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Thomson Scattering in a Strong Magnetic Field

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The effect of a strong magnetic field on neutron stars or white dwarfs is calculated for Thomson scattering in a fully ionized collisionless plasma. The photon mean free path can be greatly extended for propagation nearly parallel or, for the extraordinary mode, nearly perpendicular to the field.

I. INTRODUCTION

ENORMOUSLY strong magnetic fields seem to be characteristic of collapsed stars. Exterior fields of 10^{12} G or even greater are required by models which try to relate pulsar observations to rotating neutron stars.[1] The recent discovery of polarized light from a white dwarf suggests a surface magnetic field in such a star which is in excess of 10^8 G.[2] Such huge magnetic fields can greatly affect the motion of electrons in the outer parts of the star by tending to contain their motion perpendicular to the magnetic field. The interaction of photons with electrons will be greatly altered whenever the magnetic field is sufficiently strong to restrict the electron motion to orbits parallel to the field. Cameron[3] has pointed out that the magnetic field may possibly greatly reduce the opacity in such stars so that they may cool much more rapidly than is indicated by those calculations which ignore it. A second effect which may be especially relevant in understanding the observations in the magnetic white dwarfs is a possible large temperature variation over the surface.

There are three main interactions which may limit the mean free path of a photon in the outer parts of a star.

(a) Photoelectric absorption. Here a sufficiently strong magnetic field increases the stability of atoms,[4] and alters both the important frequencies and the cross section for photon absorption.

(b) Free-free transitions, i.e., inverse bremsstrahlung.[5] The magnetic feld as well as the Coulomb field can absorb electron momentum.

(c) Thomson scattering.[6] If the electron moves significantly only parallel to the magnetic field, photons whose electric vector remains perpendicular to the field even in the presence of rapid Faraday rotation will have a greatly reduced scattering cross section and thus an anomalously high mean free path.

In this paper we shall present only those cross sections relevant to the reduction of the Thomson scattering in a strong field. Roughly, the criterion for the magnetic field to substantially affect the canonical Thomson cross section is that the photon (angular) frequency ω be less than the electron cyclotron frequency ω_H where

$$\hbar\omega_H = e\hbar H/mc \simeq 10^{-8} H \text{ (eV)}.$$

For $H \sim 10^{12}$ G, photons at temperatures below 10^8 °K have a reduced Thomson cross section. For $H \sim 10^8$ G, "visible" light will be similarly affected. The greatest reduction in cross section comes for photons directed either parallel to the field, where despite Faraday rota-

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[3] A. G. W. Cameron (private communication).
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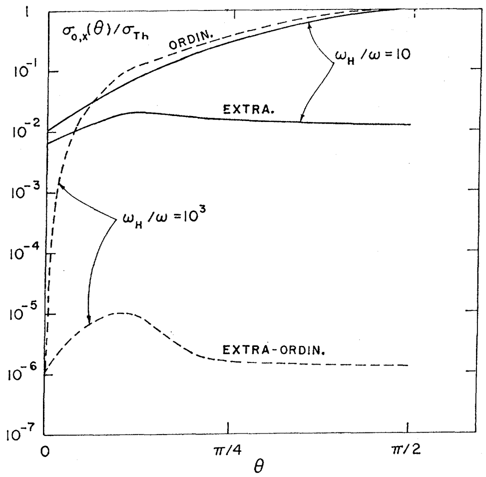

FIG. 1. Cross section, Eq. (21) vs θ for different $u^2=\omega_H/\omega$. The curves are very insensitive to the specific values of v so long as $v\ll 1$.

tion, the photon's rotating electric vector remains perpendicular to the stellar H field or the extraordinary mode of the plane polarized photon moving perpendicular to H with its E also perpendicular to H. This latter mode does not have its polarization plane altered by the magnetized plasma.

The description of an otherwise free charged particle, confined by a magnetic field is exactly analogous to that of bound states in a (two-dimensional) harmonic oscillator potential. As in that case, the scattering cross section is the same quantum mechanically and classically as long as $\hbar\omega\ll mc^2$. In Sec. II we exhibit the normal modes of electromagnetic waves in a (collisionless) plasma with a magnetic field. Section III contains the scattering cross section for these modes. The expected identity between classical and quantum mechanical cross sections is shown in Appendix B. The application of these results to calculations of stellar opacities will be published elsewhere.

