of Ref. 8. Obviously the additional restriction of setting $\lambda=1$ would make the χ^2 values still larger. In particular the ρ+cuts^V model would give a χ^2 value even larger than 257. Thus the conspiracy model is also shown to be favored by the data over the weak-cut model of Ref. 8.

In conclusion we find that Veneziano-type residues do improve the agreement of the cut model considerably. However, the data still favor the $\rho+\rho'$ conspiracy model over the various cut models. We would also like to point out that integrating numerically we are able to use any kind of amplitudes. In particular we could have used a suitable Veneziano formula instead of Eqs. (1)–(4). However, in the region where we are using these formulas it would make very little difference whether we used Eqs. (1)–(4) or a full Veneziano amplitude.

Note added in proof. Recently an article[9] has considered the same reaction with a different pole-cut model using a pair of complex-conjugate poles. Although no χ^2 values are given, the fits appear to be as good as the best fit with the absorptive-cut model. However, 11 free parameters were needed as opposed to four free parameters in the present calculations.

ACKNOWLEDGMENT

The author is indebted to Professor A. Ahmadzadeh for suggesting this problem, and for discussion and encouragement.

[8] Richard C. Arnold and Maurice L. Blackmon, Phys. Rev. **176**, 2082 (1968).

[9] Bipin R. Desai *et al.*, Phys. Rev. Letters **25**, 1389 (1970).

PHYSICAL REVIEW D VOLUME 3, NUMBER 9 1 MAY 1971

Asymptotic Behavior of Particle Distributions in Hadron Collisions*

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(Received 22 December 1970)

The distribution functions for the "inclusive" production of N specified particles plus anything else are treated from a J-plane point of view. The variables relevant to the exhibition of the asymptotic behavior of these distributions are chosen during a group-theoretic discussion of the matrix elements involved. After the variables are located in this fashion, a crossed-channel partial-wave analysis is carried out to exploit the $SO(1,3)$ symmetry of the production cross sections, and in the context of this partial-wave structure the multi-Regge asymptotics are presented. Such features as pionization and limiting fragmentation are treated, as are certain phenomena involving the approach to limiting distributions, including the rate of approach and specific dependences on certain variables related to longitudinal momenta. Single- and double-particle production is treated in detail, and then a set of numerical estimates is made for proton-proton collisions with incident lab momenta of about 200–500 GeV/c to give an indication where many of the phenomenological results might be tested. A mathematical appendix is provided for those interested in group theory.

I. INTRODUCTION

THE study of the momentum distribution of selected secondary particles in hadron collisions characterized by $a+b\to$ (N detected objects)+(anything else) offers the opportunity to probe the detailed structure of hadronic wave functions and provides the hadronic model builder with a source for determining various parameters of the model as well as a direct challenge to the fundamental features of the model itself. The multiplicity of models is easily as great as that of produced particles, and one would like to establish at least a common kinematical framework in which we might examine the individual candidates.

The first task of the present paper is to analyze the differential cross sections for the "inclusive" production[1] of N particles from a group-theoretical point of view, in order to identify variables which may prove useful in the consideration of various dynamical constructs. Essentially we take advantage of the observation that when the undetected particles are summed over in the process $a+b\to 1+2+\cdots+N+$anything, the differential cross section is related to a piece (but only a piece) of the forward absorptive part of a $(2+N)$-to-$(2+N)$ amplitude. The appropriate symmetry to be exploited in a group-theoretical analysis is then that of the little group of the respective null momentum transfers, namely, $SO(1,3)$, between particles with the same label.

The variables we choose for parametrizing the various momenta, and (by construction) the various little-group elements on which the transition matrix element depends, are not the usual boost in the z direction followed by a three-dimensional rotation; for, although they would be adequate, they do not bring out very clearly many of the interesting features of the secondary distribution. Instead we use a set of parameters strongly suggested by and intimately related to those introduced

* Research sponsored by the U. S. Atomic Energy Commission under Contract No. AT(30-1)-4159.

† Alfred P. Sloan Foundation Research Fellow.

[1] This name was introduced by R. P. Feynman in his lecture contained in *High Energy Collisions*, edited by C. N. Yang *et al.* (Gordon and Breach, New York, 1969), p. 237.

by DeTar and Wilson.[2] This choice labels each momentum in the problem by

$$p_k = R_z(\varphi_k)B_z(\theta_k)B_x(\beta_k)(m_k,0,0,0) \quad (1)$$

$$= m_k(\cosh\beta_k \cosh\theta_k,\ \sinh\beta_k \cos\varphi_k,\ \sinh\beta_k \sin\varphi_k,\ \cosh\beta_k \sinh\theta_k)\,, \quad (2)$$

where $B_n(\theta)$ is a boost through θ along the n axis and $R_z(\varphi)$ is a rotation about the z axis by φ. We call θ_k the longitudinal boost angle (following DeTar) and designate β_k the transverse boost angle. This group-theoretical decoupling of the longitudinal and transverse kinematic degrees of freedom is strongly supported by the apparent dynamic decoupling exhibited by experiments. The usefulness of the parametrization is seen as we find that much information derives from the relative position of the particles in a plot of θ_k's.

The group theory, having proved a medium in which a variable selection is easily made, is next exploited to make a multiple crossed-channel partial-wave analysis in the spirit of Toller and Bali, Chew, and Pignotti.[3] This is done with the approaching complex J-plane analysis fully in mind and also with a view toward the relation of the partial-wave amplitudes thus defined to dynamical equations for their determination.

Next we turn to a discussion of the asymptotic behavior of the inclusive differential cross sections within the framework of Regge-pole phenomenology.[4] Specifically, we use the rather well-established fact of power behavior in invariant energies and the relation of the powers to singularities in the appropriate complex J planes. The connection of the singularities that appear here with those in two to two collisions are motivated by and (hopefully) generalized from the multiperipheral model.[5]

We then look in more detail at the one- and two-particle distributions and give a discussion of such limiting phenomena as "fragmentation"[6] and "pionization"[7] and also of various results encountered on the way to the limit. A numerical example of proton-proton collisions at $p_{\rm lab} \approx 200$–500 GeV/c is worked out to give some orientation toward the possibility of experimentally testing the ideas here.

For a certain amount of completeness, and possibly some general interest, bits of the group theory lurking behind the partial-wave analysis are contained in the Appendix.

Readers wishing to skip the discussions of group theory and to go directly to the results can do so by beginning with Secs. II A and II C and then proceeding to the discussion of single-particle distributions in Sec. IV, leaving out the multiperipheral arguments. Section V contains the two-particle production phenomenology. Of possible interest to the experimenter are the numerical estimates in Sec. VI.

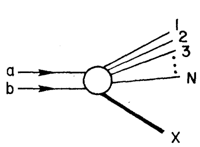

FIG. 1. Pictorial representation of the amplitude for the collision of hadrons $a+b$ producing N detected particles plus an undetected X.

II. KINEMATICAL CONSIDERATIONS

A. Differential Cross Section

We concentrate on the differential cross section for the collision of two hadrons a and b with momenta p_a and p_b, respectively, which yields n detected hadrons of momenta $p_1, p_2, \ldots, p_N$ and anything else, which we refer to as X. Such a process is depicted in Fig. 1. The matrix element for this production is written in the usual Lehmann-Symanzik-Zimmermann (LSZ) reduced form[8]

$$\frac{i^N(2\pi)^4\delta^4(p_X+\sum_{j=1}^{N} p_j-p_a-p_b)}{(2E_1\cdots 2E_N)^{1/2}} \times\int d^4y_2\cdots d^4y_N \exp(i\sum_{j=2}^{N} p_j\cdot y_j)\,\vec{K}_{y_2}\cdots\vec{K}_{y_N} \times\langle X\ \text{out}\,|\,(J_1(0)\phi_2(y_2)\cdots\phi_N(y_N))_+\,|\,a,b\ \text{in}\rangle\,, \quad (3)$$

where E_j is the energy of particle j, p_X is the momentum of X, K_y is the Klein-Gordon or Dirac or other appropriate invariant wave operator which accompanies the field $\phi_j(y)$, and $J_1(0)$ is the source density for particle 1: $K_y\phi_1(y)=J_1(y)$. Spin and isospin wave functions have been omitted, but can be added as needed.

From this amplitude the differential cross section in the barycentric frame of a and b is readily deduced to be

$$\Delta^{1/2}(s,m_a^2,m_b^2)d\sigma(a+b\to 1+2+\cdots+N+X) = \prod_{j=1}^{N}\frac{d^3p_j}{\pi E_j m_j^2}\mathfrak{M}(p_a,p_b,p_1,\ldots,p_N)\,, \quad (4)$$

where

$$\Delta(x,y,z)=(x+y-z)^2-4xy\,, \quad (5)$$

[2] C. DeTar, Phys. Rev. D **3**, 128 (1971); K. Wilson, Cornell Report No. CLNS-131, 1970 (unpublished).
[3] M. Toller, Nuovo Cimento **37**, 631 (1965); N. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967).
[4] Using $SO(1,2)$ symmetry and usual Regge theory, A. Mueller has discussed the one-particle distribution [Phys. Rev. D **2**, 2963 (1970)].
[5] The review article by D. Amati *et al.* [Nuovo Cimento **26**, 896 (1962)] contains most of the modern ideas on single-particle spectra and a thorough discussion of multiperipheral models.
[6] J. Benecke *et al.*, Phys. Rev. **188**, 2159 (1969).
[7] Yash Pal and B. Peters, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **33**, No. 15 (1964).
[8] We use many of the conventions given by S. D. Drell and J. D. Bjorken, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1964), Chap. 16.

and the dynamics is contained in

$$\mathfrak{M}(p_a,p_b,p_1,\ldots,p_N)$$
$$=\frac{1}{2(16\pi^2)^N}\int d^4y_1\cdots d^4y_N d^4z_2\cdots d^4z_N$$
$$\times\exp(-i\sum_{j=1}^{N} p_j\cdot y_j)\,\vec{K}_{y2}\cdots\vec{K}_{yN}$$
$$\times(2E_a2E_b)^{1/2}\langle ab\,\text{in}\,|\,(J_1(y_1)\cdots\phi_N(y_N))$$
$$+(J_1(0)\cdots\phi_N(z_N))_+|\,ab\,\text{in}\rangle$$
$$\times(2E_a2E_b)^{1/2}\overleftarrow{K}_{z2}\cdots\overleftarrow{K}_{zN}. \quad (6)$$

$\mathfrak{M}$ is a Lorentz covariant whose nature would be explicit if the spin indices it might carry were indicated. For the most part we average over all spins in which instance $\mathfrak{M}$ is a Lorentz invariant depending on $3N$ independent scalars formed out of the $N+2$ momenta p_a, p_b, p_1, …, p_N.[9]

B. Group-Theoretic Analysis

The differential cross section consists of one piece of the absorptive part of the $(N+2)$-to-$(N+2)$ *forward* scattering amplitude as indicated in Fig. 2. (This piece is reminiscent of the so-called z graphs much studied in current-algebra contexts.) Although it is only a piece of the forward absorptive amplitude, it is useful to continue the kinematical discussion as one would for the whole amplitude. We thus exhibit in Fig. 3 the kind of kinematic tree graph which is valuable in the group-theoretical analysis to follow.

Between each line labeled by momentum p_j is a wiggly line "carrying" null momentum transfer. This kind of momentum transfer encountered in a 2-to-2 amplitude is the usual signal for a crossed-channel partial-wave analysis based on the little group for null momentum, namely, $SO(1,3)$.[10] In the two-particle case we consider the scattering amplitude as a function of the elements of the little-group transformation which takes us from one of the vectors involved in the scattering to the other.

