Rigorous Bound on K_{l3} Decay Amplitudes*

LING-FONG LI AND HEINZ PAGELS
Rockefeller University, New York, New York 10021
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We derive a rigorous bound on the K_{l3} decay form factors $f_\pm(t)$. With $\xi\equiv f_-(0)/f_+(0)$ and $\lambda_+\equiv m_\pi^2 f_+'(0)/f_+(0)$, our bound is $|\xi+(m_K^2-m_\pi^2)\lambda_+/m_\pi^2|\leqslant 8\Delta^{1/2}(0)I^{1/2}/\sqrt{3}m_K^2|f_+(0)|$, where $I=0.584$ and $\Delta(0)$ is the propagator of the divergence of the strangeness-changing current at zero momentum. If one further assumes for the Hamiltonian density $H(x)=H_0(x)+\epsilon_0 S_0(x)+\epsilon_8 S_8(x)$, with $H_0(x)$ $SU(3)\times SU(3)$ invariant and $S_{0,8}$ transforming like members of the $(\bar{3},3)+(3,\bar{3})$ representation, then one can prove that $\Delta(0)=3\epsilon_8\langle S_8\rangle_0/4$, so $\Delta(0)$ is proportional to the product of $SU(3)$ violations in the Hamiltonian times those in the vacuum, and is hence expected to be very small. We estimate $\Delta^{1/2}\sim m_\pi f_\pi$, and our bound implies $|\xi+12.3\lambda_+|\leqslant 0.29$. With $\lambda_+=0.06$ as indicated by recent data, we have $-1.03<\xi<-0.45$, which serves to restrict the ξ value considerably. We also remark on the relation of this bound to the perturbative theorem of Dashen and Weinstein.

I. INTRODUCTION

RECENTLY there has been considerable interest in the K_{l3} decay process as a probe to experimentally test various explicit theoretical models of $SU(3)\times SU(3)$-symmetry breaking. This process is suitable as a probe of symmetry breaking because it involves both strange and nonstrange pseudoscalar mesons and measures a matrix element of the strangeness-changing weak current.[1] The experimental situation for this decay process has not been completely resolved,[2] but experiments in the next several years should serve to refine the measurements.

The theoretical side of this problem is also in a state of controversy.[3] Using standard current-algebra methods, one can establish the theorem of Callan and Treiman[4] and Mathur, Okubo, and Pandit[5] (CTMOP), which is the statement $f_+(m_K^2)+f_-(m_K^2)=f_K/f_\pi$ about the K_{l3} form factors $f_\pm(t)$ at momentum transfer $t=m_K^2$ in an $SU(2)\times SU(2)$-symmetric world with a Goldstone pion. To make contact with experimental numbers, the theoretical result $f_+(m_K^2)+f_-(m_K^2)=f_K/f_\pi+O(m_\pi^2)$ based on partial conservation of axial-vector current (PCAC) and current algebra must be extrapolated to the neighborhood of $t=0$. Using the resulting relation, one has, for $\xi\equiv f_-(0)/f_+(0)$,

$$\xi=\left[\frac{f_K}{f_\pi f_+(0)}-\frac{f_+(m_K^2)}{f_+(0)}\right]\frac{f_-(0)}{f_-(m_K^2)}+O(m_\pi^2).$$

The K_{e3} Dalitz plot in the physical regions $m_e^2<t<(m_K-m_\pi)^2$ is fitted with a linear form for $f_+(t)=f_+(0)[1+\lambda_+(t/m_\pi^2)]+O(t^2)$, so that $f_+(m_K^2)/f_+(0)=1+(m_K^2/m_\pi^2)\lambda_+$. The data also indicate that $f_-(t)$ is a slowly varying function, so $f_-(0)/f_-(m_K^2)\simeq 1$. We then have

$$\xi\simeq\frac{f_K}{f_\pi f_+(0)}-1-\frac{m_K^2}{m_\pi^2}\lambda_+=0.28-\frac{m_K^2}{m_\pi^2}\lambda_+,$$

with $f_K/f_\pi f_+(0)=1.28$. The recent value of $\lambda_+\simeq 0.06$ gives in this analysis $\xi\simeq -0.52$, which is not in conflict with the bound we derive in Sec. III. Thus there is no conflict between our bound, the CTMOP relation, and experiments, if one accepts the extrapolations assumed valid above and particularly the value of λ_+.

