lh) or extraordinary (right-handed, rh):

(lh, lh):

$$M=\delta_n{}^{n'}\left[1-\omega_H\left(\frac{n+1}{\omega_H-\omega}-\frac{n}{\omega_H-\omega'}\right)\right],\qquad \text{(B16)}$$

(rh, rh):

$$M=\delta_n{}^{n'}\left[1-\omega_H\left(\frac{n+1}{\omega_H+\omega}-\frac{n}{\omega_H+\omega'}\right)\right],\qquad \text{(B17)}$$

(rh, lh):

$$M=\omega_H[n(n-1)]^{1/2}\left(\frac{1}{\omega-\omega_H}-\frac{1}{\omega'+\omega_H}\right)\delta_{n'}{}^{n-2}$$
$$=0 \quad \text{since} \quad \omega'=\omega+(n-n')\omega_H=\omega-2\omega_H,\qquad \text{(B18)}$$

(lh, rh):

$$M=\omega_H[(n+1)(n+2)]^{1/2}\left(\frac{1}{\omega_H-\omega'}+\frac{1}{\omega_H+\omega}\right)\delta_{n'}{}^{n-2}$$
$$=0 \quad \text{since} \quad \omega'=\omega+2\omega_H.\qquad \text{(B19)}$$

In the first two equations the first term comes from the so-called seagull diagram which does not exist in the "mixed" cases.

The result is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{rh,lh}}=r_0{}^2\frac{\omega^2}{(\omega\pm\omega_H)^2},\qquad \text{(B20)}$$

which is the classical result.

An exactly similar calculation confirms the classical result for $\theta=\frac{1}{2}\pi$.

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Spectrum of High-Energy Electrons Undergoing Klein-Nishina Losses

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A solution is presented for the spectrum of high-energy electrons confined to a region where the main energy-loss mechanism is Compton scattering in the extreme Klein-Nishina limit. The solution is valid in the steady-state limit assuming the region under consideration is optically thin to the emitted photons.

I. INTRODUCTION

For many astrophysical applications it is desirable to be able to solve for the spectrum of high-energy particles undergoing energy loss via certain physical mechanisms. Blumenthal and Gould,[1] however, have recently emphasized that when particles lose energy in discrete amounts as opposed to continuously, then it is appropriate to describe their distribution function by an integrodifferential equation as opposed to the ordinary continuity equation in energy space. If $n(\gamma,t)$ is the number (or density) of particles with Lorentz factor $\gamma=(1-\beta^2)^{-1/2}$ between γ and $\gamma+d\gamma$ at time t, then the distribution function satisfies an equation of the form

$$\frac{\partial n(\gamma,t)}{\partial t}+\frac{\partial}{\partial\gamma}[\dot{\gamma}n(\gamma,t)]+n(\gamma,t)\int_1^\gamma P(\gamma,\gamma')d\gamma' -\int_\gamma^\infty d\gamma' n(\gamma',t)P(\gamma',\gamma)=q(\gamma).\qquad (1)$$

Here $\dot{\gamma}$ represents the total energy loss[2] due to continuous processes where the energy change per collision is small:

$$|\dot{\gamma}/\gamma|\ll\nu,\qquad (2)$$

ν being the collision frequency. The $P(\gamma,\gamma')d\gamma'$ in Eq. (1) represents the total probability per unit time that a particle of energy γ will lose an amount of energy between $\gamma-\gamma'$ and $\gamma-\gamma'-d\gamma'$ via all processes for which condition (2) is not satisfied. Finally, the $q(\gamma,t)$ in Eq. (1) represents possible sources and sinks of particles corresponding to creation or annihilation.

For many energy-loss mechanisms for high-energy electrons of astrophysical interest, condition (2) is amply satisfied, and the resulting continuity equation can be readily solved.[3] However, this is no longer true for electrons losing energy by either bremsstrahlung or Compton scattering in the extreme Klein-Nishina (KN) limit.[1] Indeed, for the latter process, the spectrum of emitted photons ϵ_1 is strongly peaked near $\epsilon_1\approx\gamma$. These processes must then be included in the integral terms in Eq. (1), thus making exact analytic solutions much more difficult to obtain even if time dependence

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[1] G. R. Blumenthal and R. J. Gould, Rev. Mod. Phys. **42**, 237 (1970).

[2] Since this paper deals with high-energy electrons, all energies are expressed in units of mc^2.

