Characteristics of a Regge Trajectory with a Finite Asymptotic Phase*

DENNIS SIVERS†

Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California 94720

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The hypothesis that the width of a resonance on a leading Regge trajectory is proportional to its mass is compared with the alternative that high-spin resonances become very stable. When the width grows with the mass, it is observed that the kinematics allow decay channels to have low orbital angular momentum. The dynamical implications of this observation are discussed.

I. INTRODUCTION

TAKING the limit of zero-width resonances is an extremely delicate process. In spite of the attractive simplicity of the Veneziano model,[1] detailed analyses of its predictions always conclude that a description of hadronic scattering processes involving meromorphic amplitudes is strained and artificial.[2] As an example of the subtle nature of the narrow-resonance limit, consider a Regge trajectory with a small but finite asymptotic phase.

$$\delta(\infty)=\lim_{t\to\infty}\arctan[\mathrm{Im}\alpha(t)/\mathrm{Re}\alpha(t)]. \tag{1.1}$$

A high-spin particle on such a trajectory can communicate with open channels of low orbital angular momentum, while if the asymptotic phase is zero, these channels are closed kinematically.[3]

It is entirely possible that the dynamics of two cases will be completely different. Observations based on coupled-channel unitarity approximations suggest strongly that the dynamics are dominated by low-orbital-angular-momentum channels. In order to maintain itself, an asymptotically real Regge trajectory could require a rich spectrum of nonleading singularities.[4] In contrast, the dynamics of a leading trajectory with finite asymptotic phase can be compatible with a model proposed by Carruthers[5] where a particle of spin J on the leading trajectory is primarily a bound state in the system formed by a particle of spin $J-1$ on the *same* trajectory and an arbitrary, low-mass particle. In this sort of model, there is no need for all nonleading Regge singularities to rise to high values of J.

Experimental evidence on baryon widths is consistent with these states lying on trajectories with finite asymptotic phase. There is also some indication that this notion is applicable to meson trajectories even though it implies that the very narrow peaks in the CERN missing-mass spectrometer[6] should not be identified with recurrences of the ρ-f trajectory. Instead, these states would represent fine structure in the meson spectrum, perhaps connected with the dip in the A_2 peak.[7]

II. PARAMETRIZATION OF PHYSICAL REGGE TRAJECTORY

Consider a Regge trajectory which, at $t=0$, is the leading J-plane singularity in a nonvacuum channel. For simplicity, ignore the inessential complications of signature and assume the channel has baryon number zero. A parametrization which is particularly convenient for a discussion of the asymptotic behavior of the trajectory function is the phase representation.[8,9]

Let

$$\alpha(t\pm i0)=|\alpha(t)|e^{\pm i\delta(t)},\quad t\geq 4m^2 \tag{2.1}$$

where $\delta(t)$ is the phase of the Regge trajectory above the physical cut beginning at $t=4m^2$. The phase representation for $\alpha(t)$ is then given by

$$\alpha(t)=P(t)\exp\left\{\frac{t}{\pi}\int_{4m^2}^{\infty}\frac{\delta(x)dx}{x(x-t)}\right\}, \tag{2.2}$$

where $P(t)$ is a polynomial. The representation (2.2) is valid if[10] (a) $\alpha(t)$ is analytic in t except for a cut along the real axis, (b) $\alpha(t)$ is real, in the sense that $\alpha^*(t)=\alpha(t^*)$, (c) $\alpha(t)$ is bounded as $|t|\to\infty$ on the physical sheet by a finite polynomial in t, and (d) the phase above the physical cut, $\delta(t)$, goes to a finite limit as $t\to\infty+i0$.

Conditions (a) and (b) are quite generally true for meson trajectories[11] if the singularity surface represented by $\alpha(t)$ does not collide with any other singularity in the partial-wave amplitude. For fermion trajectories, the analyticity condition for the trajectory function is usually derived in terms of the variable $W=t^{1/2}$, but the situation is complicated by the cuts which seem to be present to shield the parity partners implied by Mac-

* Work supported in part by the U. S. Atomic Energy Commission.

† NSF Predoctoral Fellow. Present address: Argonne National Laboratory, Argonne, Ill.

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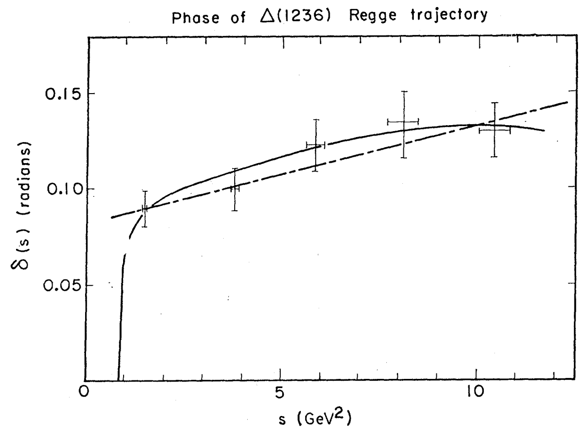

FIG. 1. Phase of the Δ trajectory calculated from the masses and widths of the $I=\frac{3}{2}$ baryons listed in the Particle Data Group (Ref. 21) compilations. A linear extrapolation of the phase gives an approximate lower limit of the energy at which this trajectory might turn over to be $s=160$ GeV^2, but the data are perfectly consistent with an indefinitely rising trajectory.