II. ELECTROMAGNETIC MODES IN MAGNETIZED PLASMA

In a collisionless plasma in a uniform magnetic field we approximate the dielectric tensor by the canonical form for electrons of mass m and plasma frequencies ω_p and ions of charge Ze and mass M,[5]

$$\epsilon_{\alpha\beta}=\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix},$$

with

$$S=\tfrac{1}{2}(R+L),\quad D=\tfrac{1}{2}(R-L),$$

$$R=1-\frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega-\omega_H}-\frac{\Omega_p^2}{\omega^2}\frac{\omega}{\omega+\Omega_H},$$

$$L=1-\frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega+\omega_H}-\frac{\Omega_p^2}{\omega^2}\frac{\omega}{\omega-\Omega_H},$$

$$P=1-\omega_p^2/\omega^2-\Omega_p^2/\omega^2,$$

$$\Omega_p^2=(m/M)Z^2\omega_p^2,\quad \Omega_H=(m/M)Z\omega_H.$$

We consider the propagation of a plane monochromatic electromagnetic wave in a plasma with its wave vector k along the z axis. Let H be a uniform static magnetic field making an angle θ with z, and lying in the yz plane. In the xy plane the electric field vector of the two normal modes, E_α ($\alpha=1$, 2) rotates on an ellipse. The ratio of the components $E_{y,\alpha}$ and $E_{x,\alpha}$ in the waves of both modes (neglecting absorption) is[7]

$$E_{y,\alpha}/E_{x,\alpha}=iK_\alpha\equiv iC/(B-n_\alpha^2),\qquad(1)$$

where

$$B=\frac{u(1-v)-(1-v)^2}{u-(1-v)-uv\cos^2\theta}\qquad(2)$$

and

$$C=\frac{u^{1/2}v(1-v)\cos\theta}{u-(1-v)-uv\cos^2\theta},\qquad(3)$$

with

$$u^{1/2}=\omega_H/\omega=eH/mc\omega\qquad(4)$$

and

$$v=\omega_p^2/\omega^2=4\pi N_e e^2/m\omega^2.\qquad(5)$$

The two indices of refraction, n_α, are defined by[7]

$$n_\alpha^2=1-\frac{2v(1-v)}{2(1-v)-u\sin^2\theta+(-)^\alpha[u^2\sin^4\theta+4u(1-v)^2\cos^2\theta]^{1/2}}.\qquad(6)$$

The $\alpha=1$ mode is conventionally designated the "extraordinary wave"; the $\alpha=2$ mode is the "ordinary wave."

There is, generally, also an electric field in the z direction given by

$$E_{z,\alpha}=\frac{-iu^{1/2}v\sin\theta E_{x,\alpha}+uv\cos\theta\sin\theta E_{y,\alpha}}{u-(1-v)-uv\cos^2\theta}$$

$$\equiv iL_\alpha(\theta)E_{x,\alpha},\qquad(7)$$

where

$$iL_\alpha(\theta)=\frac{-iu^{1/2}v\sin\theta+iuv\sin\theta\cos\theta K_\alpha(\theta)}{u-(1-v)-uv\cos^2\theta}.\qquad(8)$$

[7] V. L. Ginzburg, *The Propagation of Electromagnetic Waves in Plasmas* (Pergamon, New York, 1964).

The component $E_{z,\alpha}$ is in phase with $E_{y,\alpha}$ and $\frac{1}{2}\pi$ out of phase with $E_{x,\alpha}$. We note that for H parallel to k, $K_1(0)=-1$ and $K_2(0)=+1$, i.e., the two waves are circularly polarized with opposite helicities. When k is perpendicular to H, $K_1(\frac{1}{2}\pi)=0$ and $K_2(\frac{1}{2}\pi)=-\infty$, i.e., $E_{x,2}=0$ and $E_{y,1}=0$, giving linear polarization in the xy plane.

In the following we assumed $\omega\gg\Omega_H$. To calculate the total cross section $\sigma(\theta)$ for each mode, we first resolve the $\mathbf{E}$ vector into an elliptically polarized wave in the plane perpendicular to H and a component E_z along H. H defines the z' axis of a new Cartesian system obtained by rotating the original system xyz about x through an angle θ. The new system x', y', $z'\equiv xy'z'$ will have the plane xy' perpendicular to H. In terms of E_x, E_y, E_z, the components $E_{x'}$, $E_{y'}$, $E_{z'}$ are given by

$$\begin{aligned} E_{x'} &= E_x\,, \\ E_{y'} &= E_y\cos\theta - E_z\sin\theta\,, \\ E_{z'} &= E_y\sin\theta + E_z\cos\theta\,. \end{aligned} \tag{9}$$