Gaillard[1] was the first to emphasize that the CTMOP relation depended on the assumed symmetry-breaking scheme. Brandt and Preparata,[6] using another value $\lambda_+\simeq 0.03$ (in which case the CTMOP relation and the above analysis implied $\xi\simeq -0.1$ in conflict with experiments which suggested $\xi\simeq -1.0$), were led to abandon the theoretical assumptions on which the CTMOP relation was based and develop an alternative theory of symmetry breaking and the use of PCAC.

Also relevant to this discussion is the theorem of Dashen and Weinstein,[7] who showed that if the $SU(3)\times SU(3)$ symmetry is realized by an octet of Goldstone pseudoscalar bosons, then up to terms of second order in the symmetry breaking, the parameter ξ is determined in terms of $\lambda_+=m_\pi^2 f_+'(0)/f_+(0)$. In terms of these parameters, the theorem reads

$$\xi+(m_K^2-m_\pi^2)\lambda_+/m_\pi^2=\tfrac{1}{2}(f_K/f_\pi-f_\pi/f_K)+O(\lambda^2).$$

This theorem leaves undetermined the magnitude of the terms $O(\lambda^2)$, of higher order in the symmetry breaking, and we will comment upon this point in the light of the bound we derive.

The rigorous bound that we obtain on the K_{l3} decay

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parameters is

$$\left|\xi+\frac{m_K^2-m_\pi^2}{m_\pi^2}\lambda_+\right| \leqslant \frac{8I^{1/2}(m_\pi^2/m_K^2)}{\sqrt{3}m_K^2|f_+(0)|}\Delta^{1/2}(0), \quad (1.1)$$

where $I(m_\pi^2/m_K^2)$ is an explicit function of the ratio $m_\pi^2/m_K^2=0.075$ and has the numerical value $I(m_\pi^2/m_K^2)=0.584$ and

$$\Delta(0)=\int_0^\infty \frac{dt\,\rho(t)}{t},$$

where $\rho(t)$ is the spectral function associated with the Lehmann representation for $\langle 0|[\partial_\mu V_\mu{}^4(x),\partial_\nu V_\nu{}^4(0)]|0\rangle$, with $V_\mu{}^4(x)$ a component of the strangeness-changing vector current.

If we further assume that the Hamiltonian density is given by[8,9]

$$H(x)=H_0(x)+\epsilon_0 S_0(x)+\epsilon_8 S_8(x),$$

where $S_{0,8}(x)$ are members of the $(\bar{3},3)+(3,\bar{3})$ representation of $SU(3)\times SU(3)$ and $H_0(x)$ is $SU(3)\times SU(3)$ invariant, then one can show from a Ward identity

$$\Delta(0)=\tfrac{3}{4}\epsilon_8\langle S_8(0)\rangle_0.$$

Many estimates of the vacuum expectation value of $S_8(x)$ suggest that it is small,[9] corresponding to the approximate $SU(3)$ invariance of the vacuum state, and it is this feature of symmetry breaking that makes the bound (1.1) rather effective in establishing the numerical values for ξ in terms of λ_+.

In this same model of symmetry breaking, one has a sum rule

$$\tfrac{1}{4}\Delta_\pi(0)+\tfrac{3}{4}\Delta_\eta(0)-\Delta_K(0)=\Delta(0),$$

where $\Delta_{\pi,\eta,K}(0)$ are the Lehmann propagators at zero momentum for the divergences of axial-vector currents transforming like π, η, and K. An estimate of these quantities[10] and use of the sum rule leads to the estimate $\Delta^{1/2}(0)\simeq 1.006 m_\pi f_\pi$ which, along with an estimate of $f_+(0)=0.845$, implies from our bound

