ask if a very simple model can be constructed to fit the p waves (assuming that the nonpole current-algebra result fits the s waves). The simplest model might contain the $\frac{1}{2}^+$ octet trajectory by itself. In this case, only the u channel contributes. If only $SU(3)$ Clebsch-Gordan coefficients are used, all the p-wave amplitudes stand correctly in relation to one another when reasonable d/f ratios are used for the weak and strong vertices. However, if the magnitude *and* sign of $(d/f)_{\text{weak}}$ is taken from the current-algebra results, the relative sign of the s waves and the p waves is incorrect. Thus it may be that a number of trajectories are required to fit the data. Our criterion (1.2) may be useful in limiting this number.

We would like to remark that several authors have proposed "duality" models[11] for nonleptonic hyperon decays where the s-wave amplitude is given by the K^*-trajectory exchange in the t channel. Essentially these models claim to pick up the K^* Regge-pole term so that our result would have very serious consequences for them if it is assumed that any result which is true in the Van Hove model is true in any Regge model.

Finally, we note that a more detailed study of the l-plane analyticity of the decay amplitudes could be useful (though rather complicated). If ideas like the one presented here turn out to be useful, they may, of course, be applied to other decay processes.

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Study of the Decay $\omega \to \pi\pi\gamma$*†

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The effect of π-π interaction in isospin-zero states on the characteristics of the $\omega \to \pi\pi\gamma$ decay is investigated by means of partial-wave dispersion relations. With the ρ-exchange contribution as the driving term and elastic unitarity for π-π scattering on the right-hand cut, we arrive at integral equations whose solutions explicitly depend on the π-π phase shift. Various scattering-length and resonance models are used to represent the π-π phase shift. The high-energy contribution to the integral equation is treated in three ways: by subtraction, adding an effective pole on the left-hand cut, and normalizing the amplitude at the f_0 resonance to a Veneziano model for $\omega \to \pi\pi\gamma$. Numerical results for rate and spectrum are discussed in the light of the possibility of discriminating by experimental results among the various models for π-π interaction.

I. INTRODUCTION

In this paper we present a detailed study of the first-order electromagnetic process $\omega \to 2\pi + \gamma$. Apart from the natural challenge of a theoretical understanding of the radiative transitions among vector and pseudoscalar mesons, there is an additional feature making this process an interesting object for experimental and theoretical study, since it provides the opportunity for investigating the dynamics of π-π interaction in even angular momentum states because of the particular final-state configuration of the pion pair.

It is well known that the analysis of π-π interaction[1] is quite difficult, mainly because of the complications arising from the presence of additional hadrons in the final states of the processes investigated. As a result, our knowledge of the details of the π-π interaction, especially in the S-wave angular momentum state, is still in a quite unsettled position, which occasions the understandable interest in any process which could contribute to advancing the study of π-π interaction.

In the process $\omega \to \pi\pi\gamma$, only even angular momentum states are allowed for the two pions, as a result of the charge-conjugation invariance of the electromagnetic interactions. Moreover, the pions are restricted to the isospin-zero state when the process is considered to first order in the fine-structure constant. This determines the relation between the two possible final charge states, namely, $\Gamma(\omega \to \pi^+\pi^-\gamma) = 2\Gamma(\omega \to \pi^0\pi^0\gamma)$.

The first theoretical estimate of this process,[2] describing it as a transition $\omega \to (\rho) + \pi \to \gamma + \pi + \pi$, predicts a relative decay rate[3] of $\Gamma(\omega \to \pi^+\pi^-\gamma + \pi^0\pi^0\gamma)/\Gamma(\omega \to \text{all}) = 1.8 \times 10^{-4}$. This number is obtained by

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[3] The values quoted for ω decay in the compilation tables [Particle Data Group, Phys. Letters **33B**, 1 (1970)] are $\Gamma(\omega \to \text{all}) = 11.9 \pm 1.3$ MeV, out of which $(90 \pm 4)\%$ is made of the 3π decay mode, while the dominant electromagnetic decay to $\pi^0\gamma$ accounts for $(9.4 \pm 1.2)\%$.

taking for $g_{\rho\pi\gamma}$ the value predicted by $SU(3)$ symmetry combined with canonical ω-ϕ mixing, namely, $g_{\rho\pi\gamma} = \frac{1}{3} g_{\omega\pi\gamma}$, which is also consistent with the presently available experimental information. However, this pole-model calculation might be inadequate due to the neglect of final-state interactions, which could significantly increase the decay rate. Yellin[4] and Renard[5] have considered this possibility and have added to the above-mentioned Born term, pole diagrams with resonances (σ,ϵ) in the π-π channel. They do indeed predict an increase in the rate by approximately one order of magnitude. However, their numerical estimates suffer from the uncertainty concerning the coupling constants of the scalar particles to ω-γ. Aviv and Nussinov[6] handled this difficulty by using finite-energy sum rules to determine the coupling-constant products $g_{\epsilon\pi\pi}g_{\epsilon\omega\gamma}$ and $g_{f\pi\pi}g_{f\omega\gamma}$ and obtained a value $\Gamma(\omega \to 2\pi^0+\gamma) \approx 0.1$ MeV.

On the experimental side, the situation is that so far only upper limits have been established for the $\omega \to \pi\pi\gamma$ modes. These amount to $\Gamma(\omega \to \pi^+\pi^-\gamma)/\Gamma(\omega \to \pi^+\pi^-\pi^0) < 5\%$[7] and $\Gamma(\omega \to \pi^0\pi^0\gamma)/\Gamma(\omega \to \text{neutrals}) < 20\%$.[8–10]

From the preceding exposition, one sees that the calculated decay rate is strongly model dependent. In light of the interest attached to this process, a more detailed investigation is called for. We undertake such a study here by considering the analytic continuation of the decay amplitude, namely, the scattering process $\omega+\gamma \to \pi+\pi$. It is assumed that the partial-wave projections of the invariant amplitudes obey dispersion relations in the s variable (the square of the energy in the center-of-mass system of the two-pion channel). The left-hand cut will be approximated by the ρ-meson exchange contribution, while at the right-hand cut we shall use elastic unitarity for the two pions. We arrive thereby at an integral equation for the $\omega+\gamma \to \pi+\pi$ transition, with the ρ exchange as the inhomogeneous driving term.[11,12] With our procedure we do not need the knowledge of the couplings of ω-γ to scalar and tensor particles. The solution of the integral equation depends on the π-π phase shift, and by assuming for it various possibilities, we check their effect on the process.

A complication arises due to the apparent need for a subtraction in the dispersion relation for the S-wave amplitude. In order to tackle this question in greater detail, we use among other possibilities a Veneziano-type amplitude which was constructed[13] for the process $V_1+\pi \to V_2+\pi$, which now serves to normalize our amplitude. This Veneziano-type amplitude naturally has the correct asymptotic behavior at high energies and also gives the correct description of the lowest-lying resonances of the considered scattering process, namely, ρ and f^0. Thus, we are able to determine the nature of the subtraction constant by normalizing at relatively low energies to an amplitude possessing the correct high-energy behavior. As a result of our analysis, we are able to classify different forms of π-π interaction into three main groups, according to their effect on the decay width of $\omega \to \pi\pi\gamma$.

Before concluding the introductory remarks, we should like to explain the reason for choosing for our study the ω decay among the various possible processes of the type $V \to P+P'+\gamma$. The $\rho \to \pi\pi\gamma$[14] and $K^* \to K\pi\gamma$[15] decays, involving also charged particles, have contributions from both inner brehmsstrahlung and direct processes. Since the relative magnitude and phase of the two independent contributions are model dependent, the effect of the π-π interaction would be more difficult to disentangle. The only other process of this kind which has only a direct decay mechanism is $\phi \to \pi\pi\gamma$.[4,5] Since the Born term is proportional to the $\phi\gamma\pi$ coupling, which is supposedly very small, one expects this decay to be of insignificant magnitude compared to the other ϕ decays and hence even more difficult to study experimentally than the $\omega \to \pi\pi\gamma$ transition.