III. CROSS SECTION FOR THOMSON SCATTERING IN MAGNETIZED PLASMA

For nonrelativistic electrons subject to a time-varying $E(t)$ in a plasma imbedding a uniform H, the equation of motion of the electrons is

$$m\dot{\mathbf{v}} = e\mathbf{E}(t) + (1/c)e\mathbf{v}\times\mathbf{H}\,. \tag{10}$$

In the x', y', z' coordinate system we define the electric field components of the electromagnetic wave by

$$\mathbf{E}(t) = (E_{x'}{}^0\hat{e}_{x'} + iE_{y'}{}^0\hat{e}_{y'} + iE_{z'}{}^0\hat{e}_{z'})e^{-i\omega t} \tag{11}$$

and the induced electron velocity by

$$\mathbf{v}(t) = (v_{x'}{}^0\hat{e}_{x'} + iv_{y'}{}^0\hat{e}_{y'} + iv_{z'}{}^0\hat{e}_{z'})e^{-i\omega t}\,. \tag{12}$$

Then, from Eq. (9) it follows that

$$v_{x'} = [ie\omega/m(\omega_H{}^2-\omega^2)](-E_{x'} + u^{1/2}E_{y'})\,, \tag{13}$$

$$v_{y'} = [ie\omega/m(\omega_H{}^2-\omega^2)](-E_{y'} + u^{1/2}E_{x'})\,, \tag{14}$$

$$v_{z'} = (ie/m\omega)E_{z'}\,. \tag{15}$$

The time-averaged power per unit solid angle in the θ'' direction radiated by the electron accelerated by the incident wave moving at angle θ is (α mode, $\alpha=1, 2$)

$$\left\langle\frac{dp_\alpha(\theta,\theta'')}{d\Omega''}\right\rangle = \frac{e^2\omega^2}{8\pi c^3}\sum_{\beta=1}^{2} |(\mathbf{v}_\alpha{}^*\cdot\hat{\mathbf{E}}_\beta)\hat{\mathbf{E}}_\beta|^2 n_\beta(\theta'')\cos\delta_\beta''\,. \tag{16}$$

Here $\mathbf{v}_\alpha$ is the velocity vector whose components are given in Eqs. (12)–(14), when the incident electromagnetic wave has the components given in Eqs. (1) and (7). The index $n_\beta(\theta'')$ is the index of refraction of Eq. (6) for a radiated β-mode wave whose wave vector is in the θ'' direction. The unit vector $\hat{E}_\beta$ is the unit electric field vector of this same mode. Since the α mode can scatter into both ordinary and extraordinary modes, the sum over β includes the two possible scattered waves. The angle δ_β'' is the angle between the scattered Poynting vector S_β'' and the wave vector k'', and is given by[7]

$$\tan\delta_\beta'' = -\frac{1}{2n_\beta{}^2(\theta'')}\frac{\partial n_\beta{}^2(\theta'')}{\partial\theta''}\,. \tag{17}$$

The differential cross section for scattering from θ to θ'' is

$$\frac{d\sigma_\alpha(\theta,\theta'')}{d\Omega''} = \frac{1}{\langle S_\alpha\rangle}\left\langle\frac{dP_\alpha(\theta,\theta'')}{d\Omega}\right\rangle, \tag{18}$$

where the incident power flux $\langle S_\alpha\rangle$ is given by

$$\begin{aligned}\langle S_\alpha\rangle &= (c/8\pi)\,\mathrm{Re}(\mathbf{E}^*\times\mathbf{H}) \\ &= (c/8\pi)n_\alpha(\theta)[E_x{}^2+E_y{}^2+E_z{}^2]\cos\delta_\alpha\,,\end{aligned} \tag{19}$$

where δ_α is the angle between S_α and $\mathbf{k}$ and is given by

$$\tan\delta_\alpha = -\frac{1}{2n_\alpha{}^2(\theta)}\left(\frac{\partial n_\alpha{}^2(\theta)}{\partial\theta}\right). \tag{20}$$