[3] See, e.g., N. Kardashev, Astronom. Zh. **39**, 393 (1962) [Soviet Astron. AJ **6**, 317 (1962)].

is ignored and steady-state conditions are assumed to exist.

In this paper, a solution of Eq. (1) is presented for high-energy electrons confined to a region where they lose energy predominantly through Compton scattering in the extreme KN limit. It is assumed that the region is optically thin to the emitted photons so that pair production in photon-photon collisions can be ignored as an important source of high-energy electrons.[4,5] Since the rate of energy loss due to synchrotron radiation is proportional to γ^2 while that due to bremsstrahlung is proportional to γ, it follows that at sufficiently high energies these processes can become more important than Compton scattering in the extreme KN limit. However, if the density of the ambient gas and the magnetic field are sufficiently small, these two processes need be considered only at energies higher than those considered here, and they will therefore be ignored. A detailed comparison of these energy-loss mechanisms has been presented elsewhere.[1] In Sec. II the general form of the solution is obtained, while Sec. III contains some specific examples for the electron injection spectrum and for the ambient photon spectrum.

II. GENERAL SOLUTION IN THE KLEIN-NISHINA LIMIT

When an electron with Lorentz factor γ traverses an isotropic field of ambient photons whose energy spectrum is given by $N(\epsilon)$, then Compton scattering of these photons off the electron produces a spectrum of high energy photons and causes the electron to lose energy. Furthermore, when the energy of an initial photon in the electron's rest frame

$$\epsilon'=\gamma\epsilon(1+\beta\cos\theta)\approx\gamma\epsilon \tag{3}$$

is much greater than 1, $\gamma\epsilon\gg1$, then the extreme KN formulas become applicable. For the above situation the number of photons produced from initial photons within $d\epsilon$ per unit time per unit energy (ϵ_1) of the scattered photon is given by[1,6]

$$\frac{dN_\epsilon}{dtd\epsilon_1}=\frac{2\pi r_0{}^2c}{\gamma^2}\frac{N(\epsilon)d\epsilon}{\epsilon}\left[1+\frac{1}{2}\frac{(\rho q)^2}{1+\rho q}(1-q)\right], \tag{4}$$

where $N(\epsilon)$ is the energy spectrum of ambient photons,

$$\rho=4\gamma\epsilon, \tag{5}$$

$$q=\epsilon_1/\rho(\gamma-\epsilon_1), \tag{6}$$

and where the kinematical limits are set by the condition that

$$0\approx1/4\gamma^2\leq q\leq1. \tag{7}$$

[4] R. J. Gould and G. P. Schreder, Phys. Rev. **155**, 1404 (1967).
[5] S. Hayakawa, Progr. Theoret. Phys. Suppl. (Kyoto) **37**, 594 (1966).
[6] F. C. Jones, Phys. Rev. **167**, 1159 (1968).

Equation (4) can be put into a more useful form by noting that

$$P(\gamma,\gamma')=\frac{dN_\epsilon}{dtd(\gamma-\gamma')}. \tag{8}$$

Thus, to lowest order in $(\gamma\epsilon)^{-1}\ll1$,

$$P(\gamma,\gamma')=\frac{\pi r_0{}^2c}{\gamma}\frac{N(\epsilon)d\epsilon}{\epsilon}\left(\frac{1}{\gamma'}-\frac{1}{4\epsilon\gamma'^2}+\frac{\gamma'}{\gamma^2}\right), \tag{9}$$

where $P(\gamma,\gamma')$ is nonzero only within the kinematical limit

$$4\gamma'\epsilon>1. \tag{10}$$

Of course, the total $P(\gamma,\gamma')$ is given by the integral of Eq. (9) over ϵ. Also, using Eq. (9), the average energy loss can be calculated using the fact that

$$-\dot{\gamma}=\int_{1/4\epsilon}^{\gamma}d\gamma'(\gamma-\gamma')P(\gamma,\gamma'), \tag{11}$$

to give

$$-\dot{\gamma}=\pi r_0{}^2c\int\frac{N(\epsilon)d\epsilon}{\epsilon}\left(\ln4\gamma\epsilon-\frac{11}{6}\right), \tag{12}$$

in agreement with previous results.[7]