In Sec. II we develop the necessary kinematics and define the invariant amplitudes. In Sec. III we present the integral equations for the partial-wave amplitudes. Section IV describes the various possible parametrizations we assume for the π-π amplitude. In Sec. V we present a determination of the subtraction constant from a Veneziano-type amplitude for $\omega \to \pi\pi\gamma$. Section VI contains a summary of the numerical analysis, and in Sec. VII we discuss the conclusions we can reach from our investigation.

II. INVARIANT AMPLITUDES, HELICITY AMPLITUDES, AND KINEMATICS

In order to keep our treatment more general, we define the amplitudes and kinematics for the process $V_1+\pi_i \to V_2+\pi_j$, where the V's are isosinglet $C=-1$, vector mesons. Whenever necessary they will be de-

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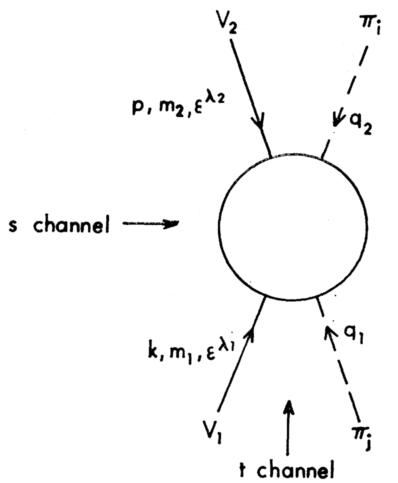

FIG. 1. Definition of kinematics.

tailed for the specific case $V_1=\gamma$, $V_2=\omega$. Let S and T be the scattering and transition matrices, respectively, connected by the relation[16]

$$S=I-\frac{(2\pi)^4 i\delta^4(p_f-p_i)}{(16E_1E_2E_3E_4)^{1/2}}T, \tag{2.1}$$

$$T_{\lambda_1\lambda_2}=\epsilon_{\lambda_1}{}^{\alpha}(k)M_{\alpha\beta}\epsilon_{\lambda_2}{}^{\beta}(p), \tag{2.2}$$

where k, $\epsilon_{\lambda_1}(k)$ and p, $\epsilon_{\lambda_2}(p)$ are the momenta and polarization vectors of V_1 and V_2, respectively. Requiring invariance under C, P, and T, the quantity $M_{\alpha\beta}$ can be described in terms of five invariant amplitudes as follows:

$$M_{\alpha\beta}=A(s,t)p_\alpha k_\beta+B(s,t)g_{\alpha\beta}+C(s,t)p_\alpha\Delta_\beta +D(s,t)\Delta_\alpha k_\beta+E(s,t)\Delta_\alpha\Delta_\beta, \tag{2.3}$$

where

$$\Delta_\mu=(q_1-q_2)_\mu, \tag{2.4}$$

and q_1, q_2 are the momenta of the pions (see Fig. 1). The invariant variables s, t, and u are defined by

$$s=(p+k)^2=(q_1+q_2)^2, \tag{2.5a}$$

$$t=(p+q_2)^2=(k+q_1)^2, \tag{2.5b}$$

$$u=(p+q_1)^2=(k+q_2)^2=m_1^2+m_2^2+2\mu^2-s-t. \tag{2.5c}$$

Parity and charge-conjugation invariance imply that $M_{\alpha\beta}$ will be symmetric under the interchange $q_1\to q_2$; therefore the amplitudes A, B, and E are symmetric, and C and D antisymmetric under the exchange $t\to u$.

In the s channel, with the z axis taken in the direction of $\mathbf{p}$, Eqs. (2.2) and (2.3) lead to the following helicity amplitudes:

$$T_{1,-1}{}^s=2(q\sin\theta)^2E, \tag{2.6a}$$

$$T_{1,0}{}^s=[(\sqrt{2}q\sin\theta)/m_2]\times[k(\sqrt{s})D-(2qp_0\cos\theta)E], \tag{2.6b}$$

[16] Throughout this article we use units in which $\hbar=c=1$.

$$T_{0,1}{}^s=[(\sqrt{2}q\sin\theta)/m_1]\times[-k(\sqrt{s})C-(2qk_0\cos\theta)E], \tag{2.6c}$$

$$T_{1,1}{}^s=B-2(q\sin\theta)^2E, \tag{2.6d}$$

$$T_{0,0}{}^s=(1/m_1m_2)[k^2sA+(p_0k_0+k^2)B +2qk(\sqrt{s})\cos\theta(k_0D-p_0C) -k_0p_0(2q\cos\theta)^2E], \tag{2.6e}$$

where

$$q=\tfrac{1}{2}(s-4\mu^2)^{1/2}, \tag{2.7a}$$

$$k=\frac{\{[s-(m_1-m_2)^2][s-(m_1+m_2)^2]\}^{1/2}}{2\sqrt{s}}, \tag{2.7b}$$

$$k_0=(s+m_1^2-m_2^2)/2\sqrt{s}, \tag{2.7c}$$

$$p_0=(s+m_2^2-m_1^2)/2\sqrt{s}, \tag{2.7d}$$

$$\cos\theta=(t-u)/4qk. \tag{2.7e}$$

q and k are the center-of-mass momenta of the pions and vector mesons, respectively, and θ is the angle between the momenta $\mathbf{q}_1$ and $\mathbf{k}$.

In the t channel, with the z axis again in the direction of $\mathbf{p}$, from Eqs. (2.2) and (2.3) the following helicity amplitudes are obtained:

$$T_{1,1}{}^t=\tfrac{1}{2}[(A-C+D-E)q_1q_2\sin^2\theta_t -(1+\cos\theta_t)B], \tag{2.8a}$$

$$T_{1,-1}{}^t=-\tfrac{1}{2}[(A-C+D-E)q_1q_2\sin^2\theta_t +(1-\cos\theta_t)B], \tag{2.8b}$$

$$T_{1,0}{}^t=[(\sin\theta_t)/\sqrt{2}m_1]\times[q_1(q_1p_0-q_2k_0{}^t\cos\theta_t)(A-C+D-E) +2q_1{}^2(\sqrt{t})(E-D)-k_0{}^tB], \tag{2.8c}$$

$$T_{0,1}{}^t=[(\sin\theta_t)/\sqrt{2}m_2]\times[q_2(-k_0{}^tq_2+p_0{}^tq_1\cos\theta_t)(A-C+D-E) -2q_2{}^2(\sqrt{t})(E+C)+p_0{}^tB], \tag{2.8d}$$

$$T_{0,0}{}^t=(1/m_1m_2)[p_0{}^tq_1\cos\theta_t-q_2k_0{}^t)\times(q_1p_0{}^t-q_2k_0{}^t\cos\theta_t)(A-C+D-E) +(q_1q_2-k_0{}^tp_0{}^t\cos\theta_t)B+4tq_1q_2E -2q_2(\sqrt{t})(q_1p_0{}^t-q_2k_0{}^t\cos\theta_t)(E+C) +2q_1(\sqrt{t})(p_0{}^tq_1\cos\theta_t-q_2k_0{}^t)(E-D)], \tag{2.8e}$$

where

$$q_1=\frac{\{[t-(m_1-\mu)^2][t-(m_1+\mu)^2]\}^{1/2}}{2\sqrt{t}}, \tag{2.9a}$$

$$q_2=\frac{\{[t-(m_2-\mu)^2][t-(m_2+\mu)^2]\}^{1/2}}{2\sqrt{t}}, \tag{2.9b}$$

$$k_0{}^t=(t+m_1^2-\mu^2)/2\sqrt{t}, \tag{2.9c}$$

$$p_0{}^t=(t+m_2^2-\mu^2)/2\sqrt{t}, \tag{2.9d}$$

$$\cos\theta_t=\frac{s-u}{4q_1q_2}+\frac{(m_1^2-\mu^2)(m_2^2-\mu^2)}{4tq_1q_2}; \tag{2.9e}$$

q_1, q_2 are the center-of-mass momenta of V_1-π_j and V_2-π_i, respectively, and θ_t is the angle between $\mathbf{p}$ and $\mathbf{q}_1$. The rest of the helicity amplitudes are derived from (2.6) and (2.8) through the relation

$$T_{-\lambda_1,-\lambda_2}=(-1)^{\lambda_1-\lambda_2}T_{\lambda_1,\lambda_2}. \tag{2.10}$$