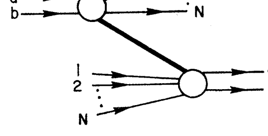

FIG. 2. Picture of the inclusive differential cross section for $a+b\to 1+2+\cdots+N+X$. This shows it to be a piece of the absorptive part of an $(N+2)$-to-$(N+2)$ process.

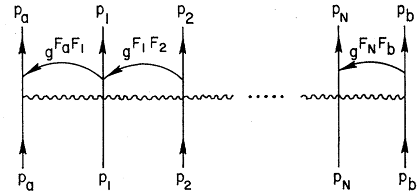

FIG. 3. Tree graph for the kinematics and group-theoretical discussion of the inclusive production $a+b\to 1+\cdots+N+X$. The lines are ordered by their longitudinal boost angles in the barycentric system of a and b with increasing values to the right: $\theta_1\le\theta_2\le\cdots\theta_N$. Also shown are the $SO(1,3)$ group transformations which take one from rest frame F_{j+1} to rest frame F_j.

As a preparation for the development to come, we recall the treatment of the 2-to-2 absorptive amplitude. We have in mind the process where a particle with momentum q and mass μ scatters forward on an object of momentum p and mass m. If we consider each of the momenta to be reached by Lorentz transformations U_p or U_q from some standard vectors p_s or q_s,

$$p=U_pp_s \quad\text{and}\quad q=U_qq_s, \quad (7)$$

then by using the Lorentz invariance of $A(p,q)$, we can write

$$A(p,q)=A(U_pp_s,U_qq_s)=A(U_q^{-1}U_pp_s,q_s)$$
$$=a(U_q^{-1}U_p). \quad (8)$$

That is, one can consider it as a function of the $SO(1,3)$ element $U_q^{-1}U_p$. Generally one has to be careful to pick sets of standard vectors and appropriate U_q's and U_p's so that $U_q^{-1}U_p$ or its analog is in a nice form, but since the full Lorentz group is involved here one may choose both p_s and q_s to be the rest vectors $(m,0,0,0)$ and $(\mu,0,0,0)$.

It is convenient to carry out the remainder of the analysis in the barycentric system, where we imagine p travels along the positive z axis at a velocity $\tanh\theta_p$ and q travels along the negative z axis with $\tanh\theta_q$. The Lorentz transformations are then

$$U_p=B_z(\theta_p) \quad (9)$$

and

$$U_q=B_z(-\theta_q), \quad (10)$$

so

$$p=m(\cosh\theta_p,0,0,\sinh\theta_p)$$

and

$$q=\mu(\cosh\theta_q,\,0,\,0,-\sinh\theta_q), \quad (11)$$

with the constraint

$$m\sinh\theta_p=\mu\sinh\theta_q. \quad (12)$$

The little-group element $U_q^{-1}U_p$ is simply $B_z(\theta_p+\theta_q)$, and the function on the little group $a(U_q^{-1}U_p)$ is a function of $\theta=\theta_p+\theta_q$ only.

We expect $a(\theta)$ to grow as θ increases since the invariant energy $s=(p+q)^2$ is given by

$$s=m^2+\mu^2+2m\mu\cosh(\theta_p+\theta_q)$$
$$=m^2+\mu^2+2m\mu\cosh\theta, \quad (13)$$

[9] The counting of independent invariants is easy from Fig. 1. It is a connected $(N+3)$-point function with the mass of X not specified. Thus we find $3(N+3)-10+1=3N$ variables.

[10] The technique of crossed-channel partial-wave analysis is reviewed by J. Boyce *et al.*, Trieste Report No. IC/67/9, 1967 (unpublished), and is exploited in a multi-Regge context by A. Mueller and I. Muzinich, Ann. Phys. (N. Y.) **57**, 20 (1970); **57**, 500 (1970); and by M. Ciafaloni, C. Detar, and M. N. Misheloff, Phys. Rev. **188**, 2522 (1969).

and power behaviors like s^α, $\alpha>0$, are commonplace. This means that one may not perform the usual partial-wave analysis[10] and expand $a(\theta)$ in harmonic functions on $SO(1,3)$, since that certainly requires α negative. Therefore the expansion is carried out by making a Laplace transform using the second kind of functions on the Lorentz group.[11,12] The lowest Laplace harmonic on the Lorentz group is

$$-2e^{-(n+1)\theta}/(n+1)\sinh\theta \equiv e_{000}{}^{0n}(\theta)\,, \qquad (14)$$

where n is the Casimir invariant of $SO(1,3)$ that directly corresponds to what we ordinarily call angular momentum. The extra indices are explained shortly. The partial-wave amplitude is then defined as

$$a_n=\int_0^\infty d\theta(\sinh\theta)^2 e_{000}{}^{0n}(\theta)a(\theta)\,, \qquad (15)$$

and the absorptive part is regained for $\theta>0$, the physical region, by integrating a_n with the ordinary lowest harmonic on the principal series of $SO(1,3)$[11]

$$\frac{\sinh(n+1)\theta}{(n+1)\sinh\theta}=d_{000}{}^{0n}(\theta)\,. \qquad (16)$$

The integration is along a path to the right of any singularities in a_n in the n plane, so if $a(\theta)\to e^{\alpha\theta}$ for large θ (i.e., s^α), choose $\mathrm{Re}c>\mathrm{Re}\alpha$ and one finds

$$a(\theta)=\int_{c-i\infty}^{c+i\infty}\frac{dn}{2\pi i}[-(n+1)^2]d_{000}{}^{0n}(\theta)a_n\,. \qquad (17)$$

The problem in the N-particle inclusive production is to locate the appropriate transformations corresponding to $U_q^{-1}U_p$ in the case just treated. To do this, we proceed along the lines indicated by Bali *et al.*[3] and attach to each particle a standard frame where it is at rest. For particle a, call this frame F_a; for particle 1, F_1, etc., until one reaches particle b. In frame F_j we denote any momentum by a superscript F_j. Thus, for example,

$$p_j{}^{F_j}=(m_j,0,0,0)\,,\quad j=a,1,\ldots,N,b\,. \qquad (18)$$

Also we denote the Lorentz transformation which brings the momentum p_j from its rest frame F_j to the barycentric (or c.m.) frame of a and b by U_j

$$p_j{}^{\mathrm{c.m.}}=U_jp_j{}^{F_j}\,. \qquad (19)$$

We describe the inclusive production cross section as a Lorentz-transformation tour from rest frame to rest frame, picking up variables as we go along. The end of the excursion on each occasion is the rest frame F_a, which also happens to be the laboratory frame. To take $p_a{}^{F_a}$ to F_a is trivial. To take $p_1{}^{F_1}$ to its value in F_a, we note

$$p_1{}^{\mathrm{c.m.}}=U_1p_1{}^{F_1}=U_ap_1{}^{F_a}\,, \qquad (20)$$

since U_a takes F_a to the c.m. system. Thus we write

$$p_1{}^{F_a}=(U_a{}^{-1}U_1)p_1{}^{F_1}=g^{F_aF_1}p_1{}^{F_1}\,, \qquad (21)$$

defining the transformation $g^{F_aF_1}$ which takes us across the leftmost wiggly line in Fig. 3. Next we bring the momentum of particle 2 into F_a in two steps by visiting F_1 as an intermediate stop. We note

$$\begin{aligned}p_2{}^{\mathrm{c.m.}}&=U_2p_2{}^{F_2}=U_ap_2{}^{F_a}\\&=U_ag^{F_aF_1}p_2{}^{F_1}=U_ag^{F_aF_1}g^{F_1F_2}p_2{}^{F_2}\,,\end{aligned} \qquad (22)$$

so that

$$U_a{}^{-1}U_2=g^{F_aF_1}g^{F_1F_2}\,, \qquad (23)$$

and using the result (21),

$$g^{F_1F_2}=U_1{}^{-1}U_2\,. \qquad (24)$$

By a series of such journeys, we make our way from any rest frame F_{j+1} to F_j and eventually to the lab frame F_a. One quickly discovers that the transformation which takes us from F_{j+1} to F_j is

$$g^{F_jF_{j+1}}=U_j{}^{-1}U_{j+1}\,. \qquad (25)$$

These Lorentz transformations g are the little-group elements on which our function $\mathfrak{M}$ depends. The number of variables involved in the $N+1$ steps from b leftwards to a can be counted as follows: In the c.m. system, choose the momenta of a and b to lie along the z axis. Specifying the magnitude of the relative momentum requires one boost in the z direction. U_a and U_b thus depend on one variable. Particle 1 usually needs three variables to label its on-shell four-momentum but we define the x-z plane in the c.m. system via it, so two suffice for U_1. Our coordinate system being fully specified, we need three more variables for each of the transformations U_2, U_3, $\ldots$, U_N, bringing us up to our full complement of $3N$.

To complete the account, we take away from our $\mathfrak{M}$, Eq. (6), the spinors which carry the usual helicity information and arrive at the c.m. M function

$$M_{s_as_1\cdots s_Ns_b}(p_a{}^{\mathrm{c.m.}},p_1{}^{\mathrm{c.m.}},\ldots,p_N{}^{\mathrm{c.m.}},p_b{}^{\mathrm{c.m.}})\,. \qquad (26)$$

The spin labels s_i designate the finite-dimensional representation of the Lorentz group according to which the fields $\phi_i(x)$ were chosen to transform. The c.m. M function is recovered from its value in F_a, to which all our transformations g carry us, by its Lorentz covariance properties

$$\begin{aligned}M_{s_a\cdots s_b}(p_a{}^{\mathrm{c.m.}},\ldots,p_b{}^{\mathrm{c.m.}})&=\sum_{s_a'\cdots s_b'}D_{s_as_a'}{}^a[U_a]D_{s_1s_1'}{}^1[U_a]\cdots D_{s_bs_b'}{}^b[U_a]\\&\quad\times M_{s_a'\cdots s_b'}(p_a{}^{F_a},p_1{}^{F_a},\ldots,p_b{}^{F_a})\,.\end{aligned} \qquad (27)$$

[11] The functions of the second kind are discussed in M. Toller and A. Sciarrino, J. Math. Phys. **8**, 1252 (1967), in Ref. 10, and by N. W. MacFayden, Carnegie-Mellon report, 1969 (unpublished).

[12] The Laplace transforms of power growth functions in $SO(1,3)$ are discussed by S. Nussinov and J. Rosner, J. Math. Phys. **7**, 1670 (1966), and H. D. I. Abarbanel and L. M. Saunders, Phys. Rev. D **2**, 711 (1970). Both groups use them to make partial-wave analyses of Bethe-Salpeter equations for the absorptive amplitudes.

The $D_{s_i' s_i}{}^i[U_a]$ are finite-dimensional representation matrices of the Lorentz group. Finally we define a function on the various little-group transformations g following Ref. 3:

$$f_{s_a \cdots s_b}(g^{F_aF_1}, g^{F_1F_2}, \ldots, g^{F_NF_b}) = \sum_{s_1' \cdots s_b'} D_{s_1s_1'}{}^1[U_1^{-1}U_a] \times D_{s_2s_2'}{}^2[U_2^{-1}U_a] \cdots D_{s_bs_b'}{}^b[U_b^{-1}U_a] \times M_{s_as_1' \cdots s_b'}[p_a{}^{F_a}, p_1{}^{F_a}, \cdots, p_b{}^{F_a}], \quad (28)$$

where the definition is motivated by noting that the product of transformations from F_k to F_a is

$$g^{F_aF_1}g^{F_1F_2}\cdots g^{F_{k-1}F_k} = U_a^{-1}U_k. \quad (29)$$

It is this function on the little-group elements which we consider in detail for the multi-crossed-channel partial-wave analysis to come.