$$|\xi+[(m_K^2-m_\pi^2)/m_\pi^2]\lambda_+|\leqslant 0.29.$$

If we use the recent value[11] of $\lambda_+=0.060\pm0.021$, this implies

$$-1.03\leqslant\xi\leqslant-0.45,$$

which serves to restrict considerably the ξ value; in particular, $\xi\approx 0$ is ruled out. Even allowing for 20% variation on our estimate for $\Delta^{1/2}(0)$ does not radically alter this bound or destroy its usefulness. Any improvement on this bound tends to drive ξ closer to

$$-[(m_K^2-m_\pi^2)/m_\pi^2]\lambda_+\simeq -0.74.$$

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[9] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2196 (1968).

[10] V. S. Mathur and S. Okubo, Phys. Rev. D **1**, 3468 (1970).

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In Sec. II we derive our bound (1.1) and in Sec. III discuss the numerical evaluation of $\Delta(0)$. In Sec. IV we discuss possible improvements of the bound and its relevance for the perturbative theorem of Dashen and Weinstein.

II. DERIVATION OF BOUND

The K_{l3} form factors are defined from the matrix elements of the strangeness-changing current

$$\langle\pi^0(p)|V_\mu{}^{K^+}(0)|K^+(k)\rangle=\tfrac{1}{2}[(k_\mu+p_\mu)f_+(t) \\ +(k_\mu-p_\mu)f_-(t)],\quad t=(p-k)^2. \quad (2.1)$$

Of particular interest are the ξ and λ_+ parameters, defined by

$$\xi=\frac{f_-(0)}{f_+(0)},\quad \lambda_+=\frac{f_+'(0)m_\pi^2}{f_+(0)}. \quad (2.2)$$

The matrix element of the divergence of the current, $\frac{1}{2}d(t)=\langle\pi^0(p)|i\partial_\mu V_\mu{}^{K^+}(0)|K^+(k)\rangle$, is $d(t)=(m_K^2-m_\pi^2)\times f_+(t)+tf_-(t)$, so that we have the following relation between the slope of $d(t)$ at $t=0$ and ξ and λ_+:

$$d'(0)=f_+(0)\left(\xi+\frac{m_K^2-m_\pi^2}{m_\pi^2}\lambda_+\right). \quad (2.3)$$

Our project is now to establish a bound for $d'(0)$.

To this end, we will make as a principal assumption the existence of a Lehmann representation for

$$\Delta(t)=\int d^4x\, e^{+iq\cdot x}\langle 0|(\partial_\mu V_\mu{}^{K^+}(x)\partial_\lambda V_\lambda{}^{K^-}(0))_+|0\rangle, \\ q^2=t. \quad (2.4)$$

Hence we have the representation

$$\Delta(t)=\int_0^\infty \frac{\rho(t')dt'}{t'-t}, \quad (2.5)$$

where $\rho(t)=\pi^{-1}\,\mathrm{Im}\Delta(t)$ is obtained from (2.4) and is given by

$$\rho(t)=(2\pi)^3\sum_n |\langle 0|\partial_\mu V_\mu{}^{K^+}(0)|n\rangle|^2\delta^4(q-p_n)\geqslant 0,$$

where the positivity is guaranteed by the causality requirement and positive norms in the Hilbert space. Each state $|n\rangle$ contributes a positive definite quantity to $\rho(t)$.

Our bound is obtained by retaining just the πK state, so we have

$$\rho(t)\geqslant\rho_{\pi K}(t)=[(t-m_\pi^2+m_K^2)^2-4tm_K^2]^{1/2}\frac{3|d(t)|^2}{16t(2\pi)^2}, \\ t\geqslant t_0=(m_K+m_\pi)^2 \quad (2.6)$$

where $d(t)$ is the matrix element of the divergence of the current as defined before, here analytically continued to $t \geqslant t_0 = (m_K + m_\pi)^2$, the physical threshold. The existence of the Lehmann representation (2.5) implies $\rho(t) \to 0$, $t \to \infty$ and hence from (2.6) we must have $d(t) \to 0$, $t \to \infty$. Assuming $d(t)$ to be analytic in the cut t plane with a cut at $t = t_0$, we may on this observation write the unsubtracted dispersion relation

$$d(t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\mathrm{Im}\, d(t') dt'}{t' - t},$$

so that

$$|d'(0)| = \left| \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\mathrm{Im}\, d(t) dt}{t^2} \right| \leqslant \frac{1}{\pi} \int_{t_0}^{\infty} \frac{|\mathrm{Im}\, d(t)| dt}{t^2}. \quad (2.7)$$