For the decay $\omega\rightarrow 2\pi+\gamma$, we set $m_2=m_\omega\equiv m$ and $m_1=0$. Gauge invariance is ensured by imposing

$$T_{0,\lambda_2}=0 \tag{2.11}$$

(or, equivalently, $k^\alpha M_{\alpha\beta}=0$), which translates into two relations between the five invariant amplitudes:

$$E=-[(s-m^2)/(t-u)]C, \tag{2.12a}$$

$$B=-\tfrac{1}{2}[(s-m^2)A+(t-u)D]. \tag{2.12b}$$

Accordingly, for the process $\omega\rightarrow\pi\pi\gamma$, the Feynman amplitude $M_{\alpha\beta}$ has the form

$$M_{\alpha\beta}'=A(p_\alpha k_\beta-k\cdot p g_{\alpha\beta})+D(\Delta_\alpha k_\beta-k\cdot\Delta g_{\alpha\beta}) +C[p_\alpha\Delta_\beta-\Delta_\alpha\Delta_\beta(s-m^2)/(t-u)]. \tag{2.13}$$

III. DISPERSION RELATIONS FOR DECAY $\omega\rightarrow 2\pi+\gamma$

We assume that the decay amplitude may be obtained from the scattering amplitude for the process $\omega+\gamma\rightarrow\pi_i+\pi_j$ by analytic continuation, for which amplitude we write a dispersion relation in the s variable. The right-hand cut is represented by elastic unitarity, and the left-hand cut is approximated by the contribution of the ρ exchange in the t and u channels.

The unitarity condition for the s-channel helicity amplitudes (2.6), after using time-reversal invariance and neglecting intermediate states with more than two pions, takes the form

$$\mathrm{Im}T_{\lambda_1,\lambda_2}{}^s(s,z_3)=\frac{1}{16\pi^2}\left(\frac{s-4\mu^2}{s}\right)^{1/2}\int F^\dagger(s,z_2) \times T_{\lambda_1,\lambda_2}{}^s(s,z_1,\phi)dz_1d\phi, \tag{3.1}$$

where F is the $I=0$ π-π scattering amplitude; z_3, z_2, and z_1 are the cosines of the angles between $\mathbf{p}_f$ and $\mathbf{p}_i$, $\mathbf{p}_n$ and $\mathbf{p}_f$, and $\mathbf{p}_i$ and $\mathbf{p}_n$, respectively; and ϕ is the angle between the two planes defined by $(\mathbf{p}_i\times\mathbf{p}_n)$ and $(\mathbf{p}_f\times\mathbf{p}_i)$.

The expansion of $T_{\lambda_1,\lambda_2}{}^s$ and F into partial waves is

$$T_{\lambda_1,\lambda_2}{}^s(s,z,\phi)=\sum_J(2J+1)D_{\lambda 0}{}^{J*}(\phi,z,0)T_{\lambda_1,\lambda_2}{}^J(s), \tag{3.2}$$

where $D_{\lambda 0}{}^J$ are the rotation matrices[17] with $\lambda=\lambda_2-\lambda_1$, and

$$F(s,z)=\sum_J(2J+1)P_J(z)f_J(s), \tag{3.3}$$

$P_J(z)$ being the Legendre polynomials.

[17] A. Edmonds, *Angular Momentun in Quantum Mechanics* (Princeton U. P., Princeton, New Jersey, 1957), p. 55.

Integrating over dz_1 and $d\phi$, the unitarity condition for $T_{\lambda_1,\lambda_2}{}^J(s)$ becomes

$$\mathrm{Im}T_{\lambda_1,\lambda_2}{}^J(s)=\frac{1}{16\pi}\left(\frac{s-4\mu^2}{s}\right)^{1/2}f_J{}^*(s)T_{\lambda_1,\lambda_2}{}^J(s), \quad 4\mu^2\leqslant s\leqslant 16\mu^2. \tag{3.4}$$

The unitarity condition for elastic π-π scattering allows us to express f_J in the form

$$f_J(s)=16\pi\left(\frac{s}{s-4\mu^2}\right)^{1/2}e^{i\delta_J(s)}\sin\delta_J(s), \tag{3.5}$$

where $\delta_J(s)$ is the $I=0$ π-π phase shift. In order to apply unitarity to the process $\omega+\gamma\rightarrow\pi_i+\pi_j$, we want the explicit form of $T_{\lambda_1,\lambda_2}{}^s$ for this case. Using Eqs. (2.11) and (2.12) together with (2.6), one has

$$T_{1,-1}{}^s=-q^2(s-m^2)C'\sin^2\theta, \tag{3.6a}$$

$$T_{1,0}{}^s=(\sqrt{2}q/m)\tfrac{1}{2}(t-u)H\sin\theta, \tag{3.6b}$$

$$T_{1,1}{}^s=-\tfrac{1}{2}(s-m^2)\left(A-2q^2C'+\frac{(t-u)^2}{2(s-m^2)^2}I\right), \tag{3.6c}$$

where

$$C'\equiv 2C/(t-u),\quad D'\equiv 2D/(t-u),$$
$$H\equiv(s-m^2)D'+(s+m^2)C',$$
$$I\equiv H+m^2C'.$$

Since C and D are antisymmetric under $t\rightarrow u$, all the above-defined functions are regular on the line $t=u$.

Expanding these amplitudes according to (3.2), the partial-wave amplitudes free of kinematical singularities are

$$t_2{}^J\equiv\frac{T_{1,-1}{}^J(s)}{q^2(s-m^2)} =-\frac{[(J-1)J(J+1)(J+2)]^{1/2}}{(2J-1)(2J+1)(2J+3)}[(2J+3)C_{J-2}' -2(2J+1)C_J'+(2J-1)C_{J+2}'], \tag{3.7a}$$

$$t_1{}^J\equiv\frac{T_{1,0}{}^J(s)}{2q^2km}=\frac{\sqrt{2}[J(J+1)]^{1/2}}{m^2(2J-1)(2J+1)(2J+3)} \times[(2J+1)^2H_J-(J-1)(2J+3)H_{J-2} -(J+2)(2J-1)H_{J+2}], \tag{3.7b}$$

$$t_0{}^J\equiv\frac{T_{1,1}{}^J(s)}{(s-m^2)}=-\left\{(A-q^2C')_J+\frac{(2qk)^2}{(s-m)^2}\times\left[\frac{J(J-1)}{(2J-1)(2J+1)}I_{J-2}+\frac{2J(J+1)-1}{(2J-1)(2J+3)}I_J +\frac{J(J+1)}{(2J+1)(2J+3)}I_{J+2}\right]\right\}, \tag{3.7c}$$

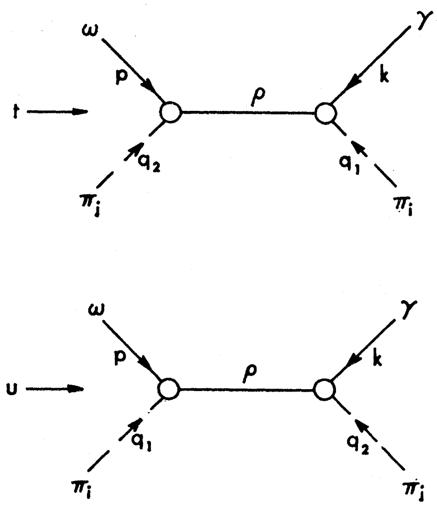

FIG. 2. Feynman diagrams of the ρ-exchange contribution to the t and u channels.

and

$$k^2=(s-m^2)^2/4s.$$

Partial waves of the invariant amplitude are defined similarly to (3.3).