Dowell symmetry.[12] It is possible that, even in meson channels, trajectory functions develop extra cuts associated with the collision of different singularity surfaces.[13,14] The representation (2.2) can be extended to include contributions from other cuts but it is a sensible first step to deal with trajectory functions having only the physical threshold cut and a discussion of possible complications due to collisions will be deferred to Sec. IV.

Condition (c) seems to be a reasonable restriction on the asymptotic behavior of physically interesting trajectory functions. Although there is no *a priori* reason to prohibit exponential growth of the trajectory function in certain sections of the t plane, such behavior is unexpected from experimental grounds. The phase $\delta(t)$ of the trajectory function is well defined along the physical cut, except, possibly at those points where $|\alpha(t)|=0$. The phase can be shown to be piecewise continuous, and it makes physical sense to limit the magnitude of discontinuities in $\delta(t)$ to be less than π,

$$\lim_{\eta\to 0} |\delta(t_0-\eta)-\delta(t_0+\eta)|<\pi. \qquad (2.3)$$

Consistency with the analytic continuation of the unitarity condition requires

$$\mathrm{Im}\alpha(t+i0)>0, \quad t>4m^2 \qquad (2.4)$$

unless at the point where $\mathrm{Im}\alpha=0$ on the physical cut, the residue function associated with $\alpha(t)$ also vanishes. In conjunction with the requirement that there can be no resonance poles on the physical sheet, it is frequently conjectured that (2.4) holds for all t above threshold.[11] Except at a point where $\mathrm{Re}\alpha=0$, this constraint then leads to the bound

$$0\leq\delta(t)\leq\pi. \qquad (2.5)$$

Condition (d) seems to be a reasonable assumption in view of (2.5). This assumption only eliminates the possibility that the phase oscillates as $t\to\infty$ without approaching a definite limit.

Using conditions (a)–(d), plus the extra assumption that the phase is consistent with the usual unitarity conditions at the elastic threshold,[15] Childers[8] was able to show that the polynomial $P(t)$ in (2.2) has one and only one zero so that it must be of order one,

$$P(t)=a+bt \quad (b\neq 0). \qquad (2.6)$$

Except for possible logarithmic factors, the asymptotic behavior of the exponential in (2.2) is determined by a single parameter $\delta(\infty)$, the asymptotic limit of the phase.[9] Making use of the result (2.6), the asymptotic behavior of the trajectory function in (2.2) is given by

$$\lim_{|t|\to\infty} \alpha(t)=be^{i\delta(\infty)}t^{1-\delta(\infty)/\pi} \qquad (2.7)$$

uniformly on the first sheet. This means, for example, that if the trajectory rises indefinitely ($\mathrm{Re}\alpha\to+\infty$ as $t\to+\infty$) faster than a power of a logarithm, then it also falls indefinitely ($\mathrm{Re}\alpha\to-\infty$ as $t\to-\infty$).

Dynamical trajectory functions found in potential scattering[16] and in those approaches such as the N/D model[17] which attempt to saturate the unitarity condition with a small number of internal channels are consistent with the prediction (2.7) in that they have

$$\delta(\infty)=\pi \qquad (2.8)$$

and they approach negative integers as $|t|\to\infty$. The strictly real, linear trajectory functions used as input in the Veneziano model are also trivially consistent with (2.7). The convenience of the phase representation for $\alpha(t)$ is seen in the fact that a wide class of possibilities are covered by the single formula (2.2). This contrasts to the parametrization of $\alpha(t)$ in terms of a dispersion relation[18,19] such as

$$\alpha(t)=\alpha(0)+t\left[d+\frac{1}{\pi}\int_{4m^2}^{\infty} dx\frac{\mathrm{Im}\alpha(x)}{x(x-t)}\right], \qquad (2.9)$$

with $\alpha(0)$, d real and $d\geq 0$, where the form changes according to whether or not $d=0$. The conditions under

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[14] J. Ball and F. Zachariasen, Phys. Rev. Letters **23**, 346 (1969).
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[16] A. Ahmadzadeh, P. G. Burke, and C. Tate, Phys. Rev. **131**, 1315 (1963).
[17] G. Chew and S. Mandelstam, Phys. Rev. **119**, 476 (1960).
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[19] D. V. Shirkov, Phys. Letters **32B**, 635 (1970); R. M. Spector, Phys. Rev. **173**, 1761 (1968).

which d can be nonzero are expressed in terms of $\mathrm{Im}(\alpha)$ and are quite complicated.[20]

Figure 1 shows a plot of the phase of the Δ trajectory against t. The points are determined from the resonance parameters given in the Particle Data Group[21] compilations. The plot is consistent with a phase which quickly approaches a small constant although, on the basis of the data, it is not possible to eliminate the situation where the phase continues to rise so that the trajectory eventually turns over. The linear extrapolation included on this graph gives an approximate lower bound on the energy where such a turnover could occur.[22] In view of the fact that the analyticity properties of fermion trajectories are expected to be more complicated than those assumed in (2.2), the relation between the phase of the Δ trajectory and the asymptotic behavior implied by (2.7) should not be taken too seriously.

