Theoretical and Experimental Consequences of a Violation of the Pomeranchuk Theorem on Total Cross Sections at High Energies

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Bounds are obtained on near-forward scattering amplitudes that violate the Pomeranchuk theorem. It is proved that these amplitudes have zeros in $C'<|t|\ln^2 s<C''$. Experimental consequences are estimated, including phases of near-forward amplitudes, growth of the forward peak and shrinkage of its width, and possible oscillations. A narrow peak in the Coulomb region cannot be excluded and may invalidate experimental estimates of total cross sections.

I. ASSUMPTIONS

THE data on total cross sections obtained at Serpukhov[1] are open to a variety of possible interpretations, which depend mainly on the type of variation that is allowed in the cross sections at energies higher than those measured. In this paper we investigate the consequences of assuming that total cross sections for particle-target and antiparticle-target collisions tend to unequal constants at asymptotic energies. This problem has previously been considered qualitatively by one of us,[2] and in a special model by Finkelstein,[3] and in a special class of models by Casella.[4] We will show that certain properties deduced in the special models can also be proved from axiomatic quantum field theory. One of our general results has also been proved independently by Kinoshita,[5] using similar assumptions. These are as follows.

(a) For a particle a_1 and its antiparticle a_2, as $s\to\infty$, $\sigma_1(\text{total}, a_1+b)\to C_1$, $\sigma_2(\text{total}, a_2+b)\to C_2$.

(b) The Pomeranchuk theorem is violated,

$$C_1\neq C_2.$$

(c) Scattering amplitudes have the analyticity and the asymptotic growth properties that have been proved from axiomatic quantum field theory.[6]

(d) Spin is neglected, but we do not believe our results depend on this.

In Sec. II we describe our theoretical method and results, and in Sec. III we indicate possible experimental consequences.

[1] J. V. Allaby *et al.*, Phys. Letters **30B**, 500 (1969).

[2] R. J. Eden, Phys. Rev. D 2, 529 (1970); see also R. J. Eden, Rev. Mod. Phys. **43**, 15 (1971).

[3] J. Finkelstein, Phys. Rev. Letters **24**, 172 (1970); **24**, 472 (1970); H. Cornille, Orsay Report Nos. IPNO/TH 178, 1970 and IPNO/TH 186, 1970 (unpublished); A. A. Anselm, G. S. Danilov, I. T. Dyatlov and E. M. Levin, Yadern. Fiz. **11**, 896 (1970) [Soviet J. Nucl. Phys. **11**, 500 (1970)]; V. N. Gribov, I. Yu. Kobsarev, V. D. Mur, L. B. Okun, and V. S. Popov, Phys. Letters **32B**, 129 (1970).

[4] R. C. Casella, Phys. Rev. Letters **24**, 1463 (1970).

[5] T. Kinoshita, Phys. Rev. D **2**, 2346 (1970).

[6] Reviewed by A. Martin, *Scattering Theory: Unitarity Analyticity and Crossing* (Springer, New York, 1969), and by R. J. Eden, Rev. Mod. Phys. **43**, 15 (1971).

II. THEORETICAL DISCUSSION AND RESULTS

$F(s,t)$ will be used to denote either F_1 or F_2, which are the elastic scattering amplitudes for particle and antiparticle, respectively. From the forward dispersion relations, using assumptions (a) and (b), it follows that, as $s\to\infty$,[7]

$$F_A(s,0)=(F_1-F_2)\sim(2s/\pi)(C_2-C_1)(\ln s-\tfrac{1}{2}i\pi)\,,\quad(1)$$

$$F_S(s,0)=(F_1+F_2)\sim is(C_2+C_1)\,.\quad(2)$$

It has been proved by Martin[8] that the partial-wave series can be truncated after L terms with negligible error as $s\to\infty$, if $|t|<t_0$; the nearest singularity:

$$F(s,t)\sim 16\pi\sum_{l=0}^{L}(2l+1)f_l(s)P_l\left(1+\frac{2t}{s}\right),\quad(3)$$

$$L=Cs^{1/2}\ln s\quad(4)$$

where, for example, $C=1/4m_\pi$ for πN scattering. We measure s and t in GeV^2 (but see Sec. III).