C. Parametrization of Momenta

One conventionally labels timelike four-momenta by a boost in the z direction $B_z(\eta_j)$ followed by a three-dimensional rotation $R_z(\alpha_j)R_y(\psi_j)$ to orient the momentum. The resulting four-vector

$$Q_j = m_j(\cosh\eta_j, \sinh\eta_j \cos\alpha_j \sin\psi_j, \sinh\eta_j \sin\alpha_j \sin\psi_j, \sinh\eta_j \cos\psi_j) \quad (30)$$

is, of course, a fine representation. What we are concerned with, however, are particles whose z-boost angles are quite far apart, since it is this distance which is a measure of the invariant energy between them. The other variables are connected to the transverse momentum and never are required, experimentally at least, to be large. For two vectors Q_i and Q_j labeled as in (30), the invariant energy

$$s_{ij} = (Q_i + Q_j)^2 = m_i^2 + m_j^2 + 2m_im_j \times (\cosh\eta_i \cosh\eta_j - \sinh\eta_i \sinh\eta_j \cos\Theta_{ij}), \quad (31)$$

with Θ_{ij} the angle between the three vectors, does not bring out directly the difference $\eta_i - \eta_j$. This fact makes it awkward, though certainly possible to discuss s_{ij} that are large in terms of the η's.

What we suggest here is a small step from DeTar and Wilson.[2] Instead of using a three-dimensional rotation to specify transverse momenta, we use a boost in the transverse direction, in particular, B_x.

We therefore give our vectors p_j in any coordinate system by an x boost through the transverse boost angle β_j followed by a z boost through the longitudinal boost angle θ_j and orientation by a z rotation by φ_j:

$$p_j = R_z(\varphi_j)B_z(\theta_j)B_x(\beta_j)(m_j, 0, 0, 0) = U_j p_j{}^{F_j} \quad (32)$$

$$= m_j(\cosh\theta_j \cosh\beta_j, \sinh\beta_j \cos\varphi_j, \sinh\beta_j \sin\varphi_j, \sinh\theta_j \cosh\beta_j). \quad (33)$$

The convenience of these variables for discussing s_{ij} is immediate:

$$s_{ij} = m_i^2 + m_j^2 + 2m_im_j(\cosh\beta_i \cosh\beta_j \cosh(\theta_i - \theta_j) - \sinh\beta_i \sinh\beta_j \cos(\varphi_i - \varphi_j)). \quad (34)$$

In physical scatterings most events have small transverse boost angles, while the longitudinal momentum $m_j \cosh\beta_j \sinh\theta_j$ can become quite large.

The order of the particles in the tree graph of Fig. 3 can now be given some meaning. In the c.m. frame we choose particle a to move along the negative z axis with velocity $\tanh\theta_a$, so

$$p_a = m_a(\cosh\theta_a, 0, 0, -\sinh\theta_a), \quad (35)$$

and thus its longitudinal boost angle is $-\theta_a$. We choose b to move along the positive z axis with θ_b,

$$p_b = m_b(\cosh\theta_b, 0, 0, \sinh\theta_b), \quad (36)$$

and

$$m_a \sinh\theta_a = m_b \sinh\theta_b. \quad (37)$$

The θ_j of the produced particles are constrained to lie essentially between $-\theta_a$ and θ_b. To see this, we examine the momentum conservation restriction

$$p_a + p_b = p_1 + p_2 + \cdots + p_N + p_X, \quad (38)$$

by looking at the 0 ± 3 components in the c.m. system:

$$m_ae^{-\theta_a} + m_be^{\theta_b} = \sum_{j=1}^N m_j \cosh\beta_j e^{\theta_j} + p_{X0} + p_{X3}, \quad (39)$$

and

$$m_ae^{\theta_a} + m_be^{-\theta_b} = \sum_{j=1}^N m_j \cosh\beta_j e^{-\theta_j} + p_{X0} - p_{X3}. \quad (40)$$

For large incident energy s,

$$s = m_a^2 + m_b^2 + 2m_am_b \cosh(\theta_a + \theta_b), \quad (41)$$

any individual θ_j is bounded by

$$-\theta_a + \ln(m_j/m_a) < \theta_j < \theta_b + \ln(m_b/m_j). \quad (42)$$

We therefore order the particles in Fig. 3 by their θ values and choose increasing θ to the right $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_N$.

Now the little group elements $g^{F_jF_{j+1}}$, on which the function $\mathfrak{M}$ depends, can be written

$$g^{F_aF_1} = U_a^{-1}U_1 = R_z(\varphi_1)B_z(\theta_1 + \theta_a)B_x(\beta_1), \quad (43)$$

$$\vdots$$

$$g^{F_jF_{j+1}} = U_j^{-1}U_{j+1} = B_x(-\beta_j)B_z(\theta_{j+1} - \theta_j) \times R_z(\varphi_{j+1} - \varphi_j)B_x(\beta_{j+1}), \quad (44)$$

$$\vdots$$

$$g^{F_NF_b} = U_N^{-1}U_b = B_x(-\beta_N)B_z(\theta_b - \theta_N)R_z(-\varphi_N). \quad (45)$$

Each z boost conveniently appears with a positive argument.

III. PARTIAL-WAVE ANALYSIS

We have already indicated how one treats the 2-to-2 absorptive part with power growth in doing a crossed-channel partial-wave expansion. For the $(N+2)$-particle case, technical complications arise because intermediate particles along the chain cannot always be oriented along the z axis as we could do in Eq. (11). This brings to our attention what I call "phases"; namely, the subgroup representation matrices required to describe the φ_j, β_j orientation of each p_j. (In the case of the ordinary helicity expansion these really are phases.) In making the partial-wave analysis, we are interested in locating those generalized angular momentum variables which are conjugate to the θ_j's and looking for Regge poles in their complex planes.

Let us label the two Casimir invariants of the Lorentz group by j_0 and n,[13] where in any representation j_0 is the minimum $O(3)$ angular momentum and in a finite-dimensional representation n is the maximum $O(3)$ angular momentum. In Naimark's notation, $n=c-1$ and $j_0=k_0$; in Toller's notation,[3] $j_0=M$ and $n=\lambda-1$. It is by now well known that n is the appropriate object for considering a complex "angular momentum," and poles in the n plane are expected to occur at the positions of the observed powers of s in forward 2-to-2 scattering. The components of a vector in any (j_0,n) representation can be given by the values of m, the eigenvalue of J_z, and by diagonalizing the $O(3)$ angular momentum $J_x^2+J_y^2+J_z^2$ or its $O(1,2)$ counterpart $K_x^2+K_y^2-J_z^2$, where K_i is the generator of a boost along the i axis. Since we are interested in the phases associated with $R_z(\varphi)$ and $B_x(\beta)$, we choose the latter. Thus the labels s_i on the M function (26) are imagined to correspond to (i) a resultant m_i for the p_i coming into and leaving the appropriate position on the tree of Fig. 3 and (ii) a resultant eigenvalue l_i for the $O(1,2)$ "angular momentum" in a finite-dimensional space spanned by the product incoming and outgoing wave functions.[14]

A final point is made before writing down the partial-wave expansion for the little-group function (28): Since the experimental behavior in the β_j is rapidly decreasing for large β_j, a fairly ordinary treatment of the $O(1,2)$ phases is in order.[15] However, to handle growth in $\theta_{j+1}-\theta_j$ we need the generalizations of the functions of the second kind appearing in Eq. (14). These are treated in the Appendix and also in the work reported in Refs. 10 and 11. We expect, then, to meet the phases $e^{im\varphi}$ associated with $R_z(\varphi)$, the transverse boost "phases" $d_{mm'}{}^l(\beta)$ associated with $B_x(\beta)$ and the second-kind functions on the Lorentz group $e_{lml'}{}^{j_0n}(\theta)$ associated with $B_z(\theta)$. The partial-wave amplitude is defined by projection with the e^{j_0n}'s as in Eq. (15), and the amplitude is recovered for positive θ by integration with the first-kind representation functions $d_{lml'}{}^{j_0n}(\theta)$.

To be precise, we define for a function $f_{lm,l'm'}(B_z(\theta))$ the partial amplitude

$$f_{l,l'}{}^{j_0n} = \sum_{m,m'} \delta_{mm'} \int_0^\infty (\sinh\theta)^2 d\theta\, e_{lml'}{}^{j_0n}(\theta) f_{lm,l'm'}(B_z(\theta)), \quad (46)$$

and then recover the function for $\theta>0$ as

$$\delta_{mm'} f_{lm,l'm'}(B_z(\theta)) = \int_{c-i\infty}^{c+i\infty} \frac{dn}{2\pi i}[j_0^2-(n+1)^2] d_{lml'}{}^{j_0n}(\theta) f_{l,l'}{}^{j_0n}, \quad (47)$$

where Rec lies to the right of any singularities of f^{j_0n} in the n plane. To discover where such singularities are, it is useful to note that for large θ [11]

$$e_{lml'}{}^{j_0n}(\theta) \to \exp[-\theta(n+2+|m-j_0|)], \quad (48)$$

so that when $f(B_z(\theta))$ is bounded by $e^{\alpha\theta}$, f^{j_0n} is analytic in the n plane to the right of Re$\alpha-|m-j_0|$. The asymptotic behavior of $d_{lml'}{}^{j_0n}(\theta)$ is $\sim\exp\theta(n-|m+j_0|)$.

To expand the little-group function (28), we proceed step by step down the chain of Fig. 3, picking up phases associated with spin labels and orientations of each momentum and e^{j_0n} functions associated with longitudinal boosts from rest frame to rest frame. The formulas are, to say the least, rather long and usually not terribly instructive, so we write only those for $N=0$ (2-to-2 absorptive part) and $N=1$ (single-particle production). For $N=0$ the formula is identical to (46) with the identifications $\theta=\theta_a+\theta_b$ and $l=l_a$, $m=m_a$ and $l'=l_b$, $m'=m_b$. The restriction that $m_a=m_b$ is just helicity conservation in the forward scattering. If all particles are spinless or a spin-averaged amplitude is involved, $l=m=l'=m'=0$ and $j_0=0$. The expansion for $\theta>0$ is then exactly that given in Eqs. (14)–(17). For $N=1$ we have one intermediate particle and desire the expansion of $f_{l_am_a,l_1m_1,l_bm_b}(g^{F_aF_1},g^{F_1F_b})$, which is (setting $\varphi_1=0$, as we may)

$$\begin{aligned} f_{l_am_a,l_1m_1,l_bm_b}&(B_z(\theta_1+\theta_a)B_x(\beta_1),B_x(-\beta_1)B_z(\theta_b-\theta_1)) \\ &= \int_{c_a-i\infty}^{c_a+i\infty} \frac{dn_a}{2\pi i} \int_{c_b-i\infty}^{c_b+i\infty} \frac{dn_b}{2\pi i}[j_{0a}^2-(n_a+1)^2] \\ &\quad \times[j_{0b}^2-(n_b+1)^2] \int d\mu(l,m) \int d\mu(l',m') \\ &\quad \times d_{l_am_al}{}^{j_{0a}n_a}(\theta_1+\theta_a) d_{m_am_1}{}^l(\beta_1) \\ &\quad \times f_{l_a,l_1m_1,l_b}{}^{j_{0a}n_a,j_{0b}n_b,lm,l'm'} d_{m'm_b}{}^{l'}(-\beta_1) \\ &\qquad \times d_{l'm_bl_b}{}^{j_{0b}n_b}(\theta_b-\theta_1), \quad (49) \end{aligned}$$

[13] M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, New York, 1954).