Figure 2 shows a plot of the phase of an exchange-degenerate ρ-f trajectory with the points again determined from the Particle Data Group compliations.[21] If the very narrow peaks, the $T(2200)$ and the $U(2375)$, found by the CERN missing-mass spectrometer,[6] are identified as recurrences of this trajectory, then the data suggest that the phase may go to zero. However, there is some reason to doubt that a missing-mass spectrometer could discriminate peaks with widths 190–220 MeV from the background in this region, so the evidence is not conclusive that there are not wide resonances in the T and U regions.[23] In fact, the $\rho(2275)$, the $N\bar{N}_{I=1}(2345)$, and the $N\bar{N}_{I=0}(2380)$ are candidates. The question then becomes one of understanding what the narrow peaks are. In view of present models for the A_2 fine structure, it seems plausible that there could exist many narrow "doorway" states in the meson spectrum which coexist with wide resonances.[7,23] Perhaps these are what the CERN spectrometer is seeing. It would certainly be an unattractive theoretical situation if the A_2 were unique, but the question has ultimately to be resolved experimentally. What is important here is that there may be two alternatives for the asymptotic phase of the ρ-f trajectory. If

$$\delta(\infty)/\pi=\epsilon\ll 1\,, \tag{2.10}$$

the states on this trajectory would grow to be very

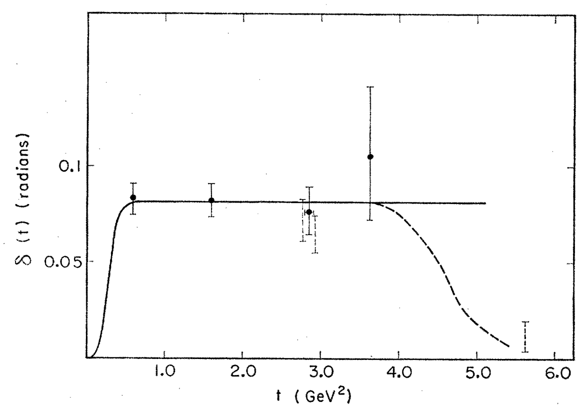

FIG. 2. Phase of an exchange-degenerate trajectory containing the $\rho(765)$, $I^GJ^P=1^+1^-$, and the $f(1260)$, 0^+2^+. The $g(1660)$ and the $\rho(1710)$ are listed separately by dashed lines and as a single $g(1690)$ state by a solid bar. The $I=0$ enhancement degenerate with the $I=1$, $\rho(1900)$ is chosen to be the recurrence of the f. The solid, straight line extrapolation predicts wide resonances in the T and U regions with $\Gamma=190$–220 MeV, while the dashed extrapolation goes through the CERN missing-mass spectrometer peaks.

wide. If

$$\delta(\infty)/\pi=0\,, \tag{2.11}$$

then the large widths of the low-spin resonances such as the ρ, f, and g are anomalous and the high-spin recurrences will become very stable. The physical consequences of the two alternatives (2.10) and (2.11) are quite distinct. In particular, a high-spin state on a trajectory with a finite phase can decay into open channels with low orbital angular momentum while in the case (2.11), a theorem due to Brower and Harte[3] shows that such decays are kinematically forbidden.

This brings up one of the anomalies of the Veneziano model[1,2] which may give a hint that the concept of an asymptotic phase has physical relevance. Although the model requires strictly real, nonphysical trajectories, it only has Regge asymptotic behavior in a region excluding a wedge of finite axis along the real axis of the t plane where there is a line of poles.[24] There may be some confusion on this point since Roskies,[24] who bases his argument on the requirement that the beta function $B[-\alpha(t),\ -\alpha(u)]$ decreases exponentially at fixed s, has quoted this requirement in the form

$$\mathrm{Im}\alpha(t)/|t|^{1+\mu}\to\infty\,,\quad \text{all}\quad \mu>0 \tag{2.12}$$

which depends on the fact that he used the dispersion relation (2.9), for $\alpha(t)$, so that $\mathrm{Re}\alpha\sim ct$. Since the argument of the beta function depends on t only parametrically through $\alpha(t)$, it is obvious that the bound

$$\mathrm{Im}\alpha(t)/|\mathrm{Re}\alpha(t)|^{1+\mu}\to\infty\,,\quad \text{all}\quad \mu>0 \tag{2.13}$$

would satisfy Roskies' constraint. Equation (2.12) is essentially the condition that the trajectory phase be

[20] The constant d in (2.9) can be nonzero only if (a) $\mathrm{Im}\alpha\to\infty$ in upper half t plane, (b) $\lim|t\alpha(t)|=\infty$ for directions not parallel to real axis, and (c) $\int_1^\infty dy\, y^{-x}\,\mathrm{Im}\alpha(iy)=\infty$ for all $x\in(0,2)$. H. Melder, Nuovo Cimento **51A**, 882 (1967), Ref. 10. See also N. Aronszan and W. F. Donaghue, J. Anal. Math. (Israel) **5**, 321 (1956–57).

[21] Particle Data Group, Phys. Letters **33B**, 1 (1970).