The left-hand cut, approximated by the ρ exchange (see Fig. 2), is given by

$$T_{\lambda_1,\lambda_2}^{(\rho)}=G\epsilon_{\alpha\beta\gamma\delta}p^\alpha\epsilon_{\lambda_2}^{\beta}(p)Q^\gamma\frac{(g^{\delta\rho}-Q^\delta Q^\rho/Q^2)}{t-m_\rho^2}\times\epsilon_{\rho\sigma\tau\eta}Q^\sigma\epsilon_{\lambda_1}^{\tau}(k)k^\eta+q_1\to q_2, \quad (3.8)$$

$$G\equiv g_{\omega\rho\pi}g_{\rho\pi\gamma}/\mu^2.$$

$\epsilon^{\alpha\beta\gamma\delta}$ is the fourth-rank antisymmetric tensor and $Q_\mu=(p+q_2)_\mu$. From (3.8), (2.2), and (2.13) we get the ρ contribution to the invariant amplitudes

$$A_\rho=-[G/4(t-m_\rho^2)][\tfrac{1}{2}(s+m^2)-2(t+\mu^2)]+t\to u, \quad (3.9a)$$

$$C_\rho=-[G/4(t-m_\rho^2)]\tfrac{1}{2}(t-u)-t\to u, \quad (3.9b)$$

$$D_\rho=[G/4(t-m_\rho^2)][\tfrac{1}{2}(t-u)+m^2]-t\to u. \quad (3.9c)$$

According to (3.7), the ρ exchange contribution to the partial-wave amplitudes is

$$t_{\rho2}^J=-\frac{G[(J-1)J(J+1)(J+2)]^{1/2}}{4qk(2J-1)(2J+1)(2J+3)}[(2J+3)Q_{J-2}(z)-2(2J+1)Q_J(z)+(2J-1)Q_{J+2}(z)], \quad (3.10a)$$

$$t_{\rho1}^J=\frac{G(m_\rho^2-\mu^2)[J(J+1)]^{1/2}}{\sqrt{2}(2qk)^2(2J+1)}\times[Q_{J-1}(z)-Q_{J+1}(z)], \quad (3.10b)$$

$$t_{\rho0}^J=-\frac{G}{8}\left\{\left[2s-m^2-4\mu^2+\left(\frac{m^2}{s-m^2}-\frac{2s}{s+x}\right)\frac{(4qk)^2z^2}{(s-m^2)}\right]\times\frac{Q_J(z)}{2qk}-\frac{8qkz}{(s-m^2)}\left(\frac{m^2}{s-m^2}-\frac{2s}{s+x}\right)\delta_{J,0}\right\}, \quad (3.10c)$$

where

$$x=2(m_\rho^2-\mu^2)-m^2,\quad z=(s+x)/4qk,$$

and Q_J are the Legendre functions of the second kind. $Q_J(z)$ may be expanded in terms of $1/z^2$ (which will simplify the solution of the resultant integral equation):

$$Q_J(z)=\pi^{1/2}\frac{\Gamma(J+1)(2z)^{-J-1}}{\Gamma(J+\frac{3}{2})}\times\left[1+\frac{(J+1)(J+2)}{2(2J+3)}\frac{1}{z^2}+\cdots\right], \quad (3.11)$$

and, since in the physical region of the decay $4\mu^2\leqslant s\leqslant m^2$ one has $|z|>3$, in the expansion we shall neglect terms of order $(1/z^2)$ when compared to 1. Then Eqs. (3.10) become

$$t_{\rho2}^2=-G\frac{2\sqrt{2}}{5\sqrt{3}}\frac{1}{s+x}, \quad (3.12a)$$

$$t_{\rho1}^2=G\frac{4}{5\sqrt{3}}\frac{m_\rho^2-\mu^2}{(s+x)^2}, \quad (3.12b)$$

$$t_{\rho0}^2=-G\frac{2q^2}{15s(s+x)^3}\left\{m^2(x+m^2)^2-2(s-m^2)^2\times\left[x+m^2+2\mu^2-\frac{3}{7}m^2\frac{s-4\mu^2}{s}\right]\right\}, \quad (3.12c)$$

$$t_{\rho0}^0=-\frac{G}{4}\left\{\frac{1}{(s+x)}\left[s-m^2+(s-4\mu^2)\left(1-\frac{m^2}{3x}\right)\right]-\frac{2}{3}\frac{(s-m^2)(s-4\mu^2)}{(s+x)^2}+\frac{m^2}{3x}\frac{(s-4\mu^2)}{s}\right\}. \quad (3.12d)$$

With the right-hand cut given by (3.4), and the left-hand cut approximation (3.10), the integral equation for the partial-wave amplitudes is

$$t_\lambda^J(s)=t_{\rho\lambda}^J+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{e^{-i\delta_J(s')}\sin\delta_J(s')\,t_\lambda^J(s')R(s')ds'}{s'-s}. \quad (3.13)$$

$R(s)$ is the inelasticity factor accounting for contributions of intermediate states of four or more pions; therefore, $R(s)=1$ for $s<16\mu^2$. Since the s channel is pure $I=0$, the four-pion state may be approximated by an equivalent 2ρ meson state which starts contributing at $s\geqslant 2m_\rho^2$, i.e., $R(s)\approx 1$ up to $s\approx 2m_\rho^2$. In calculating the decay width (where $4\mu^2\leqslant s\leqslant m^2$), we are interested in the low-energy behavior of the amplitude and therefore justified in taking $R(s)=1$ for the whole range of integration. Possible high-energy contributions to the integral (3.13) with $R(s)\neq 1$ will be taken care of by subtractions. The solution of the integral equation (3.13) is[18]

$$t_\lambda{}^J(s)=t_{\rho\lambda}{}^J(s)+e^{u_J(s)}\times\left[\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{e^{-u_J(s')}\sin\delta_J(s')\,t_{\rho\lambda}{}^J(s')ds'}{s'-s}+P(s)\right] \quad (3.14)$$

with

$$u_J(s)=\frac{s-4\mu^2}{\pi}\int_{4\mu^2}^{\infty}\frac{\delta_J(s')ds'}{(s'-4\mu^2)(s'-s)},$$

and $P(s)$ is a subtraction polynomial.

Inspection of Eqs. (3.10) reveals the asymptotic behavior of $t_{\rho\lambda}{}^J$ as $s\to\infty$:

$$t_{\rho2}{}^2\propto 1/s,\quad t_{\rho1}{}^2\propto 1/s^2,\quad t_{\rho0}{}^2\propto 1/s,\quad t_{\rho0}{}^0\propto C. \quad (3.15)$$

Therefore it is reasonable to assume that $t_\lambda{}^2$ do not require subtractions, whereas $t_0{}^0$ possibly does.

Using the approximation (3.12) for the inhomogeneous term, the equation (3.14) can be transformed[18] into

$$t_\lambda{}^J(s)=e^{u_J(s)}\left\{\sum_n e^{-u_J(s_n)}\frac{\mathrm{Res}[t_{\rho\lambda}{}^J(s)]_{s=s_n}}{s-s_n}+P(s)\right\}, \quad (3.16)$$

where s_n are the positions of the poles in $t_{\rho\lambda}{}^J$. Inserting (3.12) into (3.16), we get the amplitudes

$$t_2{}^2=-G\frac{2\sqrt{2}}{5\sqrt{3}}\frac{e^{u_2(s)-u_2(-x)}}{(s+x)}, \quad (3.17\text{a})$$

$$t_1{}^2=\frac{G4(m_\rho{}^2-\mu^2)}{5\sqrt{3}(s+x)}\left(\frac{1}{s+x}-\frac{d}{dx}\right)e^{u_2(s)-u_2(-x)}, \quad (3.17\text{b})$$