Using partial-wave unitarity and the Cauchy inequality, we have[9]

$$\frac{1}{(16\pi s)^2}|F(s,0)|^2 \leqslant\left[\frac{1}{s}\sum(2l+1)|f_l(s)|^2\right]\left[\frac{1}{s}\sum(2l+1)\right]\quad(5)$$

$$\leqslant[\pi\sigma(\text{elastic})C^2\ln^2 s]/16\pi\,.\quad(6)$$

From (1), (2), and (6) it follows that

$$\frac{1}{16\pi}\left(\frac{C_2-C_1}{\pi C}\right)^2\leqslant\sigma(\text{elastic})\leqslant\max(C_1,C_2)=C_M\,.\quad(7)$$

Using unitarity and the Cauchy inequality for $t\neq0$,

[7] See, e.g., R. J. Eden, *High Energy Collisions of Elementary Particles* (Cambridge U. P., New York, 1967).

[8] A. Martin, Phys. Rev. **129**, 1432 (1963).

[9] T. Kinoshita, in *Perspectives in Modern Physics*, edited by R. E. Marshak (Wiley, New York, 1966); R. J. Eden, Phys. Rev. Letters **16**, 39 (1964).

we obtain

$$\frac{1}{(16\pi s)^2}|F(s,t)|^2\leqslant\left[\frac{1}{s}\sum(2l+1)|f_l(s)|^2\right]\times\left[\frac{1}{s}\sum(2l+1)\left|P_l\left(1+\frac{2t}{s}\right)\right|^2\right],\quad(8)$$

provided $|t|<t_0$. For large l and s,

$$|P_l(1+2t/s)|\sim\exp(2l|t/s|^{1/2}),\quad(9)$$

$$|f(s,t)|^2\equiv\left|\frac{F(s,t)}{F(s,0)}\right|^2\leqslant\frac{16\pi s\sigma(\mathrm{el})L^2\exp(4L|t/s|^{1/2})}{|F(s,0)|^2}.\quad(10)$$

Using (4), (7), and (10) as $s\rightarrow\infty$, we obtain

$$|f(s,t)|\leqslant C_3\exp(2C|t|^{1/2}\ln s),\quad(11)$$

where C_3 denotes $4\pi^{1/2}C_M{}^{1/2}(\pi C/|C_2-C_1|)$.

We now use a theorem of Bessis,[10] which states that if a function $f(t)$ is regular in $|t|<R$, if $f(0)=1$ and $|f(t)|<M(R)$ on $|t|=R$, then $f(t)$ has no zero inside a circle of radius $r=R/M(R)$. Using (11), this theorem proves that $f(s,t)$ can have no zeros inside:

$$|t|<(C_3)^{-1}R\exp(-2CR^{1/2}\ln s).\quad(12)$$

Maximizing this value by taking $CR^{1/2}\ln s=1$, we see that $f(s,t)$ can have no zeros in

$$|t|\leqslant r_0(s)=\frac{C_4}{\ln^2 s}=\frac{1}{e^2C_3C^2\ln^2 s}.\quad(13)$$

We prove next that there must be a zero of $f(s,t)$ in a circle in the t plane whose radius r_1 is a constant multiple of $r_0(s)$. Let $f(s,t)$ have no zeros in $|t|\leqslant R(s)$. Then $\ln f(s,t)$ will be regular in this circle, and from (11) for $t<R$,

$$\mathrm{Re}[\ln f(s,t)]\leqslant 2CR^{1/2}\ln s+\ln C_3.\quad(14)$$

From (14), Caratheodory's inequality[11,12] gives

$$|\ln f(s,t)|\leqslant\frac{2|t|(2CR^{1/2}\ln s+\ln C_3)}{R-|t|}.\quad(15)$$

Hence, using (1) and (15)