The resulting differential cross section is extremely complicated and is given explicitly in Appendix A. For applications in regimes characteristic of stellar atmospheres, the two indices of refraction for the two characteristic modes are usually approximately equal; the angle δ between the wave vector and the energy flux vanishes. This occurs when either $u\ll1$ or $v\ll1$. In both cases the scattering cross section simplifies enormously. The total cross section for scattering then becomes

$$\begin{aligned}\sigma_\alpha/\sigma_{\mathrm{Th}} = {}&[1+K_\alpha{}^2(\theta)+L_\alpha{}^2(\theta)]^{-1} \\ &\times\{[1-u^{1/2}(K_\alpha(\theta)\cos\theta - L_\alpha(\theta)\sin\theta)]^2(u-1)^{-2} \\ &+[u^{1/2}-(K_\alpha(\theta)\cos\theta-L_\alpha(\theta)\sin\theta)]^2(u-1)^{-2} \\ &+[K_\alpha(\theta)\sin\theta+L_\alpha(\theta)\cos\theta]^2\}\,,\end{aligned} \tag{21}$$

with σ_{Th} the usual Thomson cross section

$$\sigma_{\mathrm{Th}} = (8\pi/3)(e^2/mc^2)^2\,. \tag{22}$$

Near $\theta=0$,

$$\sigma_1(\theta)\cong\sigma_{\mathrm{Th}}[\omega^2/(\omega_H-\omega)^2+\tfrac{1}{2}\sin^2\theta] \tag{23}$$

and

$$\sigma_2(\theta)\cong\sigma_{\mathrm{Th}}[\omega^2/(\omega_H+\omega)^2+\tfrac{1}{2}\sin^2\theta]\,. \tag{24}$$

Near $\theta=\frac{1}{2}\pi$,

$$\sigma_1(\theta)\cong\sigma_{\mathrm{Th}}[\omega^2/(\omega-\omega_H)^2+\cos^2\theta]\,, \tag{25}$$

$$\sigma_2(\theta)\cong\sigma_{\mathrm{Th}}\sin^2\theta\,. \tag{26}$$

Results for various θ and u in the limit $v\ll1$ are shown in Fig. 1.

APPENDIX A: CLASSICAL CALCULATION OF DIFFERENTIAL SCATTERING CROSS SECTION

Let H be along the z' axis of a Cartesian coordinate system $x'y'z'$ (Fig. 2). Let the scattered wave vector k'' be along the z'' axis of a new system $x''y''z''=x'y''z''$. H, k'', and $\mathbf{S}_\beta''$, the Poynting vector of the scattered wave, are all coplanar. The electric field vector of an elliptical mode in the double-primed coordinate system can be expressed in the primed system as

$$\begin{aligned}E_{x'}&=E_{x''}\cos\varphi''+i(E_{y''}\cos\theta''\sin\varphi''\\&\quad+E_{z''}\sin\theta''\sin\varphi'')\equiv A+iB\,,\\E_{y'}&=-E_{x''}\sin\varphi''+i(E_{y''}\cos\theta''\cos\varphi''\\&\quad+E_{z''}\sin\theta''\cos\varphi'')\equiv C+iD\,,\\E_{z'}&=i(-E_{y''}\sin\theta''+E_{z''}\cos\theta'')\equiv iF\,.\end{aligned}\tag{A1}$$

The unit vector $\hat{E}_\beta$ is given by

$$\hat{E}_\beta=\frac{E_{x'}\,\hat{e}_{x'}+E_{y'}\,\hat{e}_{y'}+E_{z'}\,\hat{e}_{z'}}{(\mathbf{E}\cdot\mathbf{E}^*)^{1/2}}=\frac{(A+iB)\,\hat{e}_{x'}+(C+iD)\,\hat{e}_{y'}+iF\,\hat{e}_{z'}}{(\mathbf{E}\cdot\mathbf{E}^*)^{1/2}}\,,\tag{A2}$$

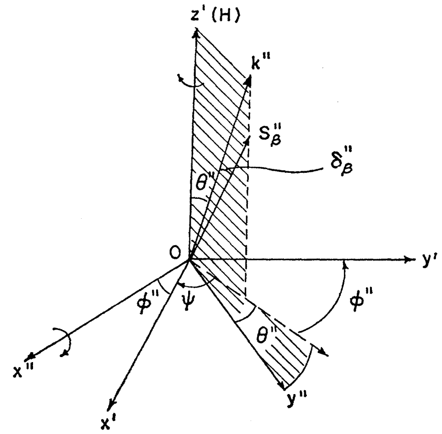

FIG. 2. Geometry of the scattering mechanism. For details see Appendix A.

where

$$\mathbf{E}\cdot\mathbf{E}^*=E_{x'}{}^2+E_{y'}{}^2+E_{z'}{}^2=E_{x''}{}^2+E_{y''}{}^2+E_{z''}{}^2\,.\tag{A3}$$