Now using (2.6), we have

$$|\mathrm{Im}\, d(t)| \leqslant |d(t)| \leqslant \frac{8\pi}{\sqrt{3}} \frac{t^{1/2} \rho^{1/2}(t)}{[(t - m_K^2 + m_\pi^2)^2 - 4 m t_K^2]^{1/2}}, \quad t \geqslant t_0 \quad (2.8)$$

and substituting in (2.7) and using the Schwarz integral inequality we have our bound on $|d'(0)|$,

$$|d'(0)| \leqslant 8 \Delta^{1/2}(0) I^{1/2}(m_\pi^2/m_K^2)/\sqrt{3} m_K^2. \quad (2.9)$$

Here

$$\Delta(0) = \int_0^{\infty} \frac{\rho(t) dt}{t}$$

and

$$I(\delta) = \int_{(1+\delta)^2}^{\infty} \frac{dx}{x^2} [(x - \delta^2 + 1)^2 - 4x]^{-1/2} = (1 - \delta^2)^{-2} \left[\frac{1 + \delta^2}{1 - \delta^2} \ln\left(\frac{1}{\delta}\right) - 1 \right], \quad (2.10)$$

so that with $\delta^2 = m_\pi^2/m_K^2 = 0.075$, $I = 0.584$.

Combining (2.9) and (2.3), we have the rigorous bound

$$\left| \xi + \frac{m_K^2 - m_\pi^2}{m_\pi^2} \lambda_+ \right| \leqslant \frac{8 \Delta^{1/2}(0) I^{1/2}(m_\pi^2/m_K^2)}{\sqrt{3} |f_+(0)| m_K^2}, \quad (2.11)$$

which is the result to be proved.

III. ESTIMATE OF $\Delta(0)$

In order to estimate the quantity $\Delta(0)$ appearing in the bound, we will assume for the Hamiltonian density[8,9]

$$H(x) = H_0(x) + \epsilon_0 S_0(x) + \epsilon_8 S_8(x), \quad (3.1)$$

where $H_0(x)$ is $SU(3) \times SU(3)$ invariant, and $S_\alpha(x)$ ($\alpha = 0, 1, \ldots, 8$) transform as members of the $(3,\bar{3}) + (\bar{3},3)$ representation of $SU(3) \times SU(3)$. Introducing the pseudoscalar densities $P^\alpha(x)$, $\alpha = 0, 1, \ldots, 8$, we have on this assumption the equal-time commutation rules

$$\begin{aligned} [F^a, S_\beta] &= i f^{a\beta\gamma} S_\gamma, & [F^a, P_\beta] &= i f^{a\beta\gamma} P_\gamma, \\ [{}^5F^a, S_\beta] &= -i d^{a\beta\gamma} P_\gamma, & [{}^5F^a, P_\beta] &= i d^{a\beta\gamma} S_\gamma, \end{aligned} \quad (3.2)$$

where F^a and ${}^5F^a$ ($a = 1, \ldots, 8$) are the generators of $SU(3) \times SU(3)$ and can be represented as space integrals over the vector and axial-vector charge densities

$$F^a = -i \int d^3x\, V_0^a(x), \quad {}^5F^a = -i \int d^3x\, A_0^a(x). \quad (3.3)$$

From the general result[12,13] $\partial_\mu J_\mu^a(x) = i[H(x), Q^a]$ and (3.1) and (3.2), one obtains

$$\begin{aligned} \partial_\mu V_\mu^a(x) &= \epsilon_8 f_{a8\beta} S_\beta(x), \\ \partial_\mu A_\mu^a(x) &= (\epsilon_0 d_{a0\beta} + \epsilon_8 d_{a8\beta}) P_\beta(x). \end{aligned} \quad (3.4)$$

