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Veneziano Secondary Terms for $\bar{p}n \to 3\pi$

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We accurately determine the form of the four-point Veneziano amplitude suitable to describe $\bar{p}n \to 3\pi$ at rest. The essential parameters are found by fitting the two-dimensional surface of the Dalitz-plot distribution directly. Excellent agreement with experiment is obtained, showing secondary terms to be essential.

I. INTRODUCTION

One of the first triumphs of Veneziano's model[1] was its adaptation by Lovelace[2] to predict the basic pattern of $\bar{p}n$ annihilation at rest into three pions[3] [Figs. 1(a) and 1(b)]. Doubts concerning the accuracy of this prediction have led to alternative prescriptions—to include secondary Veneziano terms, to use the five-point function generalization, or even to turn back to non-Veneziano resonance models.

The purpose of this paper is, firstly, to review the earlier Veneziano models, showing their detailed structure in a two-dimensional Dalitz-plot representation. By fitting directly to the full two-dimensional surface of the experimental Dalitz plot, all the available information is used. We find that secondary terms are important and very much needed—contrary to earlier Veneziano analyses. The most recent phase-shift resonance model[4] requires 14 parameters to give the features of $\bar{p}n \to 3\pi$, whereas we find that the Veneziano secondary term structure with *only four* free parameters gives excellent agreement with the data.

II. VENEZIANO MODELS

The $\bar{p}n$ system decays at rest into $\pi^+\pi^-\pi^-$ in a ${}^1S_0$, $T=1$ state; the initial state of the system then has the quantum numbers of a heavy pion. The $\pi\pi \to \pi\pi$ amplitude in a Veneziano model[5] is

$$\begin{aligned} A_s{}^0 &= \tfrac{3}{2}[A(s,t)+A(s,u)]-\tfrac{1}{2}A(t,u), \\ A_s{}^1 &= A(s,t)-A(s,u), \\ A_s{}^2 &= A(t,u), \end{aligned} \tag{1}$$

where $A_s{}^T$ is the isospin-T amplitude in the s channel, and the most general $A(s,t)$ is written as

$$A(s,t)=\sum_{n=1}^{\infty}\sum_{m=0}^{n} c_{nm}\Gamma_{nm}(s,t),$$

with

$$\Gamma_{nm}(s,t)=\frac{\Gamma(n-\alpha_s)\Gamma(n-\alpha_t)}{\Gamma(n+m-\alpha_s-\alpha_t)}. \tag{2}$$

Here α_i is the i-channel exchange-degenerate ρ-f_0 trajectory. The coefficients c_{nm} are determined from the experimental data. Equation (1) ensures that Bose statistics, crossing symmetry, and isospin conservation are satisfied; it is derived under the assumption of absence of isospin-2 resonances. Pomeranchuk contributions are ignored. The decay rate for the $\bar{p}n$ system to three pions is then given by

$$R(\bar{p}n \to \pi^+\pi^-\pi^-) \propto |A(s,t)|^2,$$

where

$$s=M^2(\pi^+,\pi_1{}^-),\quad t=M^2(\pi^+,\pi_2{}^-),\quad u=M^2(\pi_1{}^-,\pi_2{}^-).$$

To have a more physical understanding of the relative importance of the terms in Eq. (2), it is often better to think in terms of the coefficients $\tilde{c}_{nm}$ which multiply individually normalized Γ-function terms; i.e., $A(s,t)$ may be written

$$A(s,t)=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\tilde{c}_{nm}\Gamma_{nm}(s,t)\times\left[\int_{\text{Dalitz plot}}|\Gamma_{nm}(s,t)|^2 ds dt\right]^{-1/2}. \tag{3}$$

The original attempt to explain this process using a Veneziano amplitude was made by Lovelace.[2] In the expansion (2) he makes $c_{11}=1.0$ and all other c_{nm}'s zero. $A(s,t)$ in this form contains only the ϵ meson, its recurrences, and their daughters. The Regge trajectory used is

$$\alpha_s=\alpha_0+\alpha' s+i\alpha''(s-4m_\pi{}^2)^{1/2}\theta(s-4m_\pi{}^2), \tag{4}$$

where $\alpha_0=0.483$ and $\alpha'=0.885$ $(\text{GeV}/c^2)^{-2}$ are the same as for the ρ trajectory, and α'', taken to be 0.28 $(\text{GeV}/c^2)^{-1}$, is claimed to give the ϵ meson a width of 280 MeV. The Dalitz plot distribution given by this form of the amplitude is shown in Fig. 1(c). The Lovelace model predicts a depletion in the center of the Dalitz plot, but it is not steep enough to satisfy experiment [Figs. 2(d)–2(f)]. In addition, the experimental distribution shows concentrations of events along bands of constant s and t enhanced at some intersections. This