[14] A certain amount of algebra is involved in the explicit construction which backs up these statements. This is found in a forthcoming paper on the diagonalization of forward-absorptive-part equations by L. M. Saunders and me. I only hope at this point that the reader is assured that it can be done.

[15] V. Bargmann, Ann. Math. **48**, 568 (1947); M. Andrews and J. Gunson, J. Math. Phys. **5**, 1391 (1964). An extensive and lucid treatment is given by N. Ja. Vilenkin, in *Special Functions and the Theory of Group Representations* (American Mathematical Society, Providence, R. I., 1968), Vol. 22, Chap. VI.

where the integral $d\mu(l,m)$ is over the principal and discrete series of $O(1,2)$, as described in Vilenkin,[15] and the c_a, c_b lie to the right of all singularities in the n_a, n_b plane.

It is apparent how there is a notational explosion for $N=2$, ... Fortunately, as we shall see in the next sections, essentially all of the J-plane physics can be extracted without writing the whole partial-wave expansion each time. To complete this section, we write for completeness the definition of the partial amplitude in (49):

$$f_{l_a,l_1m_1,l_b}{}^{j_{0a}n_a,j_{0b}n_b,lm,l'm'}$$
$$=\sum_{m_a,m_b}\int_0^\infty d\zeta_1(\sinh\zeta_1)^2\int_0^\infty d\zeta_2(\sinh\zeta_2)^2\int_0^\infty \sinh\psi_1 d\psi_1$$
$$\times\int_0^\infty \sinh\psi_2 d\psi_2 d_{m_am}{}^l(\psi_1)^* d_{m'm_b}{}^{l'}(\psi_a)^*$$
$$\times e_{l_am_al}{}^{j_{0a}n_a}(\zeta_1)\,e_{l'm_bl_b}{}^{j_{0b}n_b}(\zeta_2)$$
$$\times f_{l_am_a,l_1m_1,l_bm_b}(B_z(\zeta_1)B_x(\psi_1),B_x(\psi_2)B_z(\zeta_2)).\quad(50)$$

Certain simplifications occur when everything is spinless; for example, helicity conservation at the central vertex requires $m=m'$.

IV. SINGLE-PARTICLE DISTRIBUTIONS AND REGGE BEHAVIOR

Having completed a long preliminary discussion, we turn to physics. In order to simplify the discussion and bring out the ideas involved in, especially, the imposition of Regge behavior, we discuss the inclusive production of a single spinless particle when spinless hadrons collide. The differential cross section is given by (4) as

$$\frac{d^2\sigma(a+b\to 1+X)}{d\theta_1 d(\cosh\beta_1{}^2)}=\frac{1}{2m_am_b\sinh(\theta_a+\theta_b)}\mathfrak{M}(\theta_1+\theta_a,\theta_b-\theta_1,\beta_1),\quad(51)$$

where we have noted that $\mathfrak{M}$ cannot depend on the azimuth of $\mathbf{p}_1$ and that

$$\Delta^{1/2}(s,m_a{}^2,m_b{}^2)=2m_am_b\sinh(\theta_a+\theta_b),$$

with s given by (41). When $m_a=m_b=m$, as it is convenient to choose for purposes of discussion, $\theta_a=\theta_b=\theta$ and we write (51) as

$$\frac{d^2\sigma(a+b\to 1+X)}{d\theta_1 d(\cosh\beta_1{}^2)}=\frac{1}{2m^2\sinh 2\theta}\mathfrak{M}(\theta+\theta_1,\theta-\theta_1,\beta_1).\quad(52)$$

At this point we consider the asymptotic behavior of $\mathfrak{M}$ for large s or θ. The form of the partial-wave expansion in (49) has been constructed to suggest that when θ is large $\theta+\theta_1$ or $\theta-\theta_1$, or both of them, also become large; then one might imagine that in the n_a and n_b planes one would encounter certain Regge poles which would in the usual manner govern the leading asymptotic behavior of $\mathfrak{M}$. It is important, however, to pause a moment and ask what basis such an n-plane hypothesis has. Since it is clear from the construction of $\mathfrak{M}$ as depicted in Fig. 2 that $\mathfrak{M}$ is not the absorptive part of a 3-to-3 amplitude but only a part of one, it is not enough to conjecture a Regge behavior for $\mathfrak{M}$ on the basis of such behavior for the whole 3-to-3 absorptive part.[16]

Lacking a general reason for the correctness of Regge behavior, I am forced to turn to the multiperipheral model since it is the strong point in the theoretical arguments for Regge asymptotics in 2-to-2 reactions. In that model, as formulated by Amati *et al.*,[5] there are three contributions to $\mathfrak{M}$: one in which the produced particle comes from the end near a, one in which it comes from the middle of the chain, and one from the end near b. If we consider single particles of mass m_1 being produced from the sides of the multiperipheral chain of mass μ with a strength g, then the $\mathfrak{M}$ of (52) is given by

$$\frac{16\pi^2}{g^2}\frac{\mathfrak{M}(\theta+\theta_1,\theta-\theta_1,\beta_1)}{\Delta^{1/2}(s,m^2,m^2)\sigma_{\rm tot}(a+b)}$$
$$=\int d^4q\,A(p_a,-q)\frac{1}{(\mu^2-q^2)^2}\delta^4(q+p_b-p_1)$$
$$+\int d^4q\,\delta^4(p_a-p_1-q)\frac{1}{(\mu^2-q^2)^2}A(q,p_b)$$
$$+\int d^4q_1d^4q_2A(p_a,-q_1)\frac{1}{(\mu^2-q_1{}^2)^2}\delta^4(q_1-p_1-q_2)$$
$$\times\frac{1}{(\mu^2-q_2{}^2)^2}A(q_2,p_b),\quad(53)$$

which is illustrated term by term in Fig. 4. The quantity $A(p,k)$ is the 2-to-2 absorptive part—off shell—for the collision $p+k\to p+k$. It satisfies a multiperipheral equation of the standard ladder variety and is related to the total cross section via

$$A(p,k)=\Delta^{1/2}((p+k)^2,p^2,k^2)\sigma_{\rm tot}.$$

Concentrate for a moment on the last term of (53), which is the contribution from Fig. 4(c). If we do the integration in frame F_1 and parametrize the q_1 and q_2 intermediate integrations by the usual $U_{q_i}=R_z(\psi_i)$

[16] Mueller (Ref. 4) also treats this point in some detail but does not make explicit reference to the multiperipheral model. His Eq. (5.2) is the type of structure such models immediately yield but could be more general.

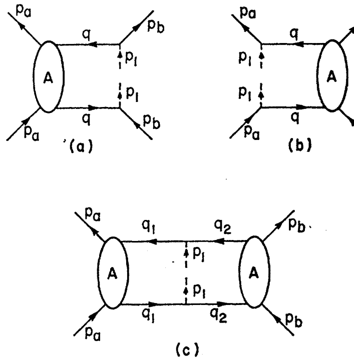

FIG. 4. Contributions to single-particle production in the multiperipheral model of Amati *et al.*, Ref. 5. (a) and (b) are fragmentation pictures while (c) gives pionization.

$\times B_z(\eta_i)B_x(\tau_i)$ operating on a standard vector, we can write for each of the $A(p,q)$'s in (53) a partial-wave representation as above:

$$A(p_a,\,-q_1) =\int_{c_a-i\infty}^{c_a+i\infty}\frac{dn_a}{2\pi i}[-(n_a+1)^2]\int d\mu(l,0)d_{00l}{}^{0n_a}(\theta_a+\theta_1) \times P_l(\cosh\beta_1)d_{l00}{}^{0n_a}(\eta_1)a^{n_a}(p_a{}^2,q_1{}^2)\,,\quad (54)$$

and a similar expression for $A(q_2,p_b)$.

We now can write the expression for $\mathfrak{M}$ in terms of $\mathfrak{M}_{ll'}{}^{n_a n_b}=f_{00,00,00}{}^{0n_a,0n_b,l0,l'0}$ coming from Eq. (49) and read off the partial amplitude arising via Fig. 4(c):

$$\mathfrak{M}_{ll'}{}^{n_a n_b}=\int d^4q_1d^4q_2\frac{a^{n_a}(p_a{}^2,q_1{}^2)d_{l00}{}^{0n_a}(\eta_1)}{(\mu^2-q_1{}^2)} \times\delta^4(q_1-p_1-q_2)\frac{d_{00l'}{}^{0n_b}(\eta_2)a^{n_b}(q_2{}^2,p_b{}^2)}{(\mu^2-q_2{}^2)}\,.\quad (55)$$

The Regge singularities appearing in n_a and n_b which govern the asymptotic behavior of the forward two-body absorptive part now make their role in determining $\mathfrak{M}$ quite explicit, namely, $\mathfrak{M}$ has its large $\theta_a+\theta_1$ or $\theta_b-\theta_1$ dependence also determined by the two-body singularities in the n_a and n_b planes.

It may well be that the identification of pole or cut positions in n_a and n_b with the familiar trajectories (at $t=0$) of two-body scattering goes beyond the multiperipheral argument given here. In any case we accept as a working hypothesis that whenever longitudinal boost angles are large, Regge asymptotic behavior with the usual singularity structure sets in. Some of the consequences of this idea have been given by Mueller[4] and DeTar,[2] and we repeat them while pointing out others.

We turn our attention, then, back to the

$$\mathfrak{M}(g^{F_aF_1},g^{F_1F_b})=\mathfrak{M}(\theta_a+\theta_1,\,\theta_b-\theta_1,\,\beta_1)$$

which enters (51). Suppose first that θ_1 is close to $-\theta_a$, which physically means that the produced particle is running along the negative z axis in the c.m. system and is, in some loose sense, associated with particle a. The invariant energy $(p_a+p_1)^2$ does not then become large as s (or equivalently $\theta_a+\theta_b$) increases. However, $\theta_b-\theta_1$ becomes large and a single asymptotic expansion is appropriate with the behavior governed by singularities in n_b. As is clear either from the multiperipheral arguments [see Fig. 4(b)] or the tree-graph picture of Fig. 3, the leading singularity can always be the vacuum trajectory (or the Pomeranchukon) with $n_b=\alpha_P\sim 1$ and the next leading singularity will lie as usual at $n_b=\alpha_{P'}\approx\frac{1}{2}$. Thus, if we hold β_1 fixed and $\theta_a+\theta_1$ fixed and "small" and let $\theta_b-\theta_1$ become large, our function $\mathfrak{M}$ behaves as[17]

$$\mathfrak{M}(\theta_a+\theta_1,\,\theta_b-\theta_1,\,\beta_1)\to e^{\alpha_P(\theta_b-\theta_1)}R_P(\theta_a+\theta_1,\,\beta_1) +e^{\alpha_{P'}(\theta_b-\theta_1)}R_{P'}(\theta_a+\theta_1,\,\beta_1)\,,\quad (56)$$

where the residues at the poles in the n_b plane have been combined with the various l integrals appearing in Eq. (49) to yield functions of the fixed variables $\theta_a+\theta_1$ and β_1 which refer to the Regge pole encountered. Before we proceed, we should point out the logical possibility that, say, $R_P(\theta_a+\theta_1,\,\beta_1)$ vanishes and that $R_{P'}$ really governs the leading behavior of $\mathfrak{M}$. This is unlikely and does not happen in any known multiperipheral model but we certainly have no general argument forbidding it. Therefore, if we assume R_P and $R_{P'}$ are nonzero, we can extract from (56) the leading asymptotic behavior of $d^2\sigma(a+b\to 1+X)/d\theta_1d(\cosh\beta_1{}^2)$ and the approach to that behavior.