$$t_0{}^2=\frac{G2q^2e^{u_2(s)}}{15s}\left[\left(\left[2(s-m^2)^2\left(x+m^2+2\mu^2-\frac{3}{7}m^2\frac{s-4\mu^2}{s}\right)-m^2(x+m^2)^2\right]\frac{1}{(s+x)^3}-\frac{24m^6\mu^2}{7x^2s}+\frac{(x+m^2)^2}{(s+x)}\right.\right.$$
$$\times\left[x+\frac{m^2}{2}+2\mu^2-\frac{3}{7}m^2\left(1+\frac{4\mu^2}{x}\right)\right]\frac{d^2}{dx^2}+\frac{x+m^2}{s+x}\left\{\left[2-\frac{x+m^2}{s+x}\right]\left[x+m^2+2\mu^2-\frac{3}{7}m^2\left(1+\frac{4\mu^2}{x}\right)\right]\right.$$
$$\left.\left.\left.+m^2(x+m^2)\left(\frac{24\mu^2}{7x^2}+\frac{1}{s+x}\right)\right\}\frac{d}{dx}\right)e^{-u_2(-x)}+\frac{24m^6\mu^2}{7x^3s}e^{-u_2(0)}\right], \quad (3.17\text{c})$$

$$t_0{}^0=\frac{G}{12}e^{u_0(s)}\left\{\frac{1}{(s+x)}\left[x+m^2+(x+4\mu^2)\left(1-\frac{m^2}{x}\right)\right.\right.$$
$$\left.\left.+2(x+m^2)(x+4\mu^2)\left(\frac{1}{s+x}-\frac{d}{dx}\right)\right]e^{-u_0(-x)}+\frac{m^24\mu^2}{xs}e^{-u_0(0)}+P(s)\right\}. \quad (3.17\text{d})$$

The exact form of the subtraction polynomial $P(s)$ is determined by the behavior of the amplitude at high energies, and we shall examine various possibilities for it, including a value suggested by a Veneziano-type amplitude for this process.[13] In this context we are using also an alternative approach to handle high-energy contributions. In order to minimize their effect on the solution of the integral equation (3.14), one can change the high-energy behavior of $t_{\rho0}{}^0$ (which is only a low-energy approximation of the left-hand cut) by adding an effective pole far to the left. To this end, let us rewrite $t_{\rho0}{}^0$ in the form

$$t_{\rho0}{}^0=\frac{G}{12}\left\{\frac{1}{s+x}\left[2x+4\mu^2\left(1-\frac{m^2}{x}\right)\right]+2\frac{(x+m^2)(x+4\mu^2)}{(s+x)^2}+\frac{4\mu^2m^2}{xs}-4\right\} \quad (3.12\text{d}')$$

and change it into

$$t'_{\rho0}{}^0=\frac{G}{12}\left\{\frac{1}{s+x}\left[2x+4\mu^2\left(1-\frac{m^2}{x}\right)\right]+\frac{2(x+m^2)(x+4\mu^2)}{(s+x)^2}+\frac{4\mu^2m^2}{xs}-\frac{4}{1+s/\Lambda}\right\}, \quad (3.18)$$

[18] J. D. Jackson, in *Dispersion Relations*, edited by G. R. Screaton (Oliver and Boyd, London 1960) p. 54.

where Λ is the position of the effective pole. Inserting (3.18) into (3.16), the solution for $t_0{}^0$ is now

$$t_0{}^0=\frac{G}{12}e^{u_0(s)}\left\{\frac{1}{s+x}\left[2x+4\mu^2\left(1-\frac{m^2}{x}\right)\right.\right.$$
$$\left.+2(x+m^2)(x+4\mu^2)\left(\frac{1}{s+x}-\frac{d}{dx}\right)\right]e^{-u_0(-x)}$$
$$\left.+\frac{4\mu^2m^2}{xs}e^{-u_0(0)}-\frac{4\Lambda}{\Lambda+s}e^{-u_0(-\Lambda)}\right\}. \quad (3.19)$$

IV. PARAMETRIZATIONS OF π-π PHASE SHIFTS

In order to compute $t_\lambda{}^J$ we need as input in Eq. (3.17) information about π-π phase shifts. In face of the poor experimental information,[1] we shall use several alternative parametrizations for δ_J and subsequently analyze their effect on the decay characteristics.

Let us start with the less controversial D wave. Experimental analysis shows that δ_2 is small at low energies and passes through $\frac{1}{2}\pi$ at the f_0 resonance with $M_f=1264\pm10$ MeV, $\Gamma_f=150\pm25$ MeV. Hence we parametrize δ_2 in two ways: (a) $\delta_2=0$ and (b) a Breit-Wigner resonance form

$$f_2(s)=\frac{(\nu/\nu_f)^2\Gamma_fM_f}{s_f-s-i\theta(\nu)\rho(\nu)(\nu/\nu_f)^2\Gamma_fM_f}, \quad (4.1)$$

with

$$\nu=(s-4\mu^2)/4\mu^2, \quad (4.2)$$

$$\rho(\nu)=\left(\frac{\nu}{\nu+1}\right)^{1/2}. \quad (4.3)$$

and $\theta(\nu)$ is a step function starting at $\nu=0$. Accordingly,

$$e^{u_2(s)}=\frac{1}{s_f-s-i\theta(\nu)\rho(\nu)(\nu/\nu_f)^2\Gamma_fM_f}. \quad (4.4)$$

For the S waves let us introduce first the notation of Chew and Mandelstam[19]:

$$f_0(s)=N_0(s)/D_0(s) \quad (4.5)$$

and δ_0 is given by

$$\left(\frac{\nu}{\nu+1}\right)^{1/2}\cot\delta_0=\frac{\mathrm{Re}D_0}{N_0}. \quad (4.6)$$

N_0 and D_0 obey the integral equations

$$N_0=a_0+\frac{\nu}{\pi}\int_{-\infty}^{-1}\frac{\mathrm{Im}f_0(\nu')D_0(\nu')}{\nu'(\nu'-\nu)}d\nu', \quad (4.7)$$

$$D_0=1-\frac{\nu}{\pi}\int_0^\infty\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{N_0(\nu')}{\nu'(\nu'-\nu)}d\nu'=e^{-u_0(s)}. \quad (4.8)$$

We present now a few approximate solutions of Eqs. (4.7) and (4.8):

a. Scattering-length approximation,[19] where N_0 is taken to be constant. We then denote the approximate N_0 and D_0 functions with subscript a,

$$N_a=a_0. \quad (4.9)$$

Insertion of (4.9) into (4.8) gives for D_0

$$D_a=1+a_0[H(\nu)-i\theta(\nu)\rho(\nu)], \quad (4.10)$$

where $H(\nu)$ is defined by

$$H(\nu)=(2/\pi)\rho(\nu)\ln[\nu^{1/2}+(\nu+1)^{1/2}]. \quad (4.11)$$

Insertion of D_a into the solution of $t_0{}^0$ (3.17d) shows that as $s\to\infty$ and $P(s)=0$, $t_0{}^0\to C/s\ln s$, while for $P(s)=b$, $t_0{}^0\to b/\ln s$.

b. Resonance approximation. Evidently (4.9), which contains only one parameter, is reliable only at low energies. A better representation of $f_0(s)$ will contain more parameters. Here this will be achieved in three ways: (I) a twice-subtracted integral equation for D_0, (II) N_0 represented by a two-parameter function, and (III) a combination of (I) and (II). For each case we denote the N and D functions with the appropriate subscript.

(I) D_0 is written in the form[20]

$$D_0=1-\frac{\nu}{\nu_0}-\frac{\nu^2}{\pi}\int_0^\infty\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{N_0(\nu')}{\nu'^2(\nu'-\nu)}d\nu', \quad (4.12)$$

where now the influence of N_0 on D_0 at high energies is reduced, and it will be sufficient to approximate it by N_a:

$$D_{(\mathrm{I})}=1-(\nu/\nu_R)[1+a_0H(\nu_R)]+a_0[H(\nu)-i\theta(\nu)\rho(\nu)],$$
$$\nu_0\equiv\frac{\nu_R}{1+a_0H(\nu_R)}. \quad (4.13)$$

For appropriate values of a_0 and ν_0, $D_{(\mathrm{I})}$ will become a resonating amplitude at ν_R with a width Γ_R given by

$$\frac{\Gamma_R}{\mu}=2\nu_R{}^{3/2}\left[\frac{\nu_R+1}{a_0}-\frac{\nu_R}{\pi}+\frac{H(\nu_R)}{2}(2\nu_R+1)\right]^{-1}. \quad (4.14)$$

Inserting $D_{(\mathrm{I})}$ into (3.17d), we get the high-energy behavior of $t_0{}^0$. For $P(s)=0$, $t_0{}^0\sim C/s^2$, and for $P(s)=b$, $t_0{}^0\sim b/s$, as $s\to\infty$.