σ(elastic)

$$\geqslant\left(\frac{C_2-C_1}{\pi}\right)^2\frac{\ln^2 s}{16\pi}\int_{-\frac{1}{2}R}^{0}dt|f(s,t)|^2\quad(16)$$

$$\geqslant\left(\frac{C_2-C_1}{\pi}\right)^2\frac{\ln^2 s}{16\pi}\int_{-\frac{1}{2}R}^{0}dt\exp(16CtR^{-1/2}\ln s)\quad(17)$$

$$\geqslant\left(\frac{C_2-C_1}{\pi}\right)^2\frac{R^{1/2}\ln s}{256\pi C}[1-\exp(-8CR^{1/2}\ln s)].\quad(18)$$

We know from (13) that $R>r_0$, so the exponential in (18) is not equal to 1. From (18) we find

$$R(s)\leqslant r_1(s)=C_5/\ln^2 s.\quad(19)$$

This proves that $f(s,t)$ has at least one zero in $|t|\leqslant r_1(s)$. Suppose now that there is just one zero of $f(s,t)$ at $t_1(s)$ in $|t|<R_2(s)$. Apply the inequalities (14) and (15) to the function $f_1(s,t)=f(s,t)(1-t/t_1)^{-1}$ and repeat the steps leading to (19). We find only a change in the constant C_5. The same conclusion holds for any finite number N of zeros. Given N, we can find a constant D_{N} such that there must be N zeros in

$$|t|<D_{\mathrm{N}}/\ln^2 s.\quad(20)$$

This proves the general result on zeros that was first noticed in the Finkelstein model.[3]

From the bounds (13) and (15), we see that $|f(s,t)|^2$ cannot decrease to less than e^{-1} when

$$|t|\leqslant\frac{R}{4\ln C_3}\leqslant r_2(s)=\frac{C_4}{4\ln C_3\ln^2 s}.\quad(21)$$

Note that we have chosen the least allowed value of R in this inequality. This sets a lower bound $r_2(s)$ on the width of the forward peak of the elastic cross section.

We can obtain an upper bound on the width of the forward peak by assuming in $0>t>-r_3$

$$|f(s,t)|^2\geqslant e^{t/r_3}.\quad(22)$$

Substituting in (16), we obtain

$$r_3(s)\leqslant\left(\frac{\pi}{C_2-C_1}\right)^2\frac{16\pi C_M}{\ln^2 s}.\quad(23)$$

The lower bound $r_2(s)$ on the width given by (21) provides a bound on the region in which Kinoshita's result[5] applies. It also proves his result that differential cross sections for elastic scattering of particles and of antiparticles become asymptotically equal if $|t|<$const $\times(\ln s)^{-2}$. This follows from our discussion because, when (21) holds, the two amplitudes will both be dominated by the real part of $F_A(s,t)$ containing an $s\ln s$ term.

Although we have located N zeros in the region (20), we have not proved that they tend to the physical region t (real)<0 as $s\rightarrow\infty$. This result holds for all zeros in the Finkelstein model,[3] but holds only for "most" zeros in the extension by Casella.[4] However, there seems no reason why it should apply to the zeros nearest $t=0$, which have the best chance of being

[10] J. D. Bessis, Nuovo Cimento **45A**, 974 (1966).

[11] See, e.g., E. C. Titchmarsh, *Theory of Functions* (Oxford U.P., New York, 1939).

[12] We are indebted to Dr. André Martin for pointing out the value of Caratheodory's inequality in studying growth rates.