Large s means that both θ_a and θ_b are large and the c.m. constraint (37) requires

$$m_ae^{\theta_a}\approx m_be^{\theta_b}\quad (57)$$

in the limit, so

$$\Delta^{1/2}(s,m_a{}^2,m_b{}^2)\approx m_b{}^2e^{2\theta_b}=m_a{}^2e^{2\theta_a}\,.\quad (58)$$

From (51) we have in the large-energy limit $\theta_a+\theta_1$, β_1 fixed,

$$\frac{d^2\sigma(a+b\to 1+X)}{d\theta_1d\,(\cosh\beta_1{}^2)}\to\left(\frac{m_a}{m_b}\right)^{\alpha_P}\frac{e^{2\theta_a(\alpha_P-1)}}{m_a{}^2} \times e^{-\alpha_P(\theta_a+\theta_1)}R_P(\theta_a+\theta_1,\beta_1) +\left(\frac{m_a}{m_b}\right)^{\alpha_{P'}}\frac{e^{2\theta_a(\alpha_{P'}-1)}}{m_a{}^2}e^{-\alpha_{P'}(\theta_a+\theta_1)}R_{P'}(\theta_a+\theta_1,\beta_1)\,.\quad (59)$$

For $\alpha_P=1$, namely, diffraction scattering, and $\alpha_{P'}\approx\frac{1}{2}$, we write (59) as a function of s, $\theta_a+\theta_1$, and β_1 by ab-

[17] Since the asymptotic behavior of $d_{lml'}{}^{j_0n}(\theta)$ is $e^{\theta(n-|m+j_0|)}$, the leading behavior comes when m and j_0 are set equal to zero. The first correction to this is one whole power of s and does not interfere with the argument of taking the correction to the Pomeranchukon to be the P' since the latter is only down by one-half power of s.

sorbing a few constants:

$$\frac{d^2\sigma(a+b\to 1+X)}{d\theta_1 d(\cosh\beta_1{}^2)} \to f_P(\theta_a+\theta_1,\beta_1) + s^{-1/2} f_{P'}(\theta_a+\theta_1,\beta_1)\,. \quad (60)$$

In the limit as $s\to\infty$, $\theta_a+\theta_1$, β_1 fixed, we see that the differential cross section becomes some unknown function, in general assumed to be nonzero, of the fixed variables. This limit is approached as $s^{-1/2}$ in traditional Regge fashion. The existence of such a limit was hypothesized by Benecke *et al.*[6] and Feynman[1] and has been discussed more or less in the above manner by DeTar[2] and Mueller.[4] By noting that the longitudinal momentum of p_1 is $m_1\cosh\beta_1\sinh\theta$ and $\sqrt{s}\sim m_a e^{\theta_a}$, one easily sees that $\theta_a+\theta_1$ is related to $\ln(p_{\rm long}/\sqrt{s})$ and the transverse momentum $|p_{\rm trans}|=\sinh\beta_1$, so that the limiting distribution can be regarded as a function of Feynman's $x=2p_{\rm long}/\sqrt{s}$ and $p_\perp$. At this stage of the argument, there is no apparent reason why the limiting function should factor into $f_\perp(p_\perp)f_x(x)$, as is occasionally supposed,[18] but it could happen.[19] The limiting behavior when $\theta_1\approx\theta_b$ and the produced particle runs along with the "fragment" from particle b is so similar to the case just discussed that I leave it to the reader to switch the a's and b's.

Instead we turn to the more interesting alternative where, as $s\to\infty$, θ_1 and β_1 are held fixed and $\theta_a+\theta_1$ and $\theta_b-\theta_1$ both become large. This is a double Regge limit and singularities in both the n_a and n_b planes come into play. If we call the contribution to $\mathfrak{M}$ of the residue in n_a at α_a and n_b at α_b, $R_{\alpha_a\alpha_b}$, the leading contributions to the double asymptotic limit described are

$$\mathfrak{M}(\theta_a+\theta_1,\theta_b-\theta_1,\beta_1)\to e^{\alpha_P(\theta_a+\theta_1)}e^{\alpha_P(\theta_b-\theta_1)}R_{PP}(\beta_1) + e^{\alpha_P(\theta_a+\theta_1)+\alpha_{P'}(\theta_b-\theta_1)}R_{PP'}(\beta_1) + e^{\alpha_{P'}(\theta_a+\theta_1)+\alpha_P(\theta_b-\theta_1)}R_{P'P}(\beta_1)\,. \quad (61)$$

With this form for $\mathfrak{M}$, the differential cross section behaves as

$$\frac{d^2\sigma(a+b\to 1+X)}{d\theta_1 d(\cosh\beta_1{}^2)} \to e^{(\alpha_P-1)(\theta_a+\theta_b)} f_{PP}(\beta_1) + e^{(\alpha_P-1)\theta_a}e^{(\alpha_{P'}-1)\theta_b}e^{(\alpha_P-\alpha_{P'})\theta_1} f_{PP'}(\beta_1) + e^{(\alpha_{P'}-1)\theta_a}e^{(\alpha_P-1)\theta_b}e^{-(\alpha_P-\alpha_{P'})\theta_1} f_{P'P}(\beta_1)\,. \quad (62)$$

Again, if $\alpha_P=1$ and $f_{PP}(\beta_1)$ are nonzero, a limiting distribution is approached, although the rate of approach is only as $s^{\frac{1}{2}(\alpha_P+\alpha_{P'})-1}\approx s^{-1/4}$, which is slower than the approach to the fragmentation limit. In the limit the leading term is a function of β_1, or the transverse momentum, only. This is called the pionization limit[7] by DeTar[2] and Mueller[4] who also discuss it.

An amusing feature of (62) is that on the way to the limit, which itself is predicted to be independent of s and θ_1, both the s and θ_1 dependence are precisely specified. In particular, suppose a and b are the same (as in pp collisions); then $f_{PP'}=f_{P'P}$, since both trajectories carry isospin zero and the behavior of (62) is

$$\frac{d^2\sigma(a+a\to 1+X)}{d\theta_1 d(\cosh\beta_1{}^2)} \to f_{PP}(\beta_1)+s^{-1/4}g(\beta_1)\cosh\tfrac{1}{2}\theta_1\,, \quad (63)$$

since $\alpha_P-\alpha_{P'}\approx\frac{1}{2}$.

An immediate application of this observation is that in the production of different members of an isomultiplet (let us imagine $p+p\to\pi^\pm+X$ for concreteness), the difference between the two distributions vanishes as $s\to\infty\,\theta_1$, β_1 fixed, since the leading $I=1$ trajectory is below the leading $I=0$ trajectory. Since $\alpha(0)\approx\frac{1}{2}$ for the ρ trajectory, one is led to expect[20]

$$s^{1/4}\left[\frac{d^2\sigma(p+p\to\pi^++X)}{d\theta_1 d(\cosh\beta_1{}^2)} - \frac{d^2\sigma(p+p\to\pi^-+X)}{d\theta_1 d(\cosh\beta_1{}^2)}\right] \to g(\beta_1)\cosh\tfrac{1}{2}\theta_1\,, \quad (64)$$

with an unknown $g(\beta_1)$.

This really completes the discussion of single-particle distributions in Regge theory, although there may be many interesting phenomena, especially with spinning particles, which have not been treated. Before we continue into two-particle production, however, we note in general that within a Regge framework alone nothing can be said about the very interesting functions of transverse boost angle which appear in, say, (64). These objects thus play a role similar to the usual residues of Regge poles which have an unknown functional dependence on invariant momentum transfers or external masses. Having determined the longitudinal dynamics from J-plane ideas, we are still left to cope with the very interesting transverse dynamics.

V. TWO-PARTICLE INCLUSIVE PRODUCTION

The two-particle inclusive differential cross section is a function of six variables which our group-theoretic analysis tells us to choose as $\theta_a+\theta_1$, $\theta_2-\theta_1$, $\theta_b-\theta_2$, β_1, β_2, and φ_2. As defined in Eq. (4), we then have for $d\sigma(a+b\to 1+2+X)$

$$\frac{d^5\sigma(a+b\to 1+2+X)}{d\varphi_2 d\theta_1 d\theta_2 d(\cosh\beta_1{}^2)d(\cosh\beta_2{}^2)} = \frac{\mathfrak{M}(\theta_a+\theta_1,\theta_2-\theta_1,\theta_b-\theta_2,\beta_1,\beta_2,\varphi_2)}{4\pi m_a m_b\sinh(\theta_a+\theta_b)}\,. \quad (65)$$

[18] N. F. Bali *et al.*, Phys. Rev. Letters **25**, 557 (1970).

[19] Actually, on the basis of their experiments with 19.2-GeV/c protons, J. V. Allaby *et al.* [CERN Report No. 70-12 (unpublished)] argue that factorization of this type indeed does not occur. Since these energies are likely to be considered pretty low when the National Accelerator Laboratory begins operating, one should probably reserve judgment on this matter.

[20] This particular result, and some of the ideas before it, have been presented in a short version of the present paper in Phys. Letters (to be published).

There are five different regions that are of interest in an asymptotic analysis. We treat the pionization-fragmentation limit first. In this limit, $\theta_1+\theta_a$ is finite while θ_2, β_1, β_2, and φ_2 are held fixed as θ_a and θ_b become large. This is the case in which particle 1 runs along with the fragment of a while particle 2 has a small velocity in the c.m. system. In this limit, $\theta_b-\theta_2$ and $\theta_2-\theta_1$ become large and we have a double Regge expansion. The leading two terms of this expansion are of the form

$$\frac{d^5\sigma(a+b\to 1+2+X)}{d\varphi_2 d\theta_1 d\theta_2 d(\cosh\beta_1{}^2)d(\cosh\beta_2{}^2)} \to e^{(\alpha_P-1)(\theta_a+\theta_b)}f_P(\theta_1+\theta_a,\beta_1,\beta_2,\varphi_2) + s^{-1/4}e^{(\alpha_P-\alpha_{P'})\theta_2}f_{PP'}(\theta_1+\theta_a,\beta_1,\beta_2,\varphi_2) + s^{-1/4}e^{-(\alpha_P-\alpha_{P'})\theta_2}f_{P'P}(\theta_1+\theta_a,\beta_1,\beta_2,\varphi_2)\,. \quad (66)$$

In this formula the $f_{\alpha\alpha'}$ are related to the residues of the Pomeranchukon and P' poles in the complex n planes conjugate to $\theta_2-\theta_1$ and $\theta_b-\theta_2$, respectively. The approach to this limit is also rather slow. The $f_{\alpha\alpha'}$ in general is expected to depend on all the indicated variables; however, the leading behavior arises from the expansion of $d_{lml'}{}^{j_0n}(\theta_2-\theta_1)$ which has $m=0$ as noted in Ref. 17. Therefore, the $e^{im\varphi_2}$ accompanying this $O(1,3)$ function is 1 in the coefficient of the leading term and no φ_2 dependence will occur. The dependence of $f_{\alpha\alpha'}$ then also factors as

$$f_{\alpha\alpha'}(\theta_1+\theta_a,\beta_1,\beta_2,\varphi_2)=g_\alpha(\theta_1+\theta_a,\beta_1)h_{\alpha'}(\beta_2)\,. \quad (67)$$

We come back to this factorization shortly.

In the second limit, the double fragmentation limit, β_1, β_2, and φ_2 are fixed and one considers both $\theta_1+\theta_a$ and $\theta_b-\theta_2$ finite while $\theta_1-\theta_2\approx\theta_a+\theta_b$ becomes large. Particle 1 is going with the fragment of a and particle 2 with the fragment of b. This is a single Regge limit whose leading contributions to $d\sigma$ are

$$f_P(\theta_1+\theta_a,\theta_b-\theta_2,\beta_1,\beta_2,\varphi_2)e^{(\alpha_P-1)(\theta_a+\theta_b)} + f_{P'}(\theta_1+\theta_a,\theta_b-\theta_a,\beta_1,\beta_2,\varphi_2)e^{(\alpha_{P'}-1)(\theta_a+\theta_b)}\,. \quad (68)$$

The approach to this limit is as $s^{-1/2}$ in normal single Regge fashion.

