient to change variables from k and k' to k and $x=k'-k$. Then

$$\langle(k'-k)^n\rangle=\int_{-k}^{\infty}dx\,x^nT(k,x;\theta)\,, \tag{A1}$$

where $T(k,x;\theta)$ is the temperature-averaged transition rate expressed in the new variables. Since it was assumed (and is borne out by numerical calculations) that S is peaked about k within $|k'-k|/k\ll1$, T will be peaked about $x=0$. This means that the lower limit of integration may be taken as $-\infty$ with negligible error.

The demonstration proceeds by noting that detailed balance gives $T(k,x;\theta)=e^{-x/\theta}T(x+k,-x;\theta)$, so that

$$\begin{aligned}\langle(k'-k)\rangle&\simeq\int_{-\infty}^{\infty}dx\,xT(k,x;\theta)\\
&=\int_{-\infty}^{\infty}dx\,xe^{-x/\theta}T(x+k,-x;\theta)\\
&\cong\int_{-\infty}^{\infty}dx\,x\left(1-\frac{x}{\theta}\right)T(k,-x;\theta)+x\frac{\partial T(k,-x;\theta)}{\partial k}\\
&\simeq\int_{-\infty}^{\infty}dx\,xT(k,-x;\theta)-\frac{1}{\theta}\int_{-\infty}^{\infty}dx\,x^2T(k,-x;\theta)\\
&\qquad+\frac{\partial}{\partial k}\int_{-\infty}^{\infty}dx\,x^2T(k,-x;\theta)\\
&=-\langle(k'-k)\rangle-\frac{1}{\theta}\langle(k'-k)^2\rangle\\
&\qquad+\frac{\partial}{\partial k}\langle(k'-k)^2\rangle.\end{aligned} \tag{A2}$$

The approximation that third- and higher-order moments of the energy exchange are negligible was made above and is in line with the approximations used to obtain the Fokker-Planck equation. From Eq. (A2) follows the desired result:

$$\frac{\partial\langle(k'-k)^2\rangle}{\partial k}\simeq\frac{1}{\theta}\langle(k'-k)^2\rangle+2\langle(k'-k)\rangle. \tag{A3}$$