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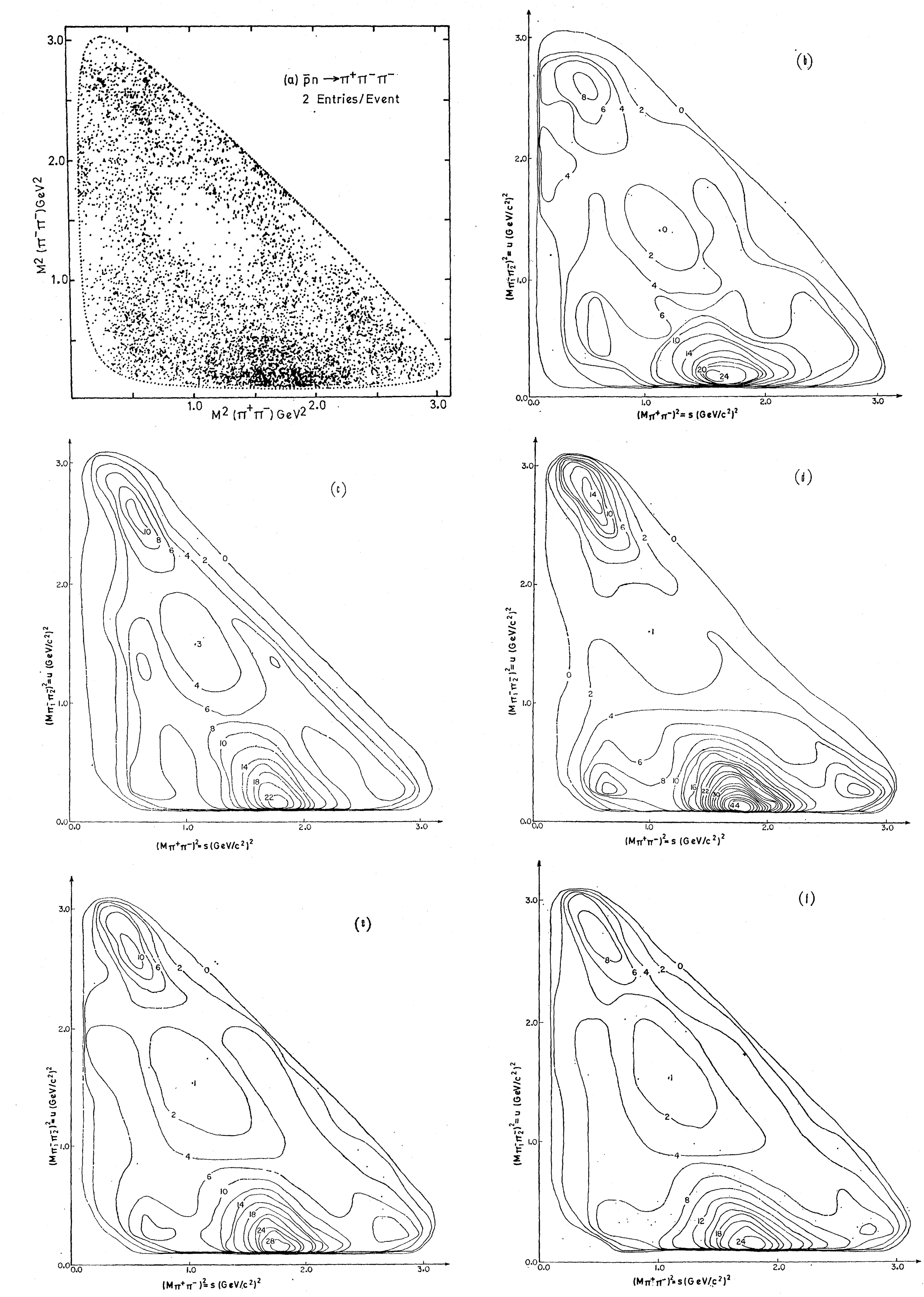

FIG. 1. $(M_{\pi^-\pi^-})^2$ vs $(M_{\pi^+\pi^-})^2$ Dalitz plots. (a) Experimental. (b) Experimental general features. (c) Lovelace's Dalitz plot distribution. (d) Rubinstein-Altarelli distribution. (e) Dalitz-plot distribution given by Veneziano secondary terms described in text with $A=0.280$. (f) Secondary terms as in text with $A=0.33$. The contour numbers in (b)–(f) represent the number of events per (0.098×0.098) $(\mathrm{GeV}/c^2)^4$ area of the Dalitz plot. The number in the center of each contour graph gives the density at the "hole" position.