Now $|(v_\alpha{}^*\cdot\hat{E}_\beta)\hat{E}_\beta|^2=(V_\alpha{}^*\cdot\hat{E}_\beta)(V_\alpha\cdot\hat{E}_\beta{}^*)$. Using Eqs. (12)–(15) and (A2), we get

$$\begin{aligned}(\mathbf{E}\cdot\mathbf{E}^*)|(v_\alpha{}^*\cdot\hat{E}_\beta)\hat{E}_\beta|^2&=|v_{x'}|^2(A^2+B^2)+|v_{y'}|^2(C^2+D^2)+|v_{z'}|^2F^2+2v_{x'}v_{y'}(BC-AD)-2v_{z'}v_{x'}AF-2v_{z'}v_{y'}FD\\&=(e/m)^2[\omega^2/(\omega_H{}^2-\omega^2)]^2[(A^2+B^2)(E_{x'}-u^{1/2}E_{y'})^2+(u^{1/2}E_{x'}-E_{y'})^2(C^2+D^2)+\omega^{-4}(\omega_H{}^2-\omega^2)^2E_{z'}{}^2F^2\\&\quad-2(BC-AD)(E_{x'}-u^{1/2}E_{y'})(E_{y'}-u^{1/2}E_{x'})+2AFE_{z'}(E_{x'}-u^{1/2}E_{y'})\omega^{-2}(\omega^2-\omega_H{}^2)\\&\quad+2FDE_{z'}(E_{y'}-u^{1/2}E_{x'})\omega^{-2}(\omega^2-\omega_H{}^2)]\,.\end{aligned}\tag{A4}$$

The components $E_{x'}$, $E_{y'}$, and $E_{z'}$ can be written in terms of E_x, E_y, and E_z using Eq. (9). The resulting averaged power scattered per unit solid angle $d\Omega''$ using Eq. (16) is given by

$$\left\langle\frac{dP_\alpha(\theta\to\theta'',\varphi'')}{d\Omega''}\right\rangle=r_0{}^2\frac{c}{8\pi}\left(\frac{\omega^2}{\omega_H{}^2-\omega^2}\right)^2\sum_{\beta=1}^{2}\{\cdots\}n_\beta(\theta'')\cos\delta_\beta''\,,\tag{A5}$$

where $r_0=e^2/mc^2$ is the classical electron radius and

$$\begin{aligned}\{\cdots\}=\{&[E_x-u^{1/2}(E_y\cos\theta-E_z\sin\theta)]^2[\cos^2\varphi''+(K_\beta(\theta'')\cos\theta''\sin\varphi''+L_\beta(\theta'')\sin\theta''\sin\varphi'')^2]\\&+[u^{1/2}E_x-(E_y\cos\theta-E_z\sin\theta)]^2[\sin^2\varphi''+(K_\beta(\theta'')\cos\theta''\cos\varphi''+L_\beta(\theta'')\sin\theta''\cos\varphi'')^2]\\&+\omega^{-4}(\omega_H{}^2-\omega^2)(E_y\sin\theta+E_z\cos\theta)^2[L_\beta(\theta'')\cos\theta''-K_\beta(\theta'')\sin\theta'']^2+2[u^{1/2}(E_y\cos\theta-E_z\sin\theta)-E_x]\\&\times[u^{1/2}E_x-(E_y\cos\theta-E_z\sin\theta)][\sin\varphi''(K_\beta(\theta'')\cos\theta''\sin\varphi''+L_\beta(\theta'')\sin\theta''\sin\varphi'')\\&+\cos\varphi''(K_\beta(\theta'')\cos\varphi''\cos\theta''+L_\beta(\theta'')\sin\theta''\cos\varphi'')]+2\omega^{-2}(\omega_H{}^2-\omega^2)(E_y\sin\theta+E_z\cos\theta)\\&\times[u^{1/2}(E_y\cos\theta-E_z\sin\theta)-E_x][L_\beta(\theta'')\cos\theta''-K_\beta(\theta'')\sin\theta'']\cos\varphi''+2\omega^{-2}(\omega_H{}^2-\omega^2)(E_y\sin\theta+E_z\cos\theta)\\&\times[u^{1/2}E_x-(E_y\cos\theta-E_z\sin\theta)][K_\beta(\theta'')\cos\theta''\cos\varphi''+L_\beta(\theta'')\sin\theta''\cos\varphi'']\\&\times[L_\beta(\theta'')\cos\theta''-K_\beta(\theta'')\sin\theta'']\}[1+K_\beta{}^2(\theta'')+L_\beta{}^2(\theta'')]^{-1}\,.\end{aligned}\tag{A6}$$