(II) Alternatively, one can feed the additional information into N_0. This will be done by approximating the left-hand cut by an effective pole; hence,

$$N_{(\mathrm{II})}=a_0z/(\nu+z),\quad z=(\Lambda+4\mu^2)/4\mu^2. \quad (4.15)$$

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[19] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[20] V. V. Serebryakov and D. V. Shirkov, Zh. Eksperim. i Teor. Fiz. **42**, 610 (1962) [Soviet Phys. JETP **15**, 425 (1962)].

The resulting D_0 obtained from (4.8) is

$$D_{(\mathrm{II})}=D_{(\mathrm{I})}/(\nu+z)\,, \tag{4.16}$$

and when the amplitude describes a resonance, its position is given by

$$\frac{\nu_R}{1+a_0H(\nu_R)}=-\frac{z}{1+a_0H(-z-1)}\,. \tag{4.17}$$

The solution of $t_0{}^0$ for $D_{(\mathrm{II})}$ is the same as for $D_{(\mathrm{I})}$, except for a change in the subtraction polynomial to $P'(s)$:

$$P'(s)=P(s)+\frac{1}{\Lambda-x}\left[x+m^2+(x+4\mu^2)\left(1-\frac{m^2}{x}\right)-2(x+m^2)(x+4\mu^2)\left(\frac{1}{\Lambda-x}+\frac{d}{dx}\right)\right]D_{(\mathrm{I})}(-x)+\frac{m^24\mu^2}{\Lambda x}D_{(\mathrm{I})}(0)\,. \tag{4.18}$$

(III) The preceding parametrizations allow δ_0 to vary at most by π. We shall now present a parametrization which allows δ_0 to vary by more than π. To this end, we take N_0 to be

$$N_{(\mathrm{III})}=a_0+b\nu\,, \tag{4.19}$$

and solve for D_0 by the twice-subtracted equation (4.12):

$$D_{(\mathrm{III})}=1-\nu/\nu_0+(a_0+b\nu)[H(\nu)-i\theta(\nu)\rho(\nu)]\,. \tag{4.20}$$

For suitable values of a_0, b, and ν_0, the form $D_{(\mathrm{III})}$ may also represent a resonance at the position ν_R, defined by the relation

$$\nu_0=\frac{\nu_R}{1+(a_0+b\nu_R)H(\nu_R)} \tag{4.21}$$

and width

$$\frac{\Gamma_R}{\mu}=2\nu_R{}^{3/2}\left[(\nu_R+1)\frac{1+a_0H(\nu_R)}{a_0+b\nu_R}-\frac{\nu_R}{\pi}-\frac{H(\nu_R)}{2}\right]^{-1}\,. \tag{4.22}$$

In terms of the three parameters the phase shift is given by

$$\left(\frac{\nu}{\nu+1}\right)^{1/2}\cot\delta_0=\frac{1-\nu/\nu_0}{a_0+b\nu}+H(\nu)\,. \tag{4.23}$$

The expression for δ_0 clearly displays the above-mentioned range of variation.

The high-energy behavior of $t_0{}^0$ based on $D_{(\mathrm{III})}$ is $t_0{}^0\sim C/s^2\ln s$ for $P(s)=0$ and $t_0{}^0\sim b/s\ln s$ for $P(s)=b$, as $s\to\infty$.

V. DETERMINATION OF HIGH-ENERGY CONTRIBUTION FROM VENEZIANO MODEL

In a previous article,[13] we developed a Veneziano-type amplitude for $V_1+\pi\to V_2+\pi$ scattering, which correctly describes both high- and low-energy behavior. As such, this amplitude can serve us in normalizing the S-wave amplitude obtained in Sec. III whose high-energy behavior is yet undetermined, as is evident from (3.17d). Although exhibiting several features of the "true amplitude," such as the presence of the lowest resonances in all channels (unaccompanied by daughters), it suffers from lack of unitarity.

Since the amplitude obtained from the dispersion-relations approach is unitary, we present here a method for matching the two. In order to implement this, we impose unitarity on the Veneziano amplitude in a restricted manner,[21] namely, at one point only. This is chosen to be[22] $\mathrm{Re}\alpha_s=2$, at which the Veneziano amplitude is fitted to represent the f_0 correctly. The resonance width fixes $\mathrm{Im}\alpha$ at this point, and imparts to the S partial wave both a real and an imaginary part, which will be used to normalize the solution (3.17d) of the dispersion integral.

One might object to this procedure since, on the one hand, the integral equation contains for its kernel the elastic π-π phase shift while, on the other hand, we normalize its solution at an energy of 1264 MeV, where the low-energy solution is not expected to hold. A partial answer to this objection was already given in Sec. III, where we pointed out that inelastic effects are small up to $s\approx 2m_\rho{}^2$ and will be included in the subtraction polynomial. Moreover, when adding an imaginary part to α at the resonance position, the Veneziano-type amplitude describes a two-pion resonating state, and thus the two amplitudes display the same physical content, the matching procedure thus being justified.

Let us now extract the S wave from the Veneziano-type amplitude. From (2.6d), and (5.16) in Ref. 13, we obtain

$$T_{1,1}=B-2(q\sin\theta)^2E=\left\{(k^2s+2q^2k\cdot p)+\frac{2k\cdot p(1-\alpha_t)(1-\alpha_u)}{(2-\alpha_t-\alpha_u)(4-\alpha_t-\alpha_u)}\left[\frac{(2-\alpha_s)}{\epsilon'}-\frac{4yq^2}{3(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\right]-\frac{(t-u)^2}{4}\left[1-\frac{k\cdot p}{2k^2}+\frac{2\epsilon's}{2-\alpha_t-\alpha_u}\left(1+\frac{z(1-\alpha_t)(1-\alpha_u)}{(3-\alpha_s-\alpha_t)(3-\alpha_s-\alpha_u)}\right)\right]\right\}F\,, \tag{5.1}$$

[21] It is known that "straightforward" unitarization by adding an imaginary part to the trajectory functions destroys some of the nice original features of the model.

[22] We use the notation defined in Ref. 13.

where ϵ' is the slope of the ρ and f_0 trajectories, taken to be degenerate,[23] and F is given by

$$F=-\tfrac{1}{8}G\epsilon'\{[B(1-\alpha_t,2-\alpha_s)+B(1-\alpha_u,2-\alpha_s)+B(1-\alpha_t,1-\alpha_u)]$$
$$+\tfrac{1}{2}(3-\epsilon)[B(2-\alpha_t,3-\alpha_s)+B(2-\alpha_u,3-\alpha_s)+B(2-\alpha_t,2-\alpha_u)]$$
$$+(3-\epsilon)[B(3-\alpha_t,4-\alpha_s)+B(3-\alpha_u,4-\alpha_s)+B(3-\alpha_t,3-\alpha_u)]$$
$$+\tfrac{1}{12}(3-\epsilon)[24-(4-\epsilon)(5-\epsilon)][B(4-\alpha_t,5-\alpha_s)+B(4-\alpha_u,5-\alpha_s)+B(4-\alpha_t,4-\alpha_u)]+\cdots\}, \quad (5.2)$$

with $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $\epsilon=\alpha_s+\alpha_t+\alpha_u$. For the mass configuration in the process $\omega\to\pi\pi\gamma$ and the value of the trajectory slope[24]

$$\epsilon'=1/2(m_\rho^2-\mu^2), \quad (5.3)$$

we get

$$\epsilon=2. \quad (5.4)$$

At the point $\mathrm{Re}\alpha_s=2$, at which $s=M_f^2=2.6m^2$ and $\nu=\nu_f=19.25$, we obtain