At this point, it should be mentioned that collisions in the extreme KN limit, $\gamma\epsilon\gg1$, are also well above threshold for energy losses due to pair production in photon-electron collisions.[8] However, since the energy loss is roughly proportional to the total cross section times the average energy lost per collision, and since $\Delta\gamma_{\rm KN}\sim\gamma$ while $\Delta\gamma_{\rm pair}\sim4/\epsilon$, one finds when the high-energy cross sections[9] are included,

$$\frac{\dot{\gamma}_{\rm pair}}{\dot{\gamma}_{\rm KN}}\sim\frac{\Delta\gamma_{\rm pair}\sigma_{\rm pair}}{\Delta\gamma_{\rm KN}\sigma_{\rm KN}}\sim4\alpha\ll1, \tag{13}$$

where $\alpha^{-1}=137$. Thus, pair production can be ignored compared to KN losses. More rigorously, however, since the momentum transfer well above threshold in pair production has an effective maximum near mc, its cross section can be well approximated by the cross section for pair production in collisions with a proton.[10] Then the results for a high-energy nucleus traversing an isotropic raidation field[11] can be compared with the energy loss given by Eq. (12). If this is done, one finds that pair production becomes important only at energies many orders of magnitude above threshold. This process is therefore ignored here.

[7] The form of this energy-loss formula for collisions with a blackbody spectrum of ambient photons is extensively discussed by F. C. Jones, Phys. Rev. **137**, B1306 (1965), and was later rederived in Ref. 1.
[8] P. Encrenaz and R. B. Partridge, Astrophys. Letters **3**, 161 (1969).
[9] J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955).
[10] R. J. Gould, Phys. Rev. **185**, 72 (1969).
[11] G. R. Blumenthal, Phys. Rev. D **1**, 1596 (1970).

Now, the solution of Eq. (1) in the steady state with $P(\gamma,\gamma')$ given by Eq. (9) can be simplified by taking $P(\gamma,\gamma')$ to be of the form

$$P(\gamma,\gamma')=G(\gamma)[b_1(\gamma')+a(\gamma)b_2(\gamma')]. \quad (14)$$

Since the limits on the ϵ integration of Eq. (9) depend only on γ', it is clear that $P(\gamma,\gamma')$ can always be expressed in this form. Then defining

$$B_j(\gamma)=\int_{1/4\epsilon}^{\gamma} d\gamma' b_j(\gamma') \quad (15)$$

for $j=1$, 2, and letting

$$f(\gamma)=G(\gamma)n(\gamma), \quad (16)$$

the integral equation becomes an equation for $f(\gamma)$,

$$f(\gamma)[B_1(\gamma)+a(\gamma)B_2(\gamma)]-b_1(\gamma)\int_\gamma^\infty dx\, f(x) -b_2(\gamma)\int_\gamma^\infty dx\, a(x)f(x)=q(\gamma). \quad (17)$$

Now, defining

$$Q(\gamma)=\int_\gamma^\infty dx\, q(x), \quad (18)$$

one can integrate Eq. (17) to obtain

$$B_1(\gamma)\int_\gamma^\infty dx\, f(x)+B_2(\gamma)\int_\gamma^\infty dx\, a(x)f(x)=Q(\gamma). \quad (19)$$

If one now solves this equation for $\int_\gamma^\infty dx\, a(x)f(x)$ and substitutes this back into Eq. (17), one obtains a first-order linear differential equation. The solution for $f(\gamma)$ is then given by

$$f(\gamma)=\beta(\gamma)+\frac{d\mu(\gamma)}{d\gamma}e^{-\mu(\gamma)}\int_\gamma^\infty dx\, e^{\mu(x)}\beta(x), \quad (20)$$

where

$$\beta(\gamma)=\frac{q(\gamma)+b_2(\gamma)Q(\gamma)/B_2(\gamma)}{B_1(\gamma)+a(\gamma)B_2(\gamma)} \quad (21)$$

and

$$\frac{d\mu(\gamma)}{d\gamma}=\frac{b_1(\gamma)-b_2(\gamma)B_1(\gamma)/B_2(\gamma)}{B_1(\gamma)+a(\gamma)B_2(\gamma)}. \quad (22)$$

The boundary condition that $f(\gamma)$ go to zero as $\gamma\to\infty$ has already been imposed to obtain this solution. Also, because of the way that $\mu(\gamma)$ enters Eq. (20), it need be determined only to within an additive constant when Eq. (22) is integrated.