[22] The condition for a maximum in $\mathrm{Re}\alpha$ is

$$\tan\delta(t)\frac{\delta(t)}{dt}=\frac{d}{dt}\ln|\alpha(t)|$$

and may take place with a phase, $\delta(t)$, anywhere between 0 and $\frac{1}{2}\pi$. For $\delta(t)\geq\frac{1}{4}\pi$ so that $\mathrm{Im}\alpha\geq\mathrm{Re}\alpha$, the identification of the point $\mathrm{Re}\alpha(t)=J$ with a spin-J resonance is uncertain.

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[24] R. Z. Roskies, Phys. Rev. Letters **21**, 1851 (1968).

finite asymptotically or fall no faster than an inverse power of a logarithm.

In terms of the interpretation of the role of small parameters in physics presented by Chew,[25] it is the smallness of ϵ which justifies the use of the Veneziano model. It is not surprising that the physics of the two cases (2.10) and (2.11) should be quite different so that a simple extrapolation procedure should fail.

III. KINEMATICS OF DECAY

Assume that the behavior of the Regge trajectory is given asymptotically by (2.7) and (2.9). This then gives the expression for the mass of the particle of spin J

$$\lim_{J\to\infty} M_J = \left[\frac{J}{b\cos(\pi\epsilon)}\right]^{1/(2-2\epsilon)}. \tag{3.1}$$

Because the phase $\pi\epsilon=\delta(\infty)$ of the trajectory is assumed to be small, so that

$$(\mathrm{Im}\alpha)^2/(\mathrm{Re}\alpha)^2 \ll 1, \tag{3.2}$$

the relation between the width of the resonance and the imaginary part of the trajectory is given by

$$\Gamma_J = \frac{\mathrm{Im}\alpha(M_J{}^2)}{[(d/dt)\,\mathrm{Re}\alpha(t)\,|_{t=M_J{}^2}]\times M_J}. \tag{3.3}$$

The reader may wonder about the validity of an expression such as (3.3), which is based on the Breit-Wigner formula, when it is used in a situation where the width it gives is large compared to the spacing of singularities. A discussion of overlapping resonances which relies on simple quasi-two-body unitarity has been presented by Coleman.[26] The situation for the case of many-body unitarity may be more complicated but it remains plausible that (3.3) should give approximately the imaginary part of the pole in the complex energy plane in spite of the spacing of nearby singularities. Of course, as the "lifetime" predicted by (3.3) becomes shorter, it becomes harder and harder to design an experiment which would measure the quantum numbers of the resonance so that the "particle" connection of the pole is gradually lost and it becomes indistinguishable from "nonresonant background."

Combining (3.3) with (2.7) gives

$$\Gamma_J = \frac{\tan(\pi\epsilon)}{1-\epsilon}\left[\frac{J}{b\cos(\pi\epsilon)}\right]^{1/(2-2\epsilon)} \equiv \gamma M_J. \tag{3.4}$$

Under the assumption that (3.4) does not hold, but instead the trajectory is asymptotically real in the sense

$$\lim_{J\to\infty} \Gamma_J/M_J = 0, \tag{3.5}$$

Brower and Harte[3] showed that kinematic constraints decouple a high-spin resonance from all available channels with low orbital angular momentum unless

$$\lim_{t\to\infty+i0} \mathrm{Re}\alpha(t) = o(t^{1/2}). \tag{3.6}$$

Of course, the behavior implied by (3.5) and (3.6) is inconsistent with the asymptotic behavior implied by the phase representation (2.2) or the disperison relation (2.9) and so, in the absence of left-hand cuts in the trajectory function, the bound (3.6) will not be considered further.

Consider the decay of a particle of spin J on the trajectory given by (2.7). Suppose it decays into two particles of spin j_1 and j_2 on the same trajectory, where

$$j_1 = \eta J \tag{3.7a}$$

and

$$j_2 = (1-\eta)J. \tag{3.7b}$$

Conservation of energy permits the decay if

$$M_J + \Gamma_J \geq M_{j1} - \Gamma_{j1} + M_{j2} - \Gamma_{j2}, \tag{3.8}$$

$$(1+\gamma)\left[\frac{J}{b\cos(\pi\epsilon)}\right]^{1/(2-2\epsilon)} \geq (1-\gamma)$$
$$\times[\eta^{1/(2-2\epsilon)} + (1-\eta)^{1/(2-2\epsilon)}]\left[\frac{J}{b\cos(\pi\epsilon)}\right]^{1/(2-2\epsilon)}, \tag{3.9}$$

$$(1+\gamma)/(1-\gamma) \geq [\eta^{1/(2-2\epsilon)} + (1-\eta)^{1/2(-2\epsilon)}]. \tag{3.10}$$

The inequality (3.10) is interesting in that it gives an idea of what size ϵ must be in order to have a "fission" decay[27] where

$$j_1 \cong j_2, \tag{3.11a}$$

$$\eta \cong \tfrac{1}{2}, \tag{3.11b}$$

so that (3.10) and (3.4) become

$$\frac{1-\epsilon+\tan\pi\epsilon}{1-\epsilon-\tan\pi\epsilon} \geq 2^{(1-2\epsilon)/(2-2\epsilon)}. \tag{3.12}$$

This inequality has the solution

$$\epsilon \geq 0.049, \tag{3.13a}$$

$$\gamma \geq 0.16, \tag{3.13b}$$

so that a "fission" decay appears impossible for a high-spin particle on the ρ-f trajectory given in Fig. 2 which has approximately

$$\epsilon_\rho \cong 0.025, \tag{3.14a}$$

$$\gamma_\rho \cong 0.081. \tag{3.14b}$$

Going back to (3.10) and assuming ϵ, γ, and η are all

[25] G. F. Chew, Phys. Rev. Letters **22**, 364 (1969).