TABLE I. Comparison of the Veneziano versions for $\bar{p}n \to 3\pi$. The c's are the unnormalized coefficients of Eq. (2), the $\tilde{c}$'s multiply the physically normalized contributions in Eq. (3). One parameter is always fixed by over-all normalization and the errors on the others do not exceed 8%. G is the goodness of fit defined by Eq. (11).[a]

	Lovelace	Altarelli-Rubinstein	$A=0.28$	$A=0.33$
c_{10}	···	1.00	1.00	1.00
c_{11}	1.00	1.89	2.55	2.90
c_{20}	···	0.00	2.96	2.14
c_{21}	···	0.00	7.80	7.31
c_{30}	···	0.57	−4.52	−3.74
$\tilde{c}_{10}$	···	1.00	1.00	1.00
$\tilde{c}_{11}$	1.00	0.78	1.05	1.18
$\tilde{c}_{20}$	···	0.00	0.70	0.53
$\tilde{c}_{21}$	···	0.00	1.04	1.02
$\tilde{c}_{30}$	···	0.00	−0.23	−0.19
$2[L_{\text{un}}(\max)-L(\max)]$	1244	1458	606	592
G	2.24	2.62	1.09	1.07

[a] *Note added in proof.* The predicted values for the ratio of the total rates $R(\bar{p}p_{T=1} \to \pi^+\pi^-\pi^0)/R(\bar{p}n \to \pi^+\pi^-\pi^-)$ for the four amplitudes above are 0.69, 1.19, 1.13, and 1.04, respectively.

model does not show quite the same enhancements away from the two major ones ($s=t=m_\rho^2$; $s=t=m_f^2$).

Altarelli and Rubinstein[6] (hereafter referred to as AR) concluded that a single term in Eq. (2) is insufficient to explain the Dalitz-plot distribution satisfactorily. Its most striking feature is the hole in the middle which occurs at values of s and t such that

$$\text{Re}(\alpha_s+\alpha_t)=3. \tag{5}$$

Owing to the pole structure in the Γ function $\Gamma(n+m-\alpha_s-\alpha_t)$, a large denominator occurs in those terms of (2) for which

$$n+m\leqslant 3. \tag{6}$$

This led them to propose

$$A(s,t)=c_{10}\frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)}+c_{11}\frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)}+c_{20}\frac{\Gamma(2-\alpha_s)\Gamma(2-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)}+c_{21}\frac{\Gamma(2-\alpha_s)\Gamma(2-\alpha_t)}{\Gamma(3-\alpha_s-\alpha_t)}+c_{30}\frac{\Gamma(3-\alpha_s)\Gamma(3-\alpha_t)}{\Gamma(3-\alpha_s-\alpha_t)}. \tag{7}$$

They fit the two invariant (π^+,π^-) and (π^-,π^-) mass-squared histogram projections with the five coefficients c_{nm} in (7). The Regge trajectory used is the same as that given by Eq. (4) for each of the five terms. This means that α'' in (4) is assigned the same value for each of the five terms. Their best values obtained imply that the first two terms completely dominate (Table I). Figure 1(d) shows the corresponding Dalitz plot distribution. The "hole" in the center is now more accurately fitted. But the distribution shows a general depletion, in contradiction with experiment, *all along* the line $u\sim 1.5$ $(\text{GeV}/c^2)^2$ which corresponds to $\text{Re}(\alpha_s+\alpha_t)=3$ [see Figs. 2(e) and 2(f)]. Moreover, the concentration close to the boundary at $s=t=m_f^2$ [around 1.7 $(\text{GeV}/c^2)^2$] is present with a density twice that in the experimental distribution [see Fig. 2(a)]. In fact, the over-all fit is worse than Lovelace's.

We believe that this failure of the AR analysis arises from the fact that the parameters c_{nm} are determined by fitting the two experimental $M^2(\pi^+,\pi^-)$ and $M^2(\pi^-,\pi^-)$ histograms. Their method thus ignores the strong correlation between these two variables. Furthermore, the use of the form $0.28(s-4m_\pi^2)^{1/2}$ for the imaginary part of the trajectory for each of the five terms is unduly restrictive. The residue at a pole in s in the representation (7) is a polynomial in t whose coefficients are functions of the five c_{nm}'s. The partial-wave decomposition of this residue implies the presence of certain particles at this pole. The imaginary part gives a finite width to this pole (it is also responsible for "ancestor" particles arising in addition to the "daughter" particles from the polynomial in t); this width relates to the widths and masses of the individual particles present. The width of the poles in the over-all amplitude (7) depends on the c_{nm}'s and the form of Imα. As such, the c_{nm}'s are related to the imaginary part of the trajectory. The more general expression $A(s-4m_\pi^2)^B$ ($B\leqslant 1$ from unitarity), with A and B as variable parameters, would treat this correlation in a better way.