III. SOME EXAMPLES OF THE SOLUTION

Probably the simplest example of the solution Eq. (20) occurs in the case that the initial photon spectrum is monoenergetic:

$$N(\epsilon)=N_0\delta(\epsilon-\epsilon_0). \quad (23)$$

This is not a bad approximation to a sharply peaked distribution. Substituting this into Eq. (9) and integrating yields to lowest order in $(\gamma\epsilon_0)^{-1}\ll 1$

$$\mu(\gamma)=2\ln\left(\frac{\ln 4\gamma\epsilon_0-\frac{1}{2}}{4\gamma\epsilon_0}\right), \quad (24)$$

and thus

$$f(\gamma)=\frac{q(\gamma)+2Q(\gamma)/\gamma}{\ln 4\gamma\epsilon_0-\frac{1}{2}}-\frac{Q(\gamma)\ln 4\gamma\epsilon_0}{\gamma}\frac{(2\ln 4\gamma\epsilon_0-3)}{(\ln 4\gamma\epsilon_0-\frac{1}{2})^3} -\frac{\gamma}{2}\frac{(3-2\ln 4\gamma\epsilon_0)}{(\ln 4\gamma\epsilon_0-\frac{1}{2})^3}\int_\gamma^\infty dx\frac{q(x)}{x^2}. \quad (25)$$

Then $n(\gamma)$ follows from Eq. (16):

$$n(\gamma)=\frac{f(\gamma)}{G(\gamma)}=\frac{\gamma\epsilon_0 f(\gamma)}{\pi r_0^2 c N_0}. \quad (26)$$

The usual astrophysical application of this result is the case of a power-law injection of electrons of the form $q(\gamma)=K\gamma^{-P}$. With this $q(\gamma)$, the steady-state spectrum (25) becomes

$$f(\gamma)=\frac{K\gamma^{-P}}{\ln 4\gamma\epsilon_0-\frac{1}{2}}\left\{\frac{P+1}{P-1} +\frac{(2\ln 4\gamma\epsilon_0-3)}{(\ln 4\gamma\epsilon_0-\frac{1}{2})^2}\left[\frac{1}{2(P+1)}-\frac{\ln 4\gamma\epsilon_0}{P-1}\right]\right\}. \quad (27)$$

Note that to a first approximation ($\ln\gamma\approx$ const), since $n(\gamma)\propto\gamma f(\gamma)$, KN losses flatten the electron spectrum index by 1. This fact could have been deduced, however, simply by considering solutions of the continuity equation with $\dot{\gamma}$ given by expression (12). However, that the solution is not a pure power law can be seen from Fig. 1, where $\gamma^P f(\gamma)/K$ is plotted. From the figure it can be seen that most of the deviation from a pure power law occurs for the very small values of $\gamma\epsilon_0$ where this whole procedure is least valid. As $\gamma\epsilon_0$ approaches 1, the scattering approaches the Thomson limit where the spectrum is expected to be steeper by a power of 2. This trend is evident in the figure.

In this connection, it is useful to discuss the need for going through all of this procedure to obtain this solution of the integral equation. If $f_c(\gamma)$ is the solution of the ordinary continuity equation with $\dot{\gamma}$ given by Eq. (12), then $f_c(\gamma)$ can be easily obtained, and a plot of $f(\gamma)/f_c(\gamma)$ for various values of P is shown in Fig. 2. From the graphs it is clear that $f_c(\gamma)$ is asymptotically a good approximation to the exact solution only for $P\approx 2$ and γ very large. Indeed for $P=2$,

$$f(\gamma)/f_c(\gamma)\to 1 \quad \text{as} \quad \ln 4\gamma\epsilon_0\to\infty. \quad (28)$$

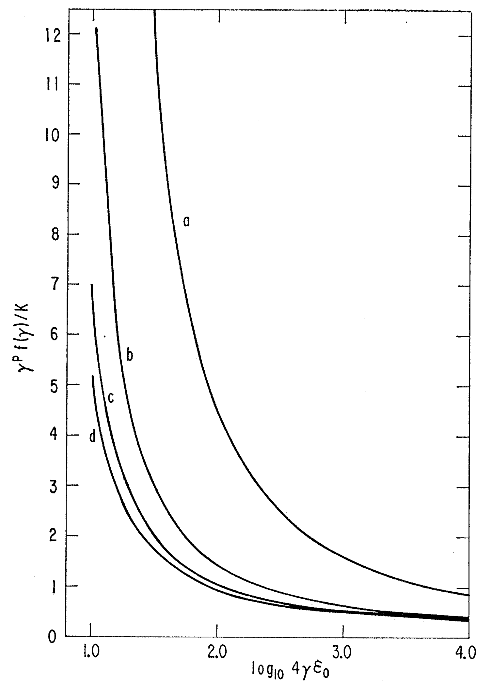

FIG. 1. Plot of the deviation from a pure power law of the steady-state electron distribution $\gamma^P f(\gamma)/K$ vs $\log_{10}4\gamma\epsilon_0$ for electrons undergoing Compton scattering in the extreme KN limit with photons of energy ϵ_0. A power-law injection of electrons is assumed with index (a) $P=1.2$, (b) $P=2.0$, (c) $P=3.0$, and (d) $P=4.0$.