[26] S. Coleman, in *Theory and Phenomenology in Particle Physics, II*, edited by A. Zichichi (Academic, New York, 1969).

[27] C. Quigg and F. von Hippel, in *Experimental Meson Spectroscopy*, edited by C. Baltay and A. Rosenfeld (Columbia U.P., New York, 1970); H. Goldberg, Phys. Rev. Letters **21**, 788 (1968).

small, the decay is permitted if

$$\gamma > \tfrac{1}{2}\eta^{1/2}. \tag{3.15}$$

Assuming that the asymptotic form (2.7) is adequate at low values of J, the constraint that $J \to (J-1) + 1[\rho(765)]$ is permitted can be found by putting $\eta = 1/J$ in (3.15) and using (3.14b)

$$0.08 > \tfrac{1}{2}J^{-1/2}, \tag{3.16a}$$

$$J > 40. \tag{3.16b}$$

Requiring very low orbital angular momentum in the final state therefore implies the decay into two particles on the *same* trajectory does not begin to take place until quite high values of spin.

At intermediate values of spin a "cascade" decay[27] into a high-mass meson of opposite G parity plus a pion is possible. Assuming the asymptotic form (2.7) holds for the leading meson trajectory of both G parities. Conservation of energy permits the cascade decay $J \to (J-1) + \pi$ if

$$M_J + \Gamma_J \geq M_{J-1} - \Gamma_{J-1} + m_\pi, \tag{3.17}$$

$$(1+\gamma)\left[\frac{J}{\cos(\pi\epsilon)}\right]^{1/(2-2\epsilon)} \geq (1-\gamma)\left[\frac{J-1}{\cos(\pi\epsilon)}\right]^{1/(2-2\epsilon)} + m_\pi. \tag{3.18}$$

To see how the theorem of Brower and Harte depends on the assumption (3.5), notice that this decay is prohibited if $\gamma = 0$,

$$\left[\frac{J}{\cos(\pi\epsilon)}\right]^{1/(2-2\epsilon)} - \left[\frac{J-1}{\cos(\pi\epsilon)}\right]^{1/(2-2\epsilon)} = O(J^{-(1-2\epsilon)/(2-2\epsilon)}). \tag{3.19}$$

A more thorough discussion of the theorem of Brower and Harte is given in the Appendix.

Goldberg[27] has also examined the kinematics of decays under the assumption (3.5) and has found that the "cascade" decay $J \to (J-L) + \pi$ dominates all other modes. It can take place provided that orbital angular momentum L is bounded by

$$L \geq 2m_\pi J^{1/2}. \tag{3.20}$$

The partial decay width of this mode then behaves like

$$\Gamma_c \sim [(2/J)\ln J]^{2m_\pi J^{1/2}} \tag{3.21}$$

and resonances become very stable unless there is a vast number of other states into which they can decay. This contrasts with the alternative (3.4) which does not involve the orbital angular momentum of the decay products growing large at any stage.

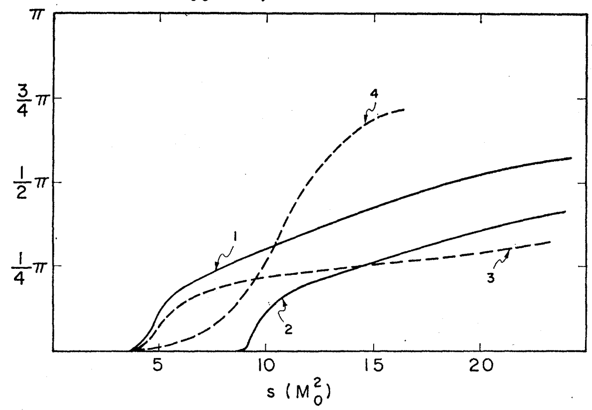

FIG. 3. Phases of output trajectories in the two-channel N/D model. In the absence of coupling, trajectory 1 is associated with the 0^+0^+ channel and trajectory 2 with the 1^-0^+ channel. When the coupling is turned on, trajectory 3 is the leadin gsingularity while trajectory 4 turns over quickly.

IV. IMPLICATIONS FOR DYNAMICAL MODELS

To this point, the discussion has been remarkably innocent of dynamics. The reduced residue functions associated with $\alpha(t)$ have nowhere appeared. Crossing has been ignored in order to discuss poles in one channel and unitarity has only been invoked to obtain the analytic structure of $\alpha(t)$ and to obtain the bound (2.5) on the phase along the physical cut.

Given the perverse, nonlinear character of the unitarity equation, it may be that a good understanding of its implications can only be obtained after the approximate nature of the resonance spectrum is known. However, it is instructive to examine a crude dynamic model based on coupled-channel unitarity which might be expected to produce a leading Regge trajectory corresponding to that given asymptotically by (2.7) and (2.10).