Again because of the requirement for the leading behavior to have $m=0$ in the magnetic quantum number sum involving $e^{im\varphi_2}$, one can expect that f_α in (68) will be independent of φ_2 and factor into $f(\theta_1+\theta_a,\beta_1)\times g(\theta_b-\theta_2,\beta_2)$.[21]

The third limit we consider here is a double pionization limit where β_1, β_2, and φ_2 are held fixed, as is $\theta_2-\theta_1$, while $\theta_1+\theta_a$ and $\theta_b-\theta_2$ become large. One carries out a by now standard double Regge limit to discover that $d\sigma(a+b\to 1+2+X)$ behaves as

$$g_{PP}(\theta_2-\theta_1,\beta_1,\beta_2,\varphi_2)e^{(\alpha_P-1)(\theta_a+\theta_b)}e^{-\alpha_P(\theta_2-\theta_1)} + g_{P'P}(\theta_2-\theta_1,\beta_1,\beta_2,\varphi_2)e^{(\alpha_P-1)\theta_b+(\alpha_{P'}-1)\theta_a} \times e^{-\frac{1}{2}(\alpha_P-\alpha_{P'})(\theta_1+\theta_2)}e^{-(\alpha_P+\alpha_{P'})/2(\theta_2-\theta_1)} + g_{PP'}(\theta_2-\theta_1,\beta_1,\beta_2,\varphi_2)e^{(\alpha_P-1)\theta_a+(\alpha_{P'}-1)\theta_b} \times e^{\frac{1}{2}(\alpha_P-\alpha_{P'})(\theta_1+\theta_2)}e^{-\frac{1}{2}(\alpha_P+\alpha_{P'})(\theta_2-\theta_1)}\,. \quad (69)$$

There are a number of comments to be made about Eq. (69). First, in the leading term involving g_{PP} we excluded the $\exp\{-[\alpha_P(\theta_2-\theta_1)]\}$ even though there is multiplying it an unknown $\theta_2-\theta_1$ dependence in g_{PP}. One can consider the function g_{PP} as having its major $\theta_2-\theta$ dependence coming from any *low-energy* resonance structure in the invariant energy $(p_1+p_2)^2$, since $\theta_2-\theta_1$ is not large. However, if two particles like K^+p or $\pi^\pm\pi^\pm$, where there is no known resonance at low energies, have been produced, g_{PP} can be expected to be a quite smooth function of $\theta_2-\theta_1$, while the dependence $e^{-(\theta_2-\theta_1)}$ for $\alpha_P=1$ makes its presence known quite dramatically. It would be very amusing to look for such an effect as one moved away from the region where $\theta_2=\theta_1$ into the region where $\theta_2>\theta_1$ but $\theta_2-\theta_1$ is not large. In any case, the leading behavior in (69) is independent of the combination $\theta_1+\theta_2$ no matter how strong the dependence of g_{PP} on $\theta_2-\theta_1$ might be. Even this would be interesting to see.

The approach to the limit is a double Regge approach behaving as $s^{-1/4}$ and governed by the P' intercept at $t=0$. In general, the double Regge limit behaves as $s^{\frac{1}{2}(\alpha_P+\alpha_{P'})-1}$, for reasons which are discussed in Sec. VII.

Since $\theta_1-\theta_2$ is now kept finite, there is no reason why the φ_2 dependence should disappear and, further, no reason for a factorization of the residue functions. In fact, if there are resonances in $(p_1+p_2)^2$, the dependences of the coordinates of particle 2 on those of particle 1 would be quite strong. As $\theta_2-\theta_1$ becomes large, however, and proceeds through a region where peripheral, then Regge, exchanges are important, the dependence on φ_2 could become insignificant, as pointed out by Treiman and Yang[22] some years ago.

The fourth limit to consider I call the Regge pionization limit. Here β_1, β_2, and φ_2 are held fixed while all three of $\theta_a+\theta_1$, $\theta_2-\theta_1$, $\theta_b-\theta_2$ become large. One may reach this case from the previous one, and I do not write down a detailed expression for $d\sigma$ but only note two striking features: (1) The $\theta_2-\theta_1$ dependence which was remarked upon after Eq. (69) goes away and $d\sigma$ becomes, in the leading asymptotic term, independent of θ_1 and θ_2. This was noted by DeTar[2] who calls a limit such as the one we are now talking about a "strong-ordering limit." (2) The approach to the Regge pionization limit is very slow because the terms coming from P and P' interference yield an $s^{-1/6}$ behavior.

Finally there is the single Regge two-particle fragmentation limit where both particles 1 and 2 run near

[21] This fact was stressed to me by A. Mueller (private communication).

[22] S. B. Treiman and C. N. Yang, Phys. Rev. Letters 8, 140 (1962).

the fragment of a. In this case we hold β_1, β_2, and φ_2 fixed as well as $\theta_a+\theta_1$ and $\theta_2-\theta_1$; that is, both θ_1 and θ_2 are "near" $-\theta_a$. As $\theta_b-\theta_2$ becomes large, we find for $d\sigma$

$$h_P(\theta_a+\theta_1,\,\theta_2-\theta_1,\,\beta_1,\,\beta_2,\,\varphi_2)e^{(\alpha_P-1)(\theta_a+\theta_b)} + h_{P'}(\theta_a+\theta_1,\,\theta_2-\theta_1,\,\beta_1,\,\beta_2,\,\varphi_2)e^{(\alpha_{P'}-1)(\theta_a+\theta_b)}, \quad (70)$$

so that this limit is approached as $s^{-1/2}$. No factorization is expected here.

Before concluding this section on limits, a few remarks are in order. (1) In each case considered, when one takes $\alpha_P=1$, the $d\sigma$ has a nonzero limit if the residue (single or multiple) of the leading pole does not vanish. Since after Eq. (56) we see no reason why it should vanish, and we shall assume that it does not. (2) One may carry out a multiperipheral-model analysis of the two-particle production cross section in order to justify, as we did for one particle production, the identification we have blithely made of the singularities in the multiple n planes with the familiar two-body singularities.

(3) With regard to the possibility of factorization as discussed after (66), one should note again that it is a phenomenon associated with the leading behavior in the $SO(1,3)$ expansion. In general, the infinite sum over magnetic quantum numbers associated with $e^{im\varphi_2}$ leads to nonfactorization. The physical reason for this is not hard to discern. Lorentz invariance requires $\mathfrak{M}$ to depend on $\varphi_2-\varphi_1$ only and not φ_2 and φ_1 separately (remembering that we set $\varphi_1=0$). Thus, independence of φ_2 and the subsequent factorization of $\mathfrak{M}$ means that no angular information had been carried between the two pieces of the diagram. This, of course, occurs when a spin-zero object is exchanged,[22] but the Pomeranchukon is composed of an infinite number of ordinary spin objects, so the φ_2 could be quite complicated in general and a factorization would not occur. However, we have here the knowledge that the leading behavior comes from the piece of the Pomeranchukon with $j_0=m=0$, that is, from special pieces of the ordinary $j=0,\,1,\,2,\,\ldots,$ angular momenta composing the Pomeranchukon. These clearly cannot carry any azimuthal information, so the dependence on φ_2 must vanish.

(4) The factorization at the Pomeranchukon pole makes it quite interesting to examine various correlation lengths along the multiperipheral chain as suggested by Wilson.[2] A particularly interesting correlation is that of two particles emitted from the central region of the chain. This would be observed in the transition from the double pionization regime to the Regge pionization limit as $\theta_2-\theta_1$ becomes large. Unfortunately, the numerical work of Sec. VI shows the total energy to be unacceptably large since Regge pionization is only reached as $s^{-1/6}$.

One may, however, examine the behavior of the correlation function[23]

$$\frac{d^4\sigma(a+b\to 1+2+X)}{d\theta_1 d(\cosh\beta_1^2)d\theta_2 d(\cosh\beta_2^2)}-\frac{1}{\sigma_{\rm tot}(a+b)}\times\frac{d^2\sigma(a+b\to 1+X)}{d\theta_1 d(\cosh\beta_1^2)}\,\frac{d^2\sigma(a+b\to 2+X)}{d\theta_2 d(\cosh\beta_2^2)} \quad (71)$$

as $\theta_1-\theta_1$ varies, in two more accessible regions: (1) the single Regge limit of double fragmentation, which can be studied at present accelerator energies, and (2) the double Regge limit of pionization fragmentation discussed by Wilson, which can probably be examined at the National Accelerator Laboratory. In each case one looks for the dependence on $\theta_2-\theta_1$ in (71) as $\theta_1+\theta_a$, $\theta_b-\theta_2$, β_1, and β_2 are held fixed. One should find, if the above analysis holds,

$$e^{-(\alpha_P-\alpha_{P'})(\theta_2-\theta_1)}\times(\text{function of fixed variables})\,. \quad (72)$$

The "correlation length" in θ space is then

$$(\alpha_P-\alpha_{P'})^{-1}\approx 2\,,$$

VI. SOME NUMERICAL OBSERVATIONS

For general orientation with respect to the phenomena discussed in Secs. IV and V, we briefly consider some numerical criteria for observing those effects. For simplicity of discussion, I concentrate on single inclusive production of pions in proton-proton collisions: $p+p\to\pi+X$. In this case, $\theta_a=\theta_b=\theta, m_a=m_b=m_p$, and $m_1=m_\pi$;

$$s=2m_p^2(1+\cosh 2\theta)=2m_p^2+2m_pE_L\,, \quad (73)$$

where the energy of the projectile in the lab is

$$E_L=m_p\cosh 2\theta\,. \quad (74)$$

We take as examples the two values $\theta=3.0$ and 3.5 or $E_L=190$ and 515 GeV, respectively, since these bracket the NAL energies. Because $\ln(m_\pi/m_p)\approx 2$, the bounds on θ_1 are in fact

$$-\theta-2\lesssim\theta_1\lesssim\theta+2\,, \quad (75)$$

so that the "overflow" of θ_1 can be a very large percentage of θ at these "low" energies. If we had considered proton production, there would have been no overflow.