The averaged incident Poynting vector $\langle\mathbf{S}_\alpha\rangle$ is given by

$$\langle S_\alpha\rangle=[cn_\alpha(\theta)/8\pi](E_x{}^2+E_y{}^2+E_z{}^2)\cos\delta_\alpha\,.\tag{A7}$$

The differential scattering cross section is

$$\frac{d\sigma_\alpha(\theta\to\theta'',\varphi'')}{d\Omega''}=\frac{1}{\langle S_\alpha\rangle}\left\langle\frac{dP_\alpha(\theta\to\theta'',\varphi'')}{d\Omega''}\right\rangle=r_0{}^2\left(\frac{\omega^2}{\omega_H{}^2-\omega^2}\right)^2\sum_{\beta=1}^{2}\frac{G_\beta(\theta,\theta'',\varphi'')n_\beta(\theta'')\cos\delta_\beta''}{[1+K_\alpha{}^2(\theta)+L_\alpha{}^2(\theta)]n_\alpha(\theta)\cos\delta_\alpha}\,,\tag{A8}$$

where $G_\beta(\theta,\theta'',\varphi'')$ is the same as Eq. (A6) with the substitution $E_x \to 1$, $E_y \to K_\alpha(\theta)$, and $E_z \to L_\alpha(\theta)$. The total cross section is obtained by integrating $(d\sigma_\alpha/d\Omega'')$ over $d\Omega''$, i.e.,

$$\sigma_{\alpha,\beta}=\int_\sigma d\Omega''\frac{d\sigma_{\alpha,\beta}}{d\Omega''}\,, \tag{A9}$$

where $d\Omega''=\sin\theta'' d\theta'' d\psi=-\sin\theta'' d\theta'' d\varphi''$, since

$$\psi=\tfrac{1}{2}\pi-\varphi''.$$

APPENDIX B: THOMSON SCATTERING IN MAGNETIC FIELD IN QUANTUM THEORY

We exhibit here, for simple cases, the quantum mechanical scattering cross section which, as it must, is that for the classical calculation of this same case.

In the nonrelativistic limit the interaction Hamiltonian of an electron with the photon field $\mathbf{A}(x)$ is given by

$$H=(e/mc)\boldsymbol{\pi}\cdot\mathbf{A}+(e^2/2m^2c^2)\mathbf{A}\cdot\mathbf{A}. \tag{B1}$$

Here $\mathbf{A}(x)$ is the photon-field vector potential operator which in a medium is[8]

$$\mathbf{A}(x)=\left(\frac{2\pi\hbar c^2}{\omega\Omega}\right)^{1/2}\left(\frac{2\,\mathrm{Tr}\lambda_{ij}}{\omega(\partial\Lambda/\partial\omega)}\right)^{1/2}\mathbf{e}\;e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\equiv\left(\frac{2\pi\hbar c^2}{\omega\Omega}\right)^{1/2}N_\gamma^{1/2}\mathbf{e}\;e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{B2}$$

where

$$\boldsymbol{\pi}=\mathbf{p}-(e/c)\mathbf{A}_{\mathrm{ext}},\quad \mathbf{A}_{\mathrm{ext}}=\tfrac{1}{2}\mathbf{r}\times\mathbf{H}. \tag{B3}$$

The quantities λ_{ij} and Λ are the cofactors and determinant of the Maxwell operator Λ_{ij},[8]

$$\Lambda_{ij}=(c^2k^2/\omega^2)(k_ik_j-\delta_{ij})+\epsilon_{ij}\,, \tag{B4}$$

where ϵ_{ij} is the dielectric constant of the medium. In conventional magneto-ionic theory, the polarization vector $\hat{\mathbf{e}}$ is given by [for details and notations, see Ref. 9]

$$\mathbf{e}=(1+\alpha^2)^{-1/2}\left[\frac{D(P-n^2\cos^2\theta)}{SP-An^2},\,i,\,-\frac{Dn^2\sin\theta\cos\theta}{SP-An^2}\right],$$

$$\alpha\equiv PD\cos\theta/(SP-An^2). \tag{B5}$$

For longitudinal and transverse propagation, we have

$$\begin{aligned}&\theta=0:\quad e(o)=2^{-1/2}[1,i,0],\quad e(x)=2^{-1/2}[-1,\,i,\,0],\\&\theta=\tfrac{1}{2}\pi:\; e(o)=[0,0,i],\qquad e(x)=[0,1,0],\end{aligned} \tag{B6}$$

where o and x stand for ordinary and extraordinary waves.

[8] D. B. Melrose, Astrophys. Space Sci. **2**, 171 (1968).