$$\mathrm{Im}\alpha_s\equiv\beta=\Gamma_f M_f\epsilon'\approx\tfrac{1}{6}, \quad (5.5)$$

$$1-\alpha_t=\epsilon'\frac{s+x}{2}\left(1-\frac{s-m^2}{s+x}\cos\theta\right)\equiv b(1-a\cos\theta), \quad (5.6a)$$

$$1-\alpha_u=\epsilon'\frac{s+x}{2}\left(1+\frac{s-m^2}{s+x}\cos\theta\right)\equiv b(1+a\cos\theta), \quad (5.6b)$$

with

$$b=1 \quad (5.7a)$$

and

$$a=0.45. \quad (5.7b)$$

We use the above values in (5.1) and expand it, retaining terms up to first order in β and third order in a, to obtain

$$T_{1,1}=-\left\{(2sk^2-\tfrac{4}{3}q^2k\cdot p)P_2(\cos\theta)+\frac{k\cdot p}{4\epsilon'}(1-\alpha_t)(1-\alpha_u)i\beta+\left[\left(1-\frac{i\beta}{1-\alpha_t}\right)^{-1}\left(1-\frac{i\beta}{1-\alpha_u}\right)^{-1}-1\right]\right.$$
$$\left.\times\left[\frac{8}{3}k\cdot pq^2+\frac{(t-u)^2}{4}\left(\frac{3s}{4q^2}-1-\epsilon' s\right)\right]\right\}F$$
$$=-k\cdot pm^2\left\{\frac{7}{9}P_2(\cos\theta)+i\beta\left[\frac{7}{15}(1-\alpha_t)(1-\alpha_u)+4\left(\frac{4}{9}+\frac{\cos^2\theta}{7}\right)\left(\frac{1}{1-\alpha_t}+\frac{1}{1-\alpha_u}\right)\right]\right.$$
$$\left.+(i\beta)^2 2\left(\frac{4}{9}+\frac{\cos^2\theta}{7}\right)\left[\frac{1}{1-\alpha_t}+\frac{1}{1-\alpha_u}+2\left(\frac{1}{(1-\alpha_t)^2}+\frac{1}{(1-\alpha_u)^2}\right)\right]\right\}F. \quad (5.8)$$

In the expansion of F, we use

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\left(\frac{1}{x}+\frac{1}{y}\right)\prod_{n=1}^{\infty}\frac{1+(x+y)/n}{(1+x/n)(1+y/n)},$$

and get

$$F=-\frac{G\epsilon'}{8}\left\{-\frac{2}{i\beta}-3+\frac{5}{3}\left(\frac{1}{1-\alpha_t}+\frac{1}{1-\alpha_u}\right)+\frac{5}{4}\left(\frac{1}{2-\alpha_t}+\frac{1}{2-\alpha_u}\right)+i\beta\left[2\left(\frac{1}{2-\alpha_t}+\frac{1}{2-\alpha_u}\right)+\frac{3}{2}\left(\frac{1}{3-\alpha_t}+\frac{1}{3-\alpha_u}\right)\right.\right.$$
$$\left.\left.+\frac{(1-\alpha_t)^2}{2(2-\alpha_t)(3-\alpha_t)}+\frac{(1-\alpha_u)^2}{2(2-\alpha_u)(3-\alpha_u)}-\frac{5}{2}+\sum_{n=3}^{\infty}\left(\frac{1}{(n-1)^2}-\frac{2}{n^2}\right)\right]\right\}. \quad (5.9)$$

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[23] The degeneracy assumption implies that all the numbers related to trajectories parameters are accurate within $\approx 5\%$.
[24] C. Lovelace, Phys. Letters **28B**, 264 (1968).

TABLE I. Value of $10^3 S_0^0$ [a] in the scattering-length approximation.[b]

P \ a_0 (μ^{-1})	-1	-0.5	$1/12.5$ [c]	0.5	1	1.5	2
-7.5	40.3	25.3		0.94	1.9	4.6	7.5
-5	16.9	6.6	0.68	2.5	6.2	10.0	13.0
-4	10.2	2.6		4.7	9.0	12.8	15.8
-3	5.3	0.54		7.6	12.3	15.9	18.7
-2	2.0	0.46		11.4	16.1	19.4	22.0
-1	0.26	2.3		16.1	20.4	23.3	25.5
0	0.15	6.1		21.6	25.3	27.6	29.3
1	1.6	11.9		27.9	30.7	32.3	33.3
2	4.7	19.6		35.1	36.6	37.3	37.6
3	9.4	29.3		43.1	43.2	42.7	42.2
4	15.7	40.9		52.0	50.2	48.5	47.1
5	24.0	55.2		61.5	57.8	54.5	52.2
7.5	50.7	97.6		90.0	79.4	71.9	66.5

[a] S_0^0 is related to the decay rate through Eq. (6.5) and it expresses the contribution to the decay rate of the S-wave amplitude t_0^0 [Eq. (3.17d)].
[b] In this approximation, the π-π scattering amplitude is defined in Eqs. (4.9)–(4.11).
[c] The particular values $a_0=(1/12.5)\mu^{-1}$ and $P=-5$ relate to the model described in Sec. V.

From (5.8) and (5.9) we derive the value of the S-wave amplitude at the matching point

$$t_0{}^0(s_f)=-\tfrac{1}{8}G(15/14)(2.32+i\beta)$$
$$=-\tfrac{1}{12}3.72G(1+i/13.9). \quad (5.10)$$

From (5.10) it is seen that $\cot\delta_0=13.9$, which gives $\delta_0=4°$ or $184°$. Among the parametrizations we have considered, only two can attain the above-mentioned values of magnitude and phase. These are (a) the scattering-length approximation (4.10), which gives a slow variation of δ_0, and (b) the resonance amplitude with three parameters (4.20), which allows δ_0 to rise beyond π.

Equating (5.10) with the solution of the integral equation (3.17d), one has

$$e^{u_0(s_f)}\left\{\frac{1}{s_f+x}\left[x+m^2+(x+4\mu^2)\left(1-\frac{m^2}{x}\right)\right.\right.$$
$$\left.+2(x+m^2)(x+4\mu^2)\left(\frac{1}{s_f+x}-\frac{d}{dx}\right)\right]e^{-u_0(-x)}$$
$$\left.+\frac{4\mu^2m^2}{s_fx}e^{-u_0(0)}+P(s_f)\right\}=\frac{-3.72}{1-i/13.9}. \quad (5.11)$$

For the scattering-length approximation, we obtain

$$a_0=(1/12.5)\mu^{-1}, \quad (5.12a)$$
$$P=-5, \quad (5.12b)$$

and for the resonance approximation

$$b=0.62(1-16.45/\nu_0), \quad (5.13a)$$
$$a_0=-\nu_f b+(1/12.5)(1-\nu_f)/\nu_0, \quad (5.13b)$$

where there is still a free parameter left.

The order of growth of δ_0 requires the following inequality:

$$\nu_R \leqslant \bar{\nu} \equiv -a_0/b \leqslant \nu_f, \quad (5.14)$$

where ν_R is the resonance position, and $\bar{\nu}$ is the point where δ_0 crosses through π. Considering (5.13), (4.21), and (5.14), we are able to restrict ν_R to the regions

$$0 \leqslant \nu_R \leqslant 2, \quad (5.15)$$
$$16.32 \leqslant \nu_R \leqslant \nu_f. \quad (5.16)$$

In the first region (5.15) Γ, turns out to be negative, and we therefore discard this alternative.