First, consider a simple N/D model for a system of two coupled channels each consisting of two natural-parity mesons. One channel has spin zero and the other has spin one. Using a simple, one-pole approximation for the left-hand cut, the N/D model gives the output trajectory functions both in the absence and the presence of coupling. In the absence of coupling, the leading singularity in each channel is a pole trajectory which turns over shortly above threshold before reaching a high value of orbital angular momentum. In the example considered, the potentials were adjusted so that a P-wave resonance was formed a short distance above threshold in each channel. When the coupling was turned on, the two output trajectories "exchanged tails" to form a leading trajectory which rises above $J=2$ ($L=1$ in the 1^--0^+ channel) and a nonleading trajectory which turns over before reaching $J=1$. Figure 3 gives the phase of the two trajectory functions in both the coupled and uncoupled cases.

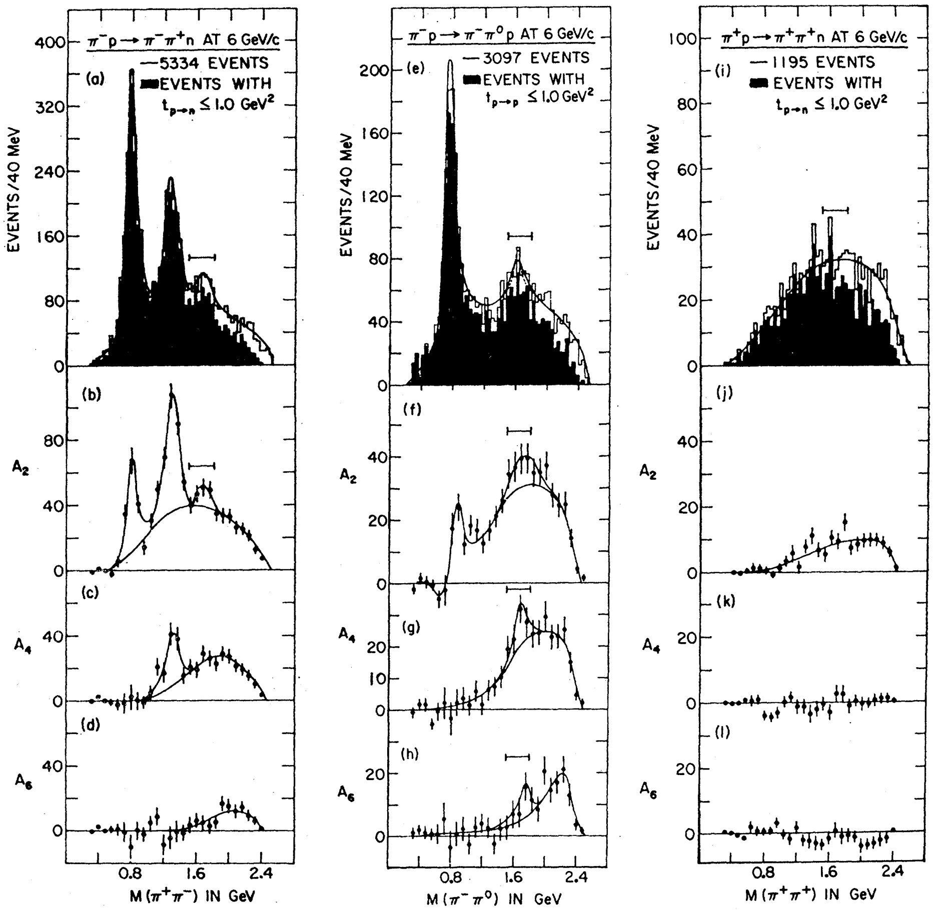

FIG. 4. Data of Crennell *et al.* (Ref. 34) on pion mass distributions for $\pi^- p \to \pi^+\pi^- n$, $\pi^- p \to \pi^-\pi^0 p$, and $\pi^+ p \to \pi^+\pi^+ n$. These data give a lower bound on the elasticity of the ρ' meson predicted by the Veneziano model.

The suggestion that an extension of this simple two-channel model which included an infinite number of channels of increasing spin could account for a leading Regge trajectory which is indefinitely rising has been frequently discussed.[5,28] Since the model at this stage ignores crossing, it is not a bootstrap attempt. The difficulties in bootstrapping the scheme by getting a self-consistent set of trajectories which provide the forces on the left-hand cut as well as containing the particles on the right will not be discussed here.[29] The goals of this discussion are more modest. The idea is that connecting the existence of high-spin resonances with the simultaneous existence of high-spin channels may have some merit whether or not it can lead immediately to a successful bootstrap. The two-channel N/D model illustrated in Fig. 3 gives some indication already that the dynamics of a trajectory is dominated by channels coupled to it with low orbital angular momentum and it seems plausible that this would be true quite generally.