[9] V. Canuto, C. Chiuderi, and C. K. Chou, Astrophys. Space Sci. **7**, 407 (1970).

Although the Compton scattering could, in principle, be computed for any angle and energy, the final form is excessively complicated. We consider only two important cases: the propagation at $\theta=0$ and $\theta=\frac{1}{2}\pi$, where θ is the angle between the propagation vector k and the magnetic field axis. For these two directions only, does the plasma wave have $\mathbf{S}$ and $\mathbf{k}$ parallel.

The differential cross section is

$$\frac{d\sigma}{d\Omega}=\tfrac{1}{2}r_0^2N_\gamma(\omega'/\omega)\times(|\mathbf{e}_1\cdot\mathbf{e}_2\langle f|e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}|i\rangle-D-E|^2). \tag{B7}$$

The uncrossed and crossed matrix elements D and E are qiven by

$$D=c\sum_I\frac{\langle f|\boldsymbol{\pi}\cdot\mathbf{e}_fe^{-i\mathbf{k}'\cdot\mathbf{x}}|I\rangle\langle I|\boldsymbol{\pi}\cdot\mathbf{e}_ie^{i\mathbf{k}\cdot\mathbf{x}}|i\rangle}{E_I-E_i-\hbar\omega}\,, \tag{B8}$$

$$E=c\sum_I\frac{\langle f|\boldsymbol{\pi}\cdot\mathbf{e}_ie^{i\mathbf{k}\cdot\mathbf{x}}|I\rangle\langle I|\boldsymbol{\pi}\cdot\mathbf{e}_fe^{-i\mathbf{k}'\cdot\mathbf{x}}|i\rangle}{E_I-E_i+\hbar\omega'}\,. \tag{B9}$$

The symbols i, f, and I specify the initial, final, and intermediate states, respectively. The quantum numbers characterizing an electron in a magnetic field are the principal quantum numbers n $(=0, 1, 2, \ldots)$, the orbital quantum numbers l $(=0, 1, 2, \ldots)$, and the momentum in the z direction p_z. The exact solution of Schrödinger's equation in a magnetic feld gives the following eigenvalues and eigenfunctions[10]:

$$E/mc^2=(P_z/mc)^2+(H/H_q)(n+\tfrac{1}{2}+\sigma)\,, \tag{B10}$$

$$\begin{aligned}&H_q=(m^2c^3/e\hbar)=4.41\times10^{13}\ \mathrm{G},\quad \sigma=\pm\tfrac{1}{2}\\&\psi(r)=L^{-1/2}e^{i\mathbf{p}\cdot z/\hbar}\Phi_{n,l}(\rho,\varphi)\,,\end{aligned} \tag{B11}$$

where

$$\Phi_{n,l}(\rho,\varphi)=\frac{e^{il\varphi}}{(2\pi)^{1/2}}(2\gamma)^{1/2}I_{n,s}(\gamma\rho^2)\,, \tag{B12}$$

$$\begin{aligned}&\gamma=\tfrac{1}{2}(H/H_q)(1/\lambda\!\!\!^{-}_c{}^2)\,,\quad \lambda\!\!\!^{-}_c=\hbar/mc\,,\quad l=n-s\\&I_{n,s}(x)=(n!s!)^{-1/2}x^{l/2}e^{-x/2}Q_s{}^l(x)\,,\end{aligned} \tag{B13}$$

and $Q_s{}^l(x)$ are Laguerre polynomials,[11]

$$Q_s{}^l(x)=x^{-l}e^x(d^s/dx^s)(x^ne^{-x})\,. \tag{B14}$$

Using the relations[11]

$$\begin{aligned}&(\pi_x+i\pi_y)|n,l\rangle\\&\quad=mc[2(n+1)H/H_q]^{1/2}|n+1,\,l+1\rangle\,,\\&(\pi_x-i\pi_y)|n,l\rangle=mc(2nH/H_q)^{1/2}|n-1,\,l-1\rangle\,,\end{aligned} \tag{B15}$$

The matrix elements M can be easily evaluated.

The various forms listed below depend on whether the initial and final waves are ordinary (left-handed,

[10] V. Canuto and H. Y. Chiu, Phys. Rev. **173**, 1210 (1968); **173**, 1220 (1968); **173**, 1229 (1968); V. Canuto, H. Y. Chiu, and L. Fassio-Canuto, Astrophys. Space Sci. **3**, 258 (1969).