VI. NUMERICAL RESULTS

The partial-decay width for the process we consider, expressed in the rest frame of the decaying particle, is given by

$$\Gamma=\frac{(2\pi)^4}{2m}\int\frac{\sum_{\lambda_1,M,M'}T_{\lambda_1,M'}\rho_{M'M}T_{\lambda_1,M}{}^\dagger}{8k_0q_{10}q_{20}}$$
$$\times\frac{d^3kd^3q_1d^3q_2}{(2\pi)^8}\delta(m-k-q_{10}-q_{20})\delta^3(\mathbf{k}+\mathbf{q}_1+\mathbf{q}_2), \quad (6.1)$$

where $\rho_{MM'}$ is the density matrix for ω production quantized along the z axis. $T_{\lambda_1,M}$ is the transition matrix from an ω with $J_z=M$ to a photon with helicity λ_1 and the rest of the quantities in (6.1), were defined in Sec. II. We evaluate the integral in (6.1) in the pions' center-of-mass system; thereby $T_{\lambda_1,M}$ transform into T_{λ_1,λ_2} by

$$T_{\lambda_1,M}=D_{M\lambda_1}{}^{s_\omega *}(\phi,\delta,0)T_{\lambda_1,\lambda_2}(s,t), \quad (6.2)$$

where ϕ and δ are the direction angles of $\mathbf{k}$ in the ω rest frame and T_{λ_1,λ_2} depend on s and t only. Integration over the rest of the variables is straightforward and we obtain

$$\Gamma=\frac{1}{3(2\pi)^3 2^5 m^3}\sum_{\lambda_1,\lambda_2}\int|T_{\lambda_1,\lambda_2}{}^s|^2 ds dt. \quad (6.3)$$

Expansion (3.2) enables us to perform the integration over t, giving

$$\Gamma=\frac{1}{3(2\pi)^3 2^4 m^3}\sum_{\lambda_1,\lambda_2,J}\int|T_{\lambda_1,\lambda_2}{}^J(s)|^2(2J+1)$$
$$\times\tfrac{1}{2}(s-m^2)\left(\frac{s-4\mu^2}{s}\right)^{1/2}ds. \quad (6.4)$$

TABLE II. Value of $10^4 S_0^0$ [a] in the effective-pole approach.[b]

Λ \ a_0 (μ^{-1})	-1	-0.5	0.5	1	1.5	2
$2x$	3.8	1.8	30.9	43.9	53.7	61.1
$4x$	1.4	2.2	22.1	30.7	37.3	42.3
$6x$	0.41	2.6	18.8	25.4	30.4	34.2
$8x$	0.06	3.0	16.9	22.3	26.4	29.4
$10x$	0.05	3.4	15.8	20.3	23.7	26.2
$12x$	0.24	3.7	14.9	18.8	21.7	23.8
$14x$	0.55	4.1	14.2	17.7	20.2	22.0
$16x$	0.93	4.2	13.7	16.8	18.9	20.6

[a] S_0^0 is related to the decay rate through Eq. (6.5).
[b] The partial-wave amplitude for decay, t_0^0, is given for this approach in Eq. (3.19), while the π-π scattering amplitude is defined in Eqs. (4.9)–(4.11).

TABLE III. Value of $10^2 S_0{}^0$ [a] in the two-parameter resonance approximation.[b]

Γ/μ / ν_R	0.6	1.2	1.8	2.4
0.5	66.1	40.4	31.5	26.9
1	54.9	33.5	26.4	22.7
1.5	43.2	27.7	21.7	18.5
2	33.6	21.9	18.0	16.0
4	12.3	10.3	9.7	9.4
5	8.0	7.6	7.6	7.7
5.5	6.6	6.7	6.9	7.1
6.5	5.1	5.5	5.8	6.1
8		4.7		5.4
10		3.8		4.5
12		3.3		3.9
14		3.0		3.6
16		2.8		3.3

[a] $S_0{}^0$ is related to the decay rate through Eq. (6.5).
[b] In this approximation, the π-π scattering amplitude is defined in Eqs. (4.13)–(4.14).

Inserting $T_{\lambda_1,\lambda_2}{}^J$ from (3.7), we rewrite (6.4) as

$$\Gamma=\frac{f^2m^5}{3\times2^7\pi^3\mu^4}\sum_{J,\lambda}\int_{4\mu^2}^{m}2I_\lambda{}^J ds$$
$$\equiv\eta\sum_{J,\lambda}S_\lambda{}^J\equiv2\eta\int_{4\mu^2}^{m}I(s)ds\,,\qquad(6.5)$$

where $f^2\equiv\mu^4G^2\equiv g_{\omega\rho\pi}{}^2g_{\rho\pi\gamma}{}^2$, $\lambda=|\lambda_1-\lambda_2|$, $I_\lambda{}^J$ is the decay spectrum for partial wave J and helicity λ, and for further reference we defined

$$S_\lambda{}^J=2\int_{4\mu^2}^{m^2}I_\lambda{}^J(s)ds\,,\quad I(s)=\sum_{J,\lambda}I_\lambda{}^J(s)\,.\qquad(6.6)$$

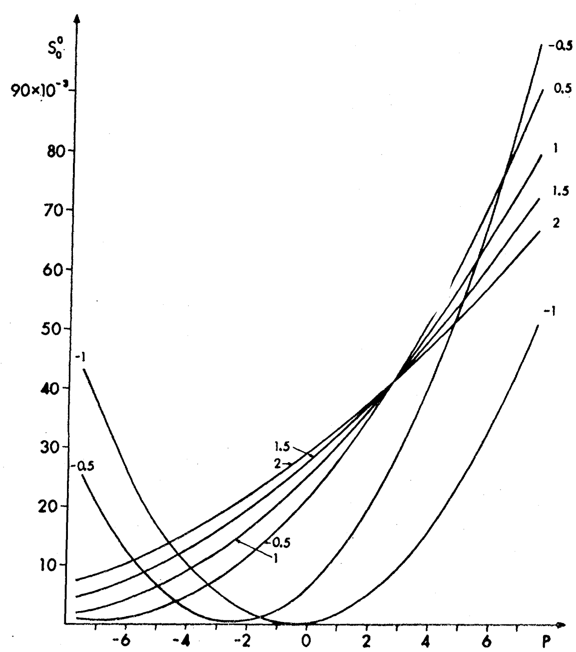

FIG. 3. $S_0{}^0$ in the scattering-length approximation, for scattering lengths a_0 (in units of μ^{-1}) $=-1$, -0.5, 0.5, 1, 1.5, 2 and subtraction parameter $-7.5\leqslant P\leqslant7.5$ (see also Table I).

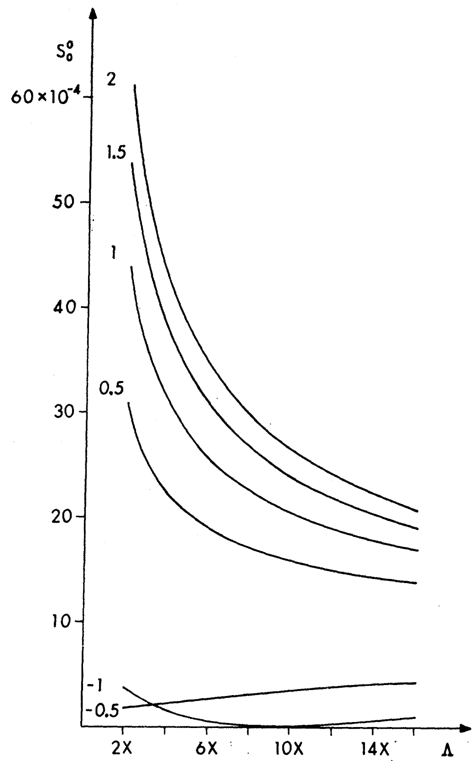

FIG. 4. $S_0{}^0$ in the effective-pole approach, for scattering lengths a_0 (in units of μ^{-1}) $=-1$, -0.5, 0.5, 1, 1.5, 2 and pole location $2x\leqslant\Lambda\leqslant16x$ (see also Table II).

We consider first the value to be used for f^2. With regard to $g_{\omega\rho\pi}$, its value is determined by assuming $\omega\to3\pi$ decay proceeds via $\omega\to(\rho\pi)\to3\pi$, which gives[25] $g_{\omega\rho\pi}{}^2/4\pi=0.6$. The