Two points must be made about the possibility that a mechanism involving coupled-channel unitarity supports a leading singularity which rises indefinitely. First, it seems unlikely that such a mechanism could produce high-spin resonances which are very stable since, once the imaginary part of the output trajectory function has become substantial, the cascade decay mode becomes available so that there is at least one decay channel with no orbital angular momentum barrier as well as an increasing number of other decay channels open as the spin is increased. From the point of view of the N/D model just discussed, the high-spin channels which keep the real part of α growing also keep the imaginary part growing and there appears to be no mechanism which would turn over the phase plotted in Fig. 4. If this type of dynamics is relevant, it provides an argument against the identification of the $U(2375)$ as a recurrence of the ρ trajectory.[18] Quite simply, the model says that if there are open decay

[28] S. Mandelstam, in *1966 Tokyo Summer Lectures in Theoretical Physics, II*, edited by G. Takeda and A. Fujii (Benjamin, New York, 1966); see also S. Mandelstam, Phys. Rev. **166**, 1539 (1968).

[29] P. D. B. Collins and R. C. Johnson, Phys. Rev. **182**, 1755 (1969).

channels of low orbital angular momentum, then resonances should have large widths.

A second point about this mechanism for producing an indefinitely rising trajectory involves the nature of the nonleading singularities. The mechanism seems to require an infinite number of nonleading singularities. The model implies the collision or near collision of the various singularity surfaces with the leading trajectory. The analytic form of the leading trajectory given by (2.2) is based on the assumption of the absence of such collisions since the collisions can introduce extra branch cuts into the function $\alpha(t)$. The existence of left-hand branch cuts destroys the relation between the asymptotic phase and the asymptotic behavior, (2.7), and changes it to (again ignoring possible logarithmic factors)

$$\lim_{t\to\infty}\alpha(t)=be^{i\delta(\infty)}t^{1-\delta(\infty)/\pi+\delta(-\infty)/\pi}, \qquad (4.1)$$

where $\delta(-\infty)$ is the asymptotic phase on the left-hand cut. The problem is that the phase on the left-hand cut is not directly measurable and there is no way to put a bound on it. Notice that if

$$\delta(\infty)=\delta(-\infty), \qquad (4.2)$$

it is possible for both the real and imaginary parts of the trajectory function to be asymptotically linear.[14] The behavior of the phase along the left-hand cuts depends crucially on the dynamic mechanism which generates the pole surfaces, $\alpha(t)$. The important thing is that the existence of open, low angular momentum decay channels depends on the width growing with the mass,

$$\Gamma_J=\gamma M_J, \qquad (4.3)$$

which holds for (4.1) as well as for (2.7). The credibility of the coupled-channel mechanism for producing rising trajectories is not destroyed. Of course, understanding the nature of all of the singularities produced by such a model, cuts as well as poles, is crucial to understanding questions such as the existence of superconvergence relations.[29]

An alternative dynamical model for indefinitely rising trajectories which has been recently discussed is the energy-dependent potential approach.[30,31] Since the output trajectories in such an approach would have cut structure implied by potential theory, much of what has been said is immediately applicable to them. However, since the mechanism which forced the trajectories to rise is at least *a priori* independent of the existence of inelastic channels it seems that such an approach could produce a finite asymptotic phase or a zero asymptotic phase equally well. This approach does have an advantage in that it can be constrained to produce a crossing-symmetric amplitude,[30] while crossing can be put into the coupled channel approach only by a generalization of the strip model.[29]

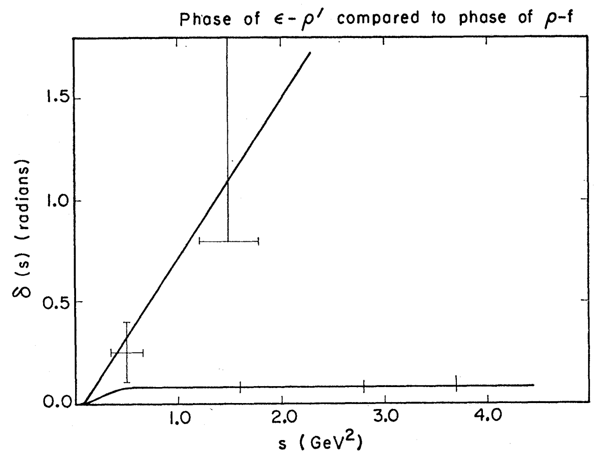

FIG. 5. Phase of the ϵ-$p'\to\rho'$ trajectory compared to the phase of the ρ-f, indicating the possibility that nonleading singularities are very wide and turn over quickly. For convenience in plotting, the real part of the ϵ-ρ' trajectory was moved by one unit to coincide with the ρ-f.

Finally, consider the problem of interpreting Veneziano-model results and extrapolating away from the limit $\mathrm{Im}\alpha(t)=0$ (often optimistically referred to as unitarizing the Veneziano model). The only internally consistent approach has been to treat the 0-width model as a Born term and construct a field theory based upon its particle spectrum.[32] If this scheme can be carried out, it will answer a lot of questions concerning the role of quarks, internal symmetries and spin in hadron scattering. It may also provide a counterexample to quash the controversial idea of the bootstrap. Without trying to anticipate the results of such a complicated endeavor involving such a large number of physicists, it might be worthwhile to speculate on the basis of the previous discussion about the two options (2.11) and (2.10) for the phase of the renormalized trajectory. If, after renormalization, the leading trajectory has a finite phase then the output particle spectrum may look considerably different than the input "bare" spectrum. Conversely, if the renormalized trajectory has zero phase then it seems unlikely that the degeneracy of the daughter singularities will be grossly broken.