Now when should we expect to see the fragmentation limit as in Eq. (60)? It is easy enough to arrange for $\theta_1+\theta$ to be small; i.e., $\theta_1\approx-\theta$ by choosing backward events in the c.m. system, and at the same time $\theta-\theta_1\approx 2\theta$ will be large. The question is really whether, when $2\theta\approx 6$ or 7, the next term in the asymptotic expansion of (60) due to the P' will be small. If we suppose that the quantities f_P and $f_{P'}$ are more or less the same order of magnitude, then the relative size of the two terms is $e^{-\theta}$, which is 5% for $\theta=3$ and 3% for $\theta=3.5$. In fact, $s^{-1/2}$ is a rapid enough decrease that we may ask when the P' term is 15%, say, of the leading Pomeranchukon term. The θ required is ≈ 1.9, leading to a lab energy of the projectile proton of 21 GeV. So one can well expect limiting fragmentation, or in general all our *single Regge*

[23] The variable φ_2 has been integrated out since the presumed factorization makes $d\sigma(a+b\to 1+2+X)$ independent of it in the leading terms.

limits to occur effectively in the range of present particle accelerators. This is entirely consistent with the observations of Bali *et al.*[18]

The pionization limit is, unfortunately, harder to reach for two reasons: (1) Small longitudinal momenta will be needed, and (2) it is only reached as $s^{-1/4}$. For this limit we require $\theta+\theta_1$ and $\theta-\theta_1$ to be large. To continue the discussion, I choose "large" to be a value of any combination of longitudinal boost angles such that exp(angles) = 10. This means that the angles are equal to 2.5. On this basis when $\theta=3.0$, we must have

$$|\theta_1|\lesssim 0.5\,, \tag{76}$$

and for $\theta=3.5$.

$$|\theta_1|\leq 1.0\,. \tag{77}$$

Since θ_1 is related to the c.m. longitudinal momentum p_l and the magnitude of the transverse momentum p_T by

$$\sinh\theta_1=p_l/(m_\pi{}^2+p_T{}^2)^{1/2}\,, \tag{78}$$

we have to look for p_l small or p_T large. There are not very many events with p_T sizeable, so let us estimate an average "large" p_T to be $\approx 2m_\pi\approx 300$ MeV/c. Then for $\theta=3.5$, $E_L\sim 500$ GeV, we need

$$|p_l|\lesssim 300\ \text{MeV}/c\,, \tag{79}$$

and for $\theta=3.0$, $E_L\sim 200$ GeV, we need

$$|p_l|\lesssim 120\ \text{MeV}/c\,. \tag{80}$$

These are, it can hardly be denied, small momenta. However, the account is not complete since we ask when the second term in the pionization asymptotic expansion, Eq. (62), is small relative to the first term. Again boldly taking the size of the residues to be about the same, the relative size of the Pomeranchukon and P' terms is governed by $e^{-\frac{1}{2}\theta}$ which is still about 20% at $E_L\sim 500$ GeV. I do not venture a guess concerning whether this is too large a background to be acceptable.

At this point I leave to the patient reader the opportunity for numerically examining the five two-particle distribution alternatives. We now turn to the conclusion.

VII. CONCLUSIONS AND OBSERVATIONS

The discussion of inclusive production in this paper was in a real sense a formalization of and generalization from lessons learned in multiperipheral or multi-Regge models.[2,4,5] The initial effort was devoted to choosing variables, which we found could be done in a manner quite similar to the discussion given in Ref. 3. The biggest help in the subsequent analysis came from the parametrization of momenta as

$$m_j(\cosh\beta_j\cosh\theta_j,\ \sinh\beta_j\cos\varphi_j,\ \sinh\beta_j\sin\varphi_j,\ \cosh\beta_j\sinh\theta_j) \tag{81}$$

and the effective separation thereby achieved between dynamics in longitudinal boost angles θ_j and transverse boost angles β_j. It is difficult to pretend that there is something unique about this choice; however, it is rather convenient and reasonably suggestive from a symmetry viewpoint.

It can correctly be asked whether the same variables can be found in a simpler fashion. For example, consider the fragmentation limit in single-particle production from a straightforward multi-Regge point of view. One begins by choosing as variables $s_a=(p_1+p_a)^2$, $s_b=(p_1+p_b)^2$ and considers $s\to\infty$ while, if particle 1 goes with particle b's fragment, $s_a\to\infty$, but s_b is fixed. In the c.m. frame for $a=b$ with mass m and each having energy E, we write

$$p_a=(E,0,0,-(E^2-m^2)^{1/2})\,, \tag{82}$$

$$p_b=(\text{E},0,0,+(E^2-m^2)^{1/2})\,, \tag{83}$$

and

$$p_1=(E_1,p_T\cos\varphi_1,p_T\sin\varphi_1,p_l)\,, \tag{84}$$

so

$$s=4E^2\,, \tag{85}$$

$$s_a=m^2+m_1{}^2+2[EE_1+p_l(E^2-m^2)^{1/2}] \tag{86}$$

and

$$s_b=m^2+m_1{}^2+2[EE_1-p_l(E^2-m^2)^{1/2}]\,. \tag{87}$$

A Regge theorist expects $d\sigma(a+b\to 1+X)$ to behave as $(s_a{}^\alpha/s)f(s,s_b)$, and is now faced with the problem of what to hold fixed. In order for $s_a\to\infty$ as $s\to\infty$ but for s_b to remain finite, it is easy to see that one holds p_T fixed (that is β_1) and p_l must grow as E or faster. If it grows just as E, $p_l/E=2p_l/\sqrt{s}$, which is Feynman's x, is finite and one is inclined, but not compelled, to hold p_T and x fixed. After such reasoning, it can be argued that since $s_a\approx s$ for $\alpha=1$, the Pomeranchukon $d\sigma$ with p_T and x held fixed has a nontrivial limit. The argument becomes slightly more torturous for two-particle production or even single-particle pionization. It is clearly a matter of taste to choose this form of reasoning over that of Sec. II.

This is also the point to recall two of the basic assumptions that entered our discussion of inclusive distributions: (1) After we located in Sec. II the appropriate complex "angular momentum" n, it was necessary to assert that the position of the singularities encountered in the n plane which govern the asymptotic behavior of the particle distributions were the selfsame $t=0$ Regge intercepts we find in two-body scattering. An attempt was made to justify this on the basis of the multiperipheral model, but it could be wrong. (2) In every case where we discussed an asymptotic limit, we took the coefficient of the leading behavior in the expansion to be nonzero and also estimated the size of the correction term by the apparent next leading behavior. This is, of course, a quite common practice and in the discussion of single-particle distributions the results of Ref. 18 would be difficult to understand if the assumption were false. Nevertheless, we had to hypothesize the existence of the limit.

Finally it is useful to note that the power of s in the approach to the various limits can be understood in the following simple manner. In the case of single Regge exchange, it is a normal $s^{\alpha_{P'}-1}\approx s^{-1/2}$. For double Regge exchange, each of the two subenergies is about $\sqrt{s}$ in scaled units, so the correction term in the asymptotic expansion is of order $(\sqrt{s})^1(\sqrt{s})^{1/2}/s=s^{-1/4}$. Similarly, multi-Regge pionization region where $N+1$ Reggeons are exchanged to produce particles 1, 2, ..., N, the limit is approached only as $s^{-1/2(N+1)}$ which is very slow indeed, as the numerical example of Sec. VI reveals.

ACKNOWLEDGMENTS

I am happy to thank G. F. Chew, D. J. Gross, A. Mueller, and S. B. Treiman for extensive discussions on various parts of the material presented here. L. M. Saunders participated extensively in the work reported in the Appendix.

APPENDIX

In this appendix[24] we indicate the role played by functions of the second kind in the group theory above and provide the source of relations such as (46) and (47). Our treatment is modeled on the discussion by Vilenkin[15] of $SL(2,R)$ as given in his Chap. 7. Many of the actual results quoted are found in a more group-theoretic context in Refs. 10 and 11. Their presentation is certainly more pedestrian and really aimed at the Laplace inversion formulas. Hopefully the repetition in our own language makes the present paper more complete.

We begin with the Gelfand representation of $SL(2,C)$, the covering group of the homogeneous Lorentz group, $SO(1,3)$. In this representation,[25] one associates with each element of $SL(2,C)$

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix},\quad ad-bc=1 \tag{A1}$$

an operation on functions $f_{j_0n}(z)$ of one complex variable

$$(T_g f_{j_0n}(z)=(bz+d)^{n_1-1}(\bar{b}\bar{z}+\bar{d})^{n_2-1} \times f_{j_0n}((az+c)/(bz+d)), \tag{A2}$$

where $j_0=\frac{1}{2}(n_2-n_1)$ and $n=\frac{1}{2}(n_1+n_2)-1$. As mentioned in the text, n and j_0 are, respectively, the maximum and minimum $O(3)$ angular momentum in a finite-dimensional representation of $SL(2,C)$.

Now Vilenkin's idea is essentially to project from $f_{j_0n}(z)$ its component corresponding to the diagonalization of some subgroup of operators in $SL(2,C)$. Our problem here is to adopt the projection properly so that a subgroup containing $R_z(\varphi)$ and $B_x(\beta)$ is simple, while operations with longitudinal boosts $B_z(\theta)$ bring forth the appropriate functions of the second kind. Since R_z and B_x along with B_y form a $SL(2,R)$ subgroup of $SL(2,C)$, what we want to do is diagonalize a noncompact subgroup of $SL(2,C)$.

To do this, we note that the complex z plane on which f_{j_0n} is defined may be projected onto the upper and lower sheets of a timelike unit hyperboloid in one time and two space dimensions. The inside of the unit circle $|z|\leq 1$, maps to the lower sheet with the point

$$z=e^{i\phi}\tanh\tfrac{1}{2}\theta \tag{A3}$$

corresponding to

$$(-\cosh\theta,\sinh\theta\cos\phi,\sinh\theta\sin\phi). \tag{A4}$$

The outside of the unit circle maps to the upper sheet with

$$z=e^{i\phi}\coth\tfrac{1}{2}\theta \tag{A5}$$

corresponding to

$$(\cosh\theta,\sinh\theta\cos\phi,\sinh\theta\sin\phi). \tag{A6}$$

The point of this geometric exercise is that we realize the operations of the $SL(2,R)$ subgroup on the $1+2$ hyperboloid or, equivalently, entirely within or entirely without the unit circle in the z plane. The $B_z(\chi)$ carries us from one section of the plane to the other with a weight which turns out to be proportional to the function of the second kind.

Therefore we make two projections from $f_{j_0n}(z)$; one for $|z|\geq 1$, which we call $F_+^{j_0n}(l,m)$, and one for $|z|\leq 1$, which we call $F_-^{j_0n}(l,m)$:

$$F_+^{j_0n}(l,m)=\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^\infty \sinh\theta d\theta(\sinh\tfrac{1}{2}\theta)^{2n} \times e^{i(m+j_0)\phi}d_{mij_0}{}^l(\theta)f_{j_0n}(z=e^{i\phi}\coth\tfrac{1}{2}\theta), \tag{A7}$$

and

$$F_-^{j_0n}(l,m)=\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^\infty \sinh\theta d\theta(\cosh\tfrac{1}{2}\theta)^{2n} \times e^{i(m+j_0)\phi}d_{m,-j_0}{}^l(\theta)f_{j_0n}(z=e^{i\phi}\tanh\tfrac{1}{2}\theta). \tag{A8}$$

In these formulas the reduced Wigner d function for $SL(2,R)$ appears[15] and the invariant measure $d\phi\,\sinh\theta d\theta$ on the $1+2$ timelike hyperboloid is present. The variables l and m will be the eigenvalues of $K_x^2+K_y^2-J_z^2$ and J_z, respectively; they and $\pm$ label the components of the $SL(2,C)$ representation (j_0,n) in its decomposition with respect to $SL(2,R)$ as given in (A7) and (A8). $F_\pm^{j_0n}(l,m)$ exist without restriction on m and with the requirement that l lie in a strip in the l plane,

$$\mathrm{Re}n\leq\mathrm{Re}l\leq-\mathrm{Re}n-1 \tag{A9}$$

or

$$\mathrm{Re}n\leq-\tfrac{1}{2}. \tag{A10}$$

We recover the function $f_{j_0n}(z)$ by inverting the $SL(2,R)$ projection with an integration along the line $\mathrm{Re}l=-\frac{1}{2}$

[24] Much of the work here was begun with L. M. Saunders in preparation for the study of Ref. 14.