[11] A. A. Sokolov and I. M. Ternov, *Synchrotron Radiation* (Akademie-Verlag, Berlin, 1968).

lh) or extraordinary (right-handed, rh):

(lh, lh):

$$M=\delta_n{}^{n'}\left[1-\omega_H\left(\frac{n+1}{\omega_H-\omega}-\frac{n}{\omega_H-\omega'}\right)\right], \tag{B16}$$

(rh, rh):

$$M=\delta_n{}^{n'}\left[1-\omega_H\left(\frac{n+1}{\omega_H+\omega}-\frac{n}{\omega_H+\omega'}\right)\right], \tag{B17}$$

(rh, lh):

$$M=\omega_H[n(n-1)]^{1/2}\left(\frac{1}{\omega-\omega_H}-\frac{1}{\omega'+\omega_H}\right)\delta_{n'}{}^{n-2}$$
$$=0 \quad \text{since} \quad \omega'=\omega+(n-n')\omega_H=\omega-2\omega_H, \tag{B18}$$

(lh, rh):

$$M=\omega_H[(n+1)(n+2)]^{1/2}\left(\frac{1}{\omega_H-\omega'}+\frac{1}{\omega_H+\omega}\right)\delta_{n'}{}^{n-2}$$
$$=0 \quad \text{since} \quad \omega'=\omega+2\omega_H. \tag{B19}$$

In the first two equations the first term comes from the so-called seagull diagram which does not exist in the "mixed" cases.

The result is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{rh,lh}}=r_0{}^2\frac{\omega^2}{(\omega\pm\omega_H)^2}, \tag{B20}$$

which is the classical result.

An exactly similar calculation confirms the classical result for $\theta=\frac{1}{2}\pi$.

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Spectrum of High-Energy Electrons Undergoing Klein-Nishina Losses

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A solution is presented for the spectrum of high-energy electrons confined to a region where the main energy-loss mechanism is Compton scattering in the extreme Klein-Nishina limit. The solution is valid in the steady-state limit assuming the region under consideration is optically thin to the emitted photons.

I. INTRODUCTION

FOR many astrophysical applications it is desirable to be able to solve for the spectrum of high-energy particles undergoing energy loss via certain physical mechanisms. Blumenthal and Gould,[1] however, have recently emphasized that when particles lose energy in discrete amounts as opposed to continuously, then it is appropriate to describe their distribution function by an integrodifferential equation as opposed to the ordinary continuity equation in energy space. If $n(\gamma,t)$ is the number (or density) of particles with Lorentz factor $\gamma=(1-\beta^2)^{-1/2}$ between γ and $\gamma+d\gamma$ at time t, then the distribution function satisfies an equation of the form

$$\frac{\partial n(\gamma,t)}{\partial t}+\frac{\partial}{\partial\gamma}[\dot{\gamma}n(\gamma,t)]+n(\gamma,t)\int_1^\gamma P(\gamma,\gamma')d\gamma' -\int_\gamma^\infty d\gamma' n(\gamma',t)P(\gamma',\gamma)=q(\gamma). \tag{1}$$

Here $\dot{\gamma}$ represents the total energy loss[2] due to continuous processes where the energy change per collision is small:

$$|\dot{\gamma}/\gamma|\ll\nu, \tag{2}$$

ν being the collision frequency. The $P(\gamma,\gamma')d\gamma'$ in Eq. (1) represents the total probability per unit time that a particle of energy γ will lose an amount of energy between $\gamma-\gamma'$ and $\gamma-\gamma'-d\gamma'$ via all processes for which condition (2) is not satisfied. Finally, the $q(\gamma,t)$ in Eq. (1) represents possible sources and sinks of particles corresponding to creation or annihilation.

For many energy-loss mechanisms for high-energy electrons of astrophysical interest, condition (2) is amply satisfied, and the resulting continuity equation can be readily solved.[3] However, this is no longer true for electrons losing energy by either bremsstrahlung or Compton scattering in the extreme Klein-Nishina (KN) limit.[1] Indeed, for the latter process, the spectrum of emitted photons ϵ_1 is strongly peaked near $\epsilon_1\approx\gamma$. These processes must then be included in the integral terms in Eq. (1), thus making exact analytic solutions much more difficult to obtain even if time dependence

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[1] G. R. Blumenthal and R. J. Gould, Rev. Mod. Phys. **42**, 237 (1970).

[2] Since this paper deals with high-energy electrons, all energies are expressed in units of mc^2.

[3] See, e.g., N. Kardashev, Astronom. Zh. **39**, 393 (1962) [Soviet Astron. AJ **6**, 317 (1962)].