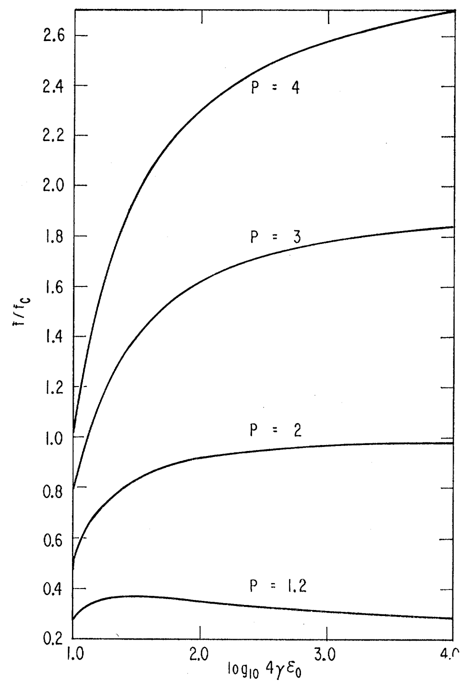

FIG. 2. Plot of $f(\gamma)/f_c(\gamma)$, the ratio of the exact energy spectrum for electrons undergoing Compton scattering in the extreme KN limit with photons of energy ϵ_0 to the solution obtained from a continuity equation in energy space.

However, for different values of the index P, the difference is significant, thus justifying the recourse to the more complicated procedure for obtaining $f(\gamma)$.

The solution just considered is valid only for a monoenergetic spectrum of ambient photons. In general, for any $N(\epsilon)$, Eq. (9) can be put into the form of Eq. (14) and therefore the solution [Eq. (20)] is valid. The only problem then is in evaluating the functions $b_j(\gamma)$, $B_j(\gamma)$, and $\mu(\gamma)$. Even if these integrals cannot be obtained in closed form, as is the case for a blackbody spectrum, the problem is nevertheless reduced to numerically calculating a few integrals as opposed to solving an integral equation. It should be mentioned, however, that if the KN limit is to hold for an arbitrary distribution $N(\epsilon)$, then a relationship of the form $\gamma\epsilon^*\gg1$ must be valid. Now, since the KN cross section $\sigma(\epsilon)\propto\epsilon^{-1}$ the collision rate is significantly greater for low-energy photons. Thus, ϵ^* above should really be given by $\langle\epsilon^{-1}\rangle^{-1}$. The KN domain therefore is involved only if $N(\epsilon)$ increases no slower than ϵ for small ϵ.

As an example of this, one can take

$$N(\epsilon)=A\epsilon\theta^{-1}U(\theta-\epsilon)\,, \qquad (29)$$

where $U(x)$ is the step function, and for the KN condition to hold, $\gamma\theta\gg1$. This distribution can be regarded as a first approximation to a Planckian. Then, substituting this into Eq. (9), for

$$G(\gamma)=\pi r_0^2cA/\gamma\,, \qquad (30)$$

the solution becomes

$$f(\gamma)=\frac{q(\gamma)+2Q(\gamma)/\gamma}{\ln4\gamma\theta+\frac{1}{2}}+\gamma\frac{(1-2\ln4\gamma\theta)}{(\ln4\gamma\theta+\frac{1}{2})^3}\times\int_\gamma^\infty dx\frac{(\ln4x\theta+\frac{1}{2})}{x^2}[q(x)+2Q(x)/x^2]. \qquad (31)$$

This solution is very similar to Eq. (27) for the δ-function distribution, and, in fact, they are identical if $\theta=\epsilon_0/2.718$. Thus, using a photon spectrum (or at least this photon spectrum) whose width is of the same order of magnitude as its average energy does not significantly alter the final solution from that obtained by assuming a monoenergetic initial photon distribution.

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