[25] I. M. Gelfand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions* (Academic, New York, 1966), Vol. 5; Ref. 13, Chap. 9.

and a sum over any discrete representations that appear.[15] The condition for us to do this turns out to be just (A10). If we denote the measure over the principal series along $\mathrm{Re}\,l=-\frac{1}{2}$ and sum on discrete series by $d\mu(l,m)$ (its form is given in Ref. 15 and is irrelevant for what we are doing here), the function $f_{j_0n}(z)$ is recovered by

$$f_{j_0n}(z)=\theta(|z|-1)(\sinh\tfrac{1}{2}\theta)^{-2n}\sum_{m=-\infty}^{+\infty}\int d\mu(l,m)\times F_+{}^{j_0n}(l,m)e^{-im\phi}d_{m,j_0}{}^{l}(\theta)e^{-ij_0\phi}+\theta(1-|z|)(\cosh\tfrac{1}{2}\theta)^{-2n}\sum_{m=-\infty}^{+\infty}\int d\mu(l,m)\times F_-(l,m)e^{-im\phi}d_{m,-j_0}{}^{l}(\theta)e^{-ij_0\phi}. \quad (A11)$$

The real test of the definitions $F_\pm{}^{j_0n}(l,m)$ comes when we apply the transformations

$$R_z(\psi)=\begin{pmatrix}e^{-i\psi/2} & 0\\ 0 & e^{i\psi/2}\end{pmatrix}, \quad (A12)$$

and

$$B_x(\beta)=\begin{pmatrix}\cosh\frac{1}{2}\beta & \sinh\frac{1}{2}\beta\\ \sinh\frac{1}{2}\beta & \cosh\frac{1}{2}\beta\end{pmatrix} \quad (A13)$$

to them. It is quite easy to see that if we define a representation of $SL(2,C)$ by any element's action on the $f_{j_0n}(z)$ in (A7) or (A8) via (A2), then

$$R_z(\psi)F_\pm{}^{j_0n}(l,m)=e^{im\psi}F_\pm{}^{j_0n}(l,m). \quad (A14)$$

It is necessary to use judiciously the addition theorem on $d_{mn}{}^l(\theta)$ given by Vilenkin to show

$$B_x(\beta)F_\pm{}^{j_0n}(l,m)=\sum_{m'=-\infty}^{+\infty}d_{mm'}{}^{l}(-\beta)F_\pm{}^{j_0n}(l,m'). \quad (A15)$$

Thus any element h of the $SL(2,R)$ subgroup formed from B_x, B_y, and R_z operates on $F_\pm$ by

$$hF_\pm{}^{j_0n}(l,m)=\sum_{m'=-\infty}^{+\infty}D_{m,m'}{}^{l}(h^{-1})F_\pm{}^{j_0n}(l,m'), \quad (A16)$$

with $D_{m,m'}{}^l(h)$ representation matrix for $SL(2,R)$.

Now the operation of $B_z(\chi)$ can take us from outside the unit circle in z to the interior or vice versa; hence it mixes $F_\pm(l,m)$. Thus, by operating with $B_z(\chi)$ on $F_\pm(l,m)$ and then reexpressing the resultant $f_{j_0n}(z)$ under the integral in terms of $F_\pm$ by means of (A11), we realize a representation of $SL(2,C)$ by integral operators. This may be expressed in matrix form

$$(R_gF_a{}^{j_0n})(l,m)=\sum_{b=\pm}\sum_{m'=-\infty}^{+\infty}\int d\mu(l',m')\times K_{ab}(l,m;l',m';j_0,n;g)F_b{}^{j_0n}(l',m'),\quad a=\pm. \quad (A17)$$

For a boost along the z axis by χ,

$$B_z(\chi)=\begin{pmatrix}e^{\chi/2} & 0\\ 0 & e^{-\chi/2}\end{pmatrix}, \quad (A18)$$

and

$$B_z(\chi)f_{j_0n}(z)=e^{-n\chi}f_{j_0n}(e^{\chi}z). \quad (A19)$$

For $\chi>0$ it is clear that $K_{+-}=0$, and some straightforward algebra reveals

$$K_{++}(l,m;l',m';j_0,n;B_z(\chi))=\delta_{mm'}\int_0^\infty \sinh\theta d\theta(\sinh\tfrac{1}{2}\theta/\sinh\tfrac{1}{2}\theta')^{2n}\times d_{mj_0}{}^{l}(\theta)d_{mj_0}{}^{l'}(\theta')e^{-n\chi} \quad (A20)$$

with

$$e^{\chi}\coth\tfrac{1}{2}\theta=\coth\tfrac{1}{2}\theta', \quad (A21)$$

and

$$K_{-+}(l,m;l',m';j_0,n;B_z(\chi))=\delta_{mm'}\int_{\coth\chi}^{\infty}d(\cosh\theta)(\cosh\tfrac{1}{2}\theta/\sinh\tfrac{1}{2}\theta_1)^{2n}\times d_{m,-j_0}{}^{l}(\theta)d_{m',j_0}{}^{l'}(\theta_1)e^{-n\chi} \quad (A22)$$

with

$$e^{\chi}\tanh\tfrac{1}{2}\theta=\coth\tfrac{1}{2}\theta_1, \quad (A23)$$

and

$$K_{--}(l,m;l',m';j_0,n;B_z(\chi))=\delta_{mm'}\int_1^{\coth\chi}d(\cosh\theta)(\cosh\tfrac{1}{2}\theta/\cosh\tfrac{1}{2}\theta_2)^{2n}\times d_{m,-j_0}{}^{l}(\theta)d_{m,-j_0}{}^{l'}(\theta_2)e^{-n\chi} \quad (A24)$$

with

$$e^{\chi}\tanh\tfrac{1}{2}\theta=\tanh\tfrac{1}{2}\theta_2. \quad (A25)$$

It will perhaps be a relief to the reader to note that for $l=m=l'=m'=j_0=0$, K_{--} becomes

$$K_{--}(0,0;0,0;0,n;B_z(\chi))=\frac{e^{-(n+1)\chi}}{(n+1)\sinh\chi}, \quad (A26)$$

which begins to establish the connection between the kernels of the integral representation and the functions of the second kind.

A few changes of variables reveals that the connection between our K_{--} and the $a^{M\lambda}$ in Ref. 11 is

$$K_{--}(l,m;l;m_jj_0,n,B_z(\chi))=2a_{mll'}{}^{j_0,n+1}(\chi)/[(2l+1)(2l'+1)]^{1/2}. \quad (A27)$$

This is quite a useful result since many of the properties of the $a^{M,\lambda}$ are tabulated by Sciarrino and Toller. In particular, the asymptotic behavior for large χ is given as

$$a_{mll'}{}^{j_0,n+1}(\chi)\to\exp[-\chi(n+2+|m-j_0|)]. \quad (A28)$$

It is this asymptotic behavior, which is nicely decreasing for Ren positive, that one takes advantage of when defining the Lorentz Laplace transform of a function of a longitudinal boost angle θ. We always have in mind the connection of an invariant subenergy s with $\cosh\theta$. A behavior like s^α leads to $e^{\alpha\theta}$, so suppose $g(\theta)$ is bounded by $e^{\alpha\theta}$. We then define the partial amplitude which is analytic in $\mathrm{Re}\,n>\mathrm{Re}\,\alpha-|m-j_0|$:

$$g_{l_1l_2}{}^{j_0n}=\sum_m\int_0^\infty(\sinh\theta)^2d\theta\frac{g_{l_1ml_2}(\theta)a_{ml_1l_2}{}^{j_0,n+1}(\theta)}{B(j_0,n;l_1,m,l_2)},\qquad(\mathrm{A29})$$

where the usual measure $(\sinh\theta)^2d\theta$ on the $SO(1,3)$ hyperboloid has entered, and B is a combinatorial coefficient which is specified shortly. By using the asymptotic behavior of the $a^{j_0,n+1}$ functions, and the connection between them and the usual representation functions $d^{j_0,n}$ on $SL(2,C)$,[11] one may show that if we define $g_{l_1l_2}{}^{j_0n}$ via (A29), we can recover $g(\theta)$ for positive θ by

$$g_{l_1ml_2}(\theta)=\int_{c-i\infty}^{c+i\infty}\frac{dn}{2\pi i}[j_0{}^2-(n+1)^2]d_{l_1ml_2}{}^{j_0,n}(\theta)g_{l_1l_2}{}^{j_0n}\qquad(\mathrm{A30})$$

when we take for $m\geqslant j_0$:

$$B(j_0,n,l_1,m,l_2)=\frac{j_0{}^2-(n+1)^2}{4(m-j_0)!(m+j_0)!}(2l_1+1)(2l_2+1)\times\frac{(l_1+j_0)!(l_2+j_0)!}{(l_1-j_0)!(l_2-j_0)!}\times\frac{[n+2-j_0]_{l_2+j_0}[-n-2m+j_0]_{l_1-j_0}}{[-n-1-m]_{l_2+m+1}[n+1+m]_{l_1+m+1}},\qquad(\mathrm{A31})$$

with

$$[a]_q=a(a+1),\ldots,(a+q-\tfrac{1}{2}).\qquad(\mathrm{A32})$$

In the text we defined $e_{l_1ml_2}{}^{j_0n}(\theta)$ as

$$e_{l_1ml_2}{}^{j_0n}(\theta)=a_{ml_1l_2}{}^{j_0,n+1}(\theta)/B(j_0,n,l_1,m,l_2),\qquad(\mathrm{A33})$$

so our transform pair is (46) and (47).

PHYSICAL REVIEW D VOLUME 3, NUMBER 9 1 MAY 1971

Study of Chiral $SU(2)\times SU(2)$ Current-Algebra Models*

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The smoothness assumptions of the chiral $SU(2)\times SU(2)$ hard-pion current-algebra method are examined in detail. A new model, the current-smoothness model, emerges as a plausible alternative to the standard hard-pion model of Schnitzer and Weinberg, and others. The two models give satisfactory predictions for the decay $A_1\to\rho+\pi$, but quite different predictions for the decay $A_1\to\pi+\gamma$ and the colliding-beam reaction $e^+e^-\to A_1{}^\pm\pi^\mp$. Other possible models are also discussed.

I. INTRODUCTION

THE hard-pion current-algebra method,[1,2] which consists of the chiral $SU(2)\times SU(2)$ current commutation relations proposed by Gell-Mann,[3] conservation of the vector current (CVC), partial conservation of the axial current (PCAC), together with certain "smoothness" assumptions, provides a useful phenomenological tool for the analysis and correlation of various strong, electromagnetic, and weak processes. In particular, it leads to relations between the pion electromagnetic form factor, A_1-meson decays,[1,2] pion-pion scattering,[4,5] and pion-nucleon scattering.[6] Extension to the $SU(3)\times SU(3)$ current algebra, with nonconservation of the strangeness-changing vector current appropriately taken into account, leads to further relations between these processes, K-π scattering,[7] and the form factors of the K_{l3} decay,[8–10] although these relations are not entirely in agreement with experiment. The method can also be applied, with rather more

* Supported in part by the National Science Foundation under Grant No. GP-12396.
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[1] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).
[2] R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Letters **19**, 1085 (1967); Phys. Rev. **174**, 1999 (1968); **174**, 2008 (1968); R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, *ibid.* **175**, 1802 (1968).
[3] M. Gell-Mann, Physics **1**, 63 (1964).
[4] I. Gerstein and H. J. Schnitzer, Phys. Rev. **170**, 1638 (1968).
[5] R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. Letters **20**, 475 (1968); Phys. Rev. **175**, 1820 (1968).
[6] H. J. Schnitzer, Phys. Rev. **158**, 1471 (1967).
[7] P. Pond, Northeastern University Ph.D. thesis, 1970 (unpublished) and unpublished report.
[8] S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968).
[9] I. Gerstein and H. J. Schnitzer, Phys. Rev. **175**, 1876 (1968).
[10] R. Arnowitt, M. H. Friedman, and P. Nath, Nucl. Phys. **B10**, 578 (1969).