If we next assume a Lehmann representation of the form

$$\langle 0 | [A_\mu^a(x), A_\nu^b(y)] | 0 \rangle = \int_0^{\infty} dt \left[\left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{t} \right) \rho_{ab}^{(1)}(t, A) - \frac{1}{t} \rho_{ab}^{(0)}(t, A) \partial_\mu \partial_\nu \right] \times \Delta(x - y, t)$$

and a similar representation for the vector current, then upon taking the divergence $\partial/\partial x_\mu$ of this expression and setting $x_0 = y_0$, one has the Ward identities

$$A_{ab} \equiv \int_0^{\infty} dt\, \rho_{ab}^{0}(t, A) = -(\epsilon_0 d_{0a\gamma} + \epsilon_8 d_{8a\gamma}) \times (s_0 d_{0b\gamma} + s_8 d_{8b\gamma}), \quad (3.5)$$

$$V_{ab} \equiv \int_0^{\infty} dt\, \rho_{ab}^{0}(t, V) = -\epsilon_8 s_8 f_{8a\gamma} f_{8b\gamma},$$

where

$$s_8 = \langle S_8 \rangle_0, \quad s_0 = \langle S_0 \rangle_0$$

are the vacuum expectation values of the scalar operators.

Following Mathur and Okubo[10] by introducing the parameters

$$a = \frac{1}{\sqrt{2}} \frac{\epsilon_8}{\epsilon_0}, \quad b = \frac{1}{\sqrt{2}} \frac{s_8}{s_0}, \quad \gamma = -\tfrac{2}{3} \epsilon_0 s_0,$$

we then have, from (3.5),

$$\begin{aligned} A_{33} &= \gamma(1 + a)(1 + b), & A_{44} &= \gamma(1 - \tfrac{1}{2}a)(1 - \tfrac{1}{2}b), \\ A_{88} &= \gamma(1 - a - b + 3ab), & V_{44} &= (9/4)\gamma ab. \end{aligned} \quad (3.6)$$

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[13] D. J. Gross and R. Jackiw, Phys. Rev. **163**, 1688 (1967).

We identify the spectral function which appears in our bound as $\rho(t)=t\rho_{44}^{(0)}(t,V)$, so that $\Delta(0)=V_{44}$. It then follows from (3.6) that

$$\Delta(0)=V_{44}=(3\epsilon_8/4)\langle S_8\rangle_0\,, \tag{3.7}$$

and thus $\Delta(0)$ is proportional to the $SU(3)$ violations in the vacuum $\langle S_8\rangle_0$ times the $SU(3)$ violation in the Hamiltonian ϵ_8. Several estimates[9,14] have $\langle S_8\rangle_0 \ll \langle S_0\rangle_0$, so the vacuum is approximately $SU(3)$ symmetric, and $\Delta(0)$ is a small quantity serving the purpose of making our bound effective. However, if $\langle S_8\rangle_0=0$ exactly, then $\Delta(0)=0$ and it follows from the Lehmann representation that $\rho(t)=0$ and the Johnson-Federbush theorem,[15] that $\partial_\mu V_\mu{}^4=0$, so we must have exact $SU(3)$ symmetry in this case. Then, of course, in the case of exact $SU(3)$ symmetry, a consistent result has $f_-(t)=0$ and $\xi=0$, the $SU(3)$-symmetric result. Hence it is important to establish more precisely the value of $\Delta(0)$ and how much it differs from zero.