To find the best parameters, we use the Dalitz plot $M_{\pi^-\pi^-}^2(=u)$ vs $M_{\pi^-\pi^+}^2(=s)$. The lower and upper limits of u and s, fixed by the pion mass and the total center-of-mass energy, are used to define a 30×30 grid across the Dalitz plot. The experimental number of events, N_i, in each square i with at least one corner within the boundary are determined. For a given set of values of the free parameters, the predicted probability distribution p_i over the significance squares $\{i\}$ is found by integrating the expression

$$d^2p/dsdu=c|A(s,t)|^2 \tag{8}$$

over the area of the square i within the boundary (c is the over-all normalization constant such that $\sum p_i=1$). The predicted distribution of events is simply $\mu_i=Np_i(N=\sum_i N_i)$. As we are compelled to use a fine grid to retain the unique features of the experimental Dalitz plot the conventional χ^2 method is not applicable in view of the small values of N_i encountered. Instead we directly maximize the likelihood of the observation to find the best parameters. The probability of the observation $\{N_i\}$ is

$$P=\prod_{i=1}^{n} p_i^{N_i}, \tag{9}$$

where n is the total number of significant squares

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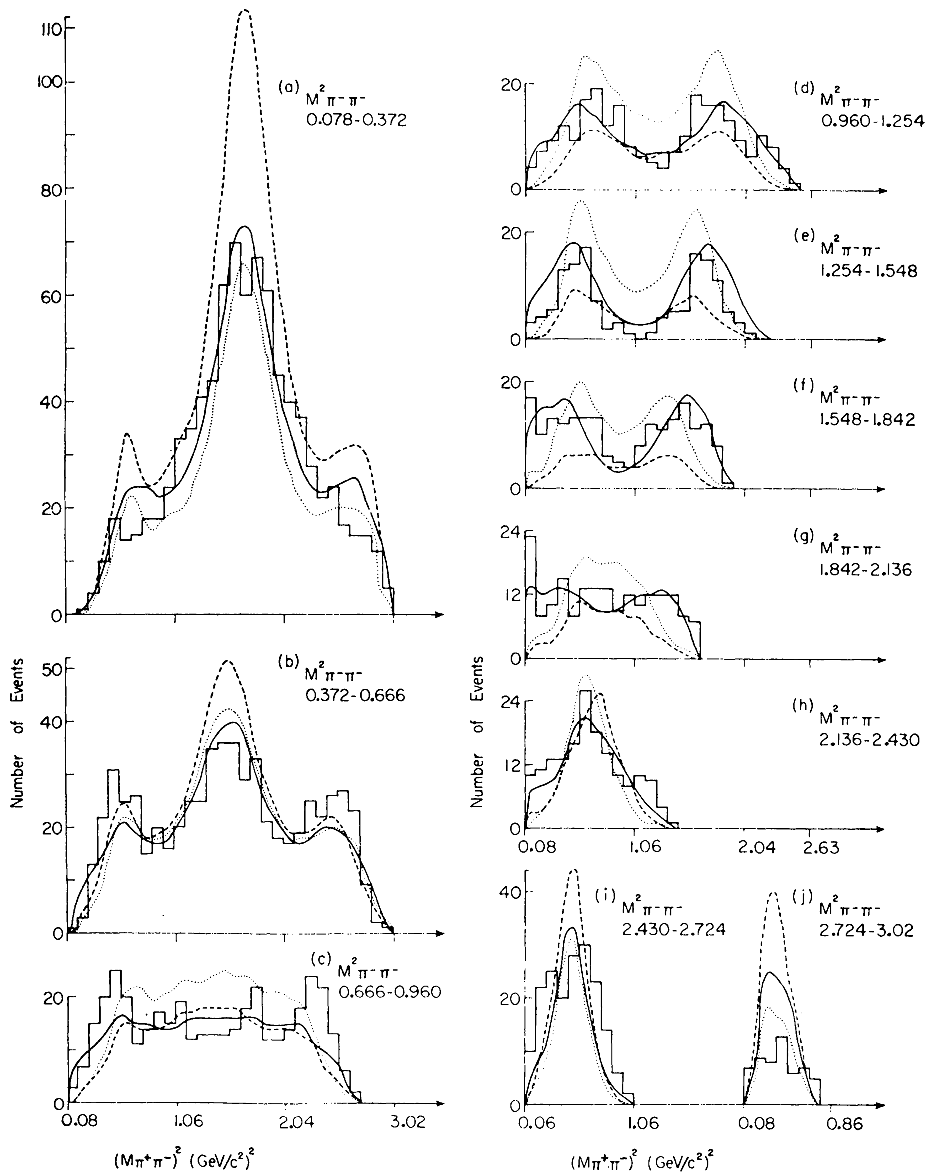