relevant to experiment. The distant zeros provide an interesting mathematical problem, which we will discuss elsewhere.[13] For example, using the method of Eden and Łukaszuk,[14] it can be shown that the number of zeros $N(s,r)$ inside $|t\ln^2 s|<r$ satisfies the inequality

$$N(s,r)<Cer^{1/2} \quad \text{for large } r. \tag{24}$$

III. EXPERIMENTAL CONSEQUENCES

We now consider experimental consequences of our results given in Sec. II based on our interpretation of the Serpukhov data[1] that was stated in Sec. I. For illustration, we take the scale factors s_0 and t_0 in $\ln(s/s_0)$ and t/t_0 to be 1 GeV^2, but we emphasize that they are not known theoretically. We also take asymptotic total cross sections to give

$$\delta=\frac{C_2-C_1}{C_2+C_1}=\frac{1}{10}. \tag{25}$$

We approximate the constants in Eqs. (13), (19), (21), and (23),

$$r_0=10r_2=10^{-3}(\ln s)^{-2}, \quad r_1=10^8(\ln s)^{-2}, \quad r_3=10^2(\ln s)^{-2}. \tag{26}$$

A. Real-to-Imaginary Ratio for Forward Amplitudes

From (1),

$$\frac{\mathrm{Re}F(s,0)}{\mathrm{Im}F(s,0)}\sim\frac{1}{15}\ln s. \tag{27}$$

This would be observable at $s\sim30$ GeV^2, provided the width of the $\ln^2 s$ contribution to the forward peak is not so narrow that the effect occurs only within the region ($t<0.002$) where Coulomb effects dominate.

B. Growth of Forward Peak at Fixed $z=t\ln^2 s$

$$\frac{d\sigma(1\text{ or }2)}{dt}\sim\frac{(C_1+C_2)^2}{32\pi}\left[(1\pm\delta)^2+D_s(z)\left(\frac{\ln s}{15}\right)^2\right], \tag{28}$$

where $D_s(z)\rightarrow1$, as $z\rightarrow0$. From (21) and (23), the width Δ of the forward peak ($1>D>e^{-1}$) satisfies

$$r_2(s)<\Delta(s)<r_3(s). \tag{29}$$

Provided $\Delta(s)$ exceeds $\frac{1}{10}(\ln s)^{-2}$, this effect may give a 10% contribution outside the Coulomb region for $s>100$ GeV^2, which could be observable unless $\Delta(s)$ oscillates with s.

C. Shrinkage of Forward Peak at Fixed t

At fixed t, the coefficient $D_s(z)$ in (28) will give a width $\Delta(s)$ that shrinks like $(\ln s)^{-2}$. Our bounds (29) allow $\Delta(s)$ to oscillate within wide limits. Only if $\Delta(s)$ is fairly large would an effect be observable; it might give a spuriously large (or small) effect due to enhancement by the oscillations of $\Delta(s)$.

D. Asymptotic Equality of Differential Cross Sections

When the last term in Eq. (28) dominates $d\sigma_1$ and $d\sigma_2$, they will be nearly equal. This is obviously outside the range of accessible energies. Even if $\ln(s)\sim100$ were accessible, the dominance could be concealed by Coulomb effects.

E. Oscillations of Differential Cross Sections due to Zeros

If the width $\Delta(s)$ exceeds $\frac{1}{10}(\ln s)^{-2}$ and if the nearest zero of $F(s,t)$ is favorably placed so that $D_s(z)$ in (28) nearly vanishes within the forward peak, an oscillation of a few percent could occur with $s\sim100$ GeV^2, and t outside the Coulomb region.

F. Interference with Coulomb Effects near $t=0$

Our bounds (29) on the width $\Delta(s)$ of the forward peak refer to the asymptotically dominant part that grows like $(\ln s)^2$ as $s\rightarrow\infty$. These bounds do not exclude the possibilities that at the relevant high energies,

(i) the $\ln^2 s$ peak may lie entirely inside the Coulomb dominant region $|t|\ll0.003$ GeV^2, or
(ii) the peak causes a rapid change in the strong-interaction amplitude in the Coulomb interference region $|t|\sim0.003$ GeV^2.

If either of these possibilities does occur, it would invalidate the usual experimental methods for evaluating total cross sections and phases of forward amplitudes.

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[13] R. J. Eden and G. D. Kaiser, Nucl. Phys. (to be published).
[14] R. J. Eden and L. Łukaszuk, Nuovo Cimento 47, 817 (1967).