It has been noted that the pseudoscalar propagators at zero momentum $\Delta_\pi=A_{33}$, $\Delta_K=A_{44}$, $\Delta_\eta=A_{88}$, and $\Delta=V_{44}$ satisfy the sum rule

$$\tfrac{1}{4}\Delta_\pi+\tfrac{3}{4}\Delta_\eta-\Delta_K=\Delta\,, \tag{3.8}$$

as follows from (3.6). Using the pole approximation for the pseudoscalar propagators, neglect of η-X mixing, and other approximations, Okubo and Mathur[10] estimate for the parameters in (3.6)

$$\begin{gathered} a=-0.89,\quad b=-0.10,\quad \gamma=5.05m_\pi{}^2f_\pi{}^2\,,\\ f_K/f_\pi=1.08,\quad f_\eta/f_\pi=1.06\,, \end{gathered} \tag{3.9}$$

so that from (3.6) and (3.7) we obtain

$$\Delta^{1/2}\approx 1.006m_\pi f_\pi\,. \tag{3.10}$$

The smallness of this parameter, as remarked already, is due to the fact that it is the product of $SU(3)$ violations in the vacuum times those in the Hamiltonian, one or both of which are small compared to the $SU(3)$-invariant terms in all determinations of symmetry breaking. Using other determinations of these parameters, we get estimates for $\Delta^{1/2}$ not differing by more than $\sim 20\%$ from (3.10). From the sum rule (3.8) we see that Δ is a measure of the deviation of the pseudoscalar propagators $\sim m^2f^2$ from the Gell-Mann–Okubo relation, which is another way to see why it is so small.

From the ratio of π_{l2} to K_{l2}, one obtains the ratio $f_K/f_\pi f_+(0)=1.28$ which, with the estimate (3.9), gives $f_+(0)=0.845$. Using this and (3.10), we obtain from our bound (2.11)

$$|\xi+12.3\lambda_+|\lesssim 0.29\,. \tag{3.11}$$

It is evident that the ξ parameter is sensitive to variations in λ_+. Using the recent value[11] $\lambda_+=0.06$, our bound is

$$-1.03\lesssim\xi\lesssim -0.45\,,$$

which would rule out $\xi\simeq 0$.

IV. DISCUSSION AND CONCLUSIONS

Our method of bounding $d'(0)$ in terms of $\Delta(0)$ leads to a practical bound, so it is of more than purely theoretical interest to see if this result can be improved to force the parameter ξ closer to $-12.3\lambda_+$. First we might remark that we have retained only πK states in bounding $\rho(t)$. However, this state is the most important because its contribution is proportional to $(m_K{}^2-m_\pi)/m_K{}^2\sim 1$; baryon states contribute proportional to $\Delta M/M\sim 0.15$ and hence are much smaller. The ηK state could also be included but we have no precise or approximate knowledge of the couplings to bound its contribution. Hence we would conclude that if improvement is sought, it should be in terms of just the πK state. We have not been successful in improving this bound beyond the simple bound given above.

Of relevance to our bound on $d'(0)$ is the perturbative theorem of Dashen and Weinstein.[7] They showed that if $H=H+\lambda H'$, where H_0 is the $SU(3)\times SU(3)$-invariant Hamiltonian, H' breaks this symmetry, and the $SU(3)\times SU(3)$ symmetry is realized by an octet of pseudoscalar bosons, then

$$d'(0)=\tfrac{1}{2}(f_K/f_\pi-f_\pi/f_K)+O(\lambda^2)\,. \tag{4.1}$$

The question this theorem raises is the validity of perturbation theory in λ, or the largeness of $O(\lambda^2)$ relative to the first term $\frac{1}{2}(f_K/f_\pi-f_\pi/f_K)\sim O(\lambda)$.

Our bound on $|d'(0)|$ is given by (2.9) or $|d'(0)|\leqslant 3.53\Delta^{1/2}(0)/m_K{}^2\simeq 0.29$, using our estimate (3.10). Suppose we completely ignore the terms of $O(\lambda^2)$ in (4.1). Then if we use our estimate (3.9), $f_K/f_\pi=1.08$, we have from (4.1) that $d'(0)=0.08$, which lies well within our bound and would imply $\xi\simeq-12.3\lambda_+$. What is usually done, however, in this perturbative treatment, is to use the value $f_K/f_\pi=1.28$ since $f_K/f_\pi f_+(0)=1.28$ and $f_+(0)=1+O(\lambda^2)$ by the Ademollo and Gatto,[16] and Berhends and Sirlin[17] theorem, in which case $d'(0)=0.25$, saturating the bound. In view of our bound, this latter estimation with $d'(0)=0.25$ is probably too large and we conclude, in the light of these observations, that any determination of the precise value of $d'(0)$ obtained from (4.1) should be viewed with caution.