FIG. 2. Histograms showing the Density of events as a function of $M_{\pi^+\pi^-}{}^2$. Each histogram averages over a 0.294-(GeV/c^2)^2-wide slice of $M_{\pi^-\pi^-}{}^2$. Dotted, broken, and full lines describe contour Figs. 1(c), 1(d), and 1(e), respectively.

($n=561$ in this case). The likelihood is defined as

$$L\equiv \ln P=\sum_{i=1}^{n} N_i \ln p_i. \qquad (10)$$

Maximizing L is equivalent to maximizing the Poisson probability $\mathcal{P}$, with means μ_i,

$$\mathcal{P}=\prod_{i=1}^{n} e^{-\mu_i}\frac{\mu^{N_i}}{(N_i)!}.$$

The maximum likelihood with unrestricted probabilities p_i is given by

$$L_{\text{un}}(\max)=\sum_i N_i \ln(N_i/N),$$

and the ratio

$$P(\max)/P_{\text{un}}(\max)=\exp\{-[L_{\text{un}}(\max)-L(\max)]\}$$

enables us to define an indication of goodness of fit G as

$$G=2\frac{L_{\text{un}}(\max)-L(\max)}{\text{number of degrees of freedom}}. \qquad (11)$$

Equation (11) is consistent with the usual definition of goodness of fit since, writing $\epsilon_i=(\mu_i-N_i)/N_i$,

$$L_{\text{un}}(\max)-L(\max)=-\sum_i N_i \ln(1+\epsilon_i). \qquad (12)$$

If ϵ_i is small, the right-hand side of Eq. (12) becomes

$$\tfrac{1}{2}\sum_i (\mu_i - N_i)^2/N_i\,,$$

which is simply one-half the usual χ^2.

With the above procedure, the four free coefficients of Eq. (7) are redetermined with the original Lovelace form for the trajectory as in Eq. (4). The best values of c_{nm} (or equivalently $\tilde{c}_{nm}$) obtained are given in Table I. No term or pair of terms is dominant, as is indicated by the relative equality of the first four $\tilde{c}_{nm}$'s and the important role of the destructive interference. The interference term has an intensity roughly equal to that of the direct contribution and its structure is complex, since the relative sign of the five terms varies over the Dalitz plot. Figure 1(e) shows the corresponding Dalitz-plot distribution. The main defects of the AR fit are remedied. The concentration of events along the line $\mathrm{Re}(\alpha_s+\alpha_t)=3$ is now reproduced correctly [see also Figs. 2(d)–2(g)]. Furthermore, the central hole and the concentration close to the boundary at $s=t=m_f^2$ now have the correct densities. These improvements are reflected in the values of G compared with those obtained using the Lovelace and AR amplitudes as shown in Table I.

Using the more general representation, $A(s-4m_\pi^2)^B$ for $\mathrm{Im}\alpha$, we find that no value of B other than 0.5 gives any significant improvement. However, a better value for A is found to be 0.33 with some slight readjustment of the c_{nm}'s as expected from their correlation to A. The corresponding distribution is shown in Fig. 1(f) and the values of c_{nm} and G are shown in Table I.

The Lovelace and AR analyses in the context of the Veneziano model were unable to explain all the features of $\bar{p}n$ annihilation at rest into three pions because the strong correlation between the two physical variables describing the process were ignored. This has led to a belief that the Veneziano model is inadequate for this process. The excellent agreement with experiment over the whole region of the Dalitz plot shown by our fit provides ample evidence to the contrary. Moreover, our values for the coefficients are in disagreement with those of AR. It is apparent that the secondary Veneziano terms allowed are all necessary to describe the data accurately. This necessity stems from the need to include the "daughter" π-π resonances into the amplitude with contributions not manageable with only one or two terms. Our disagreement with AR also invalidates those attempts to reduce B_5 terms using a pion mass extrapolation which reproduce the AR B_4 coefficients.[7]

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