One argument for the existence of an exponentially growing number of states is that, because of the kinematics of the angular momentum barrier, states on rising trajectories find it increasingly difficult to decay into any particular state.[33] If the output trajectory has a finite phase, this argument fails and the leading trajectory can be largely self-supporting. Instead of requiring nonleading trajectories which are parallel to the leading one, this suggests the possibility that

[30] L. A. P. Balázs, Phys. Rev. D 2, 999 (1970).
[31] H. H. Aly and H. J. W. Müller, Southern Illinois University report, 1970 (unpublished).
[32] A. R. Swift and R. W. Tucker, Phys. Rev. D 2, 2486 (1970).
[33] A. Kryzwicki, Phys. Rev. 187, 1964 (1969).

nonleading singularities have a completely different behavior.

The fact that this kind of behavior might have physical significance is suggested by the large width of the $\epsilon(700,0^+)$ and the absence of $\rho'(1250,1^-)$ predicted by the Veneziano model. An analysis of the data of Crennell *et al.* (Fig. 4) shows no evidence of the ρ'.[34] This is inconsistent with the predictions of the Veneziano model unless the width of this resonance is greater than 1 GeV. This suggests that the phase of the first-daughter trajectory predicted by the Veneziano model behaves something like that shown in Fig. 5, much like the behavior of the nonleading pole in the coupled-channel N/D problem in Fig. 3, so that the trajectory could turn over or become largely imaginary. If daughter trajectories acquire large imaginary parts shortly above threshold, this would reconcile some of the qualitative success of the Veneziano model with the nonappearance of those daughter states it predicts.

The possibility that daughter trajectories turn over could upset the prediction of a number of resonances which increases exponentially with mass.[4] Since Hagedorn[35] is careful to include nonresonant states in his prediction for the thermodynamic model, this type of model is not necessarily inconsistent with his predictions.

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APPENDIX: THEOREM OF BROWER AND HARTE

Consider the kinematics for the decay of a high-spin resonance on the leading Regge trajectory in a particular channel. The mass of the initial resonance is M, its width Γ, and its spin is $\mathrm{Re}\alpha_0(M^2)$. It can decay into a multiparticle channel consisting of n_f particles on each of N trajectories, α_f. The νth particle on the fth trajectory then has mass $m_\nu{}^f$, width $\Gamma_\nu{}^f$ and spin $\mathrm{Re}\alpha_f[(m_\nu{}^f)^2]$. Conservation of energy for the decay process can then be written

$$M+\Gamma \geq \sum_{f=1}^{N} \sum_{\nu=1}^{nf} (m_\nu{}^f - \Gamma_\nu{}^f). \qquad \text{(A1)}$$

If L is the total orbital angular momentum of the decay channel, conservation of angular momentum gives

$$L+\sum_{f=1}^{N} \sum_{\nu=1}^{nf} \mathrm{Re}\alpha_f[(m_\nu{}^f)^2] \geq \mathrm{Re}\alpha_0(M^2)$$
$$\geq \min\left| L+\sum_{f=1}^{N} \sum_{\nu=1}^{nf} (\pm)\mathrm{Re}\alpha_f[(m_\nu{}^f)^2]\right|. \qquad \text{(A2)}$$

Assuming that the asymptotic behavior of the real parts of the trajectory functions are given by

$$\mathrm{Re}\alpha_0(s) \sim c_0 s^{a_0} \qquad \text{(A3)}$$

and

$$\mathrm{Re}\alpha_f(s) \sim c_f s^{a_f}, \qquad \text{(A4)}$$

with

$$a_f \leq a_0, \quad f=(1,\cdots,N) \qquad \text{(A5)}$$

then the theorem of Brower and Harte[3] generalized to include the effects of resonance widths can be stated in the following form.

Theorem: If $a_0 > \frac{1}{2}$ then the decay process defined by (A1) and (A2) above must fail to satisfy one of the following conditions as $M \to \infty$:

(i) $\dfrac{L}{\mathrm{Re}\alpha_0(M^2)} \to 0$,

(ii) $\dfrac{1}{M}\left(\Gamma + \sum_{f=1}^{N} \sum_{\nu=1}^{nf} \Gamma_\nu{}^f\right) \to 0$,

(iii) $m_\nu{}^f/M < 1$.

It is apparent that if the asymptotic phase of the trajectory is not zero, condition (ii) does not hold for simple decay channels. Also, it is possible for condition (iii) to be violated so that one of the "decay products" has mass equal to or greater than the original resonance without violating (A1). On the other hand, if the asymptotic phase of the trajectories is zero, so that

$$a_0 = 1 \qquad \text{(A6)}$$

and

$$\Gamma/M \to 0, \qquad \text{(A7)}$$

condition (ii) is forced to hold. If condition (i) fails and the orbital angular momentum grows with $\alpha_0(M^2) \propto M^2$, then the result of Jones and Teplitz[36] forces the partial width of the resonance to this *particular* decay channel to decrease exponentially with M^2. If the total width is not to decrease exponentially, then there must be an exponentially growing number of available decay channels.

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[35] R. Hagedorn, Nuovo Cimento **56A**, 1027 (1968).
[36] C. E. Jones and V. Teplitz, Phys. Rev. Letters **19**, 135 (1967).