If we assume that only the sign of $d'(0)$ is given correctly by the first term of (4.1), $\frac{1}{2}(f_K/f_\pi-f_\pi/f_K)$, then since in all determinations $f_K/f_\pi\geqslant 1$, $d'(0)\geqslant 0$. If we assume $f_+(0)>0$, as is also valid perturbatively, then the absolute value sign can be removed from our bound

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(3.11); hence with $d'(0)/f_+(0)>0$, we have

$$0\leqslant \xi+12.3\lambda_+<0.29\,,$$

or, for $\lambda_+=0.06$, $-0.74\leqslant \xi\leqslant -0.45$.

What emerges most generally from this analysis is that $d'(0)$ is probably a small quantity. This feature then requires that $\xi\simeq -12.3\lambda_+$ and hence ξ is contingent on the difficultly measured parameter λ_+. Before more definite experimental values of these parameters are available and the experimental situation stabilizes, it is difficult to make theoretical pronouncements on the character of the symmetry breaking.

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Theory of Pionization*

HUNG CHENG†
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

AND

TAI TSUN WU
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138

(Received 28 August 1970)

In this paper we make a detailed study of the phenomenon of pionization. We first concentrate on processes in which the two incident particles are identical, e.g., proton-proton scattering. By assuming that the c.m. system has no special significance in high-energy scattering or, more precisely, by assuming that the distribution of pionization products has a forward-backward symmetry not only in the c.m. system but also in a class of coordinate systems called the C systems (this assumption will be called the *symmetry hypothesis*), we can get rather precise information about the dependence of pionization distribution on the longitudinal momenta. In particular, the one-particle distribution function is $f(\mathbf{p}_\perp)d^3p/E$. These results are then explicitly verified in the model of quantum electrodynamics. Next, the considerations are extended to processes in which the two incident particles are different. For such processes, we propose the *limiting hypothesis:* The distribution of the pionization products is independent of the momenta of either of the incident particles in the limit the incident momenta are both infinite. With this hypothesis we can again obtain quantitative information on the dependence of the pionization distribution on the longitudinal momenta. When applied to the special case of identical particles, the limiting hypothesis is equivalent to the symmetry hypothesis. For the more general case where the two incident particles are not identical, it follows from the limiting hypothesis that the one-particle distribution function is again $f(\mathbf{p}_\perp)d^3p/E$, but the results on the multiparticle distribution functions are slightly weaker than those for identical particles. The limiting hypothesis is then shown to be valid in various field-theory models. The process of photon-fermion scattering is studied in particular detail. We also investigate the process of bremsstrahlung and show that it gives no pionization products. Finally, the above results are examined in the context of scalar electrodynamics, and all of the qualitative features are found to be the same.

1. INTRODUCTION

ABOUT a decade ago, the alternating-gradient synchrotrons at CERN and BNL came into operation. During the intervening years, many beautiful and important experiments have been performed at these accelerators. By now we have a great deal of information about, among others, hadronic collision processes, both elastic and inelastic.[1] We should remember, however, that, at 32 BeV for proton-proton scattering, the kinetic energy of each proton in the c.m. system is only about 3 BeV. Therefore, at energies available from these synchrotrons, we are not yet in the high-energy region where the incident particles are extreme relativistic. This situation is somewhat improved by the recent operation of the Serpukhov accelerator.

In order to learn some physics for higher energies, at say, 400 BeV, before accelerators are built, we look for common features between experiments at energies below 35 BeV and those utilizing cosmic rays.[2] For example, a rather striking common feature is the presence of two

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[1] A summary of recent experiments may be found in the *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).

[2] See, for example, the rapporteur paper of M. Koshiba, in Proceedings of the Tenth International Conference on Cosmic Rays, Calgary, Canada, 1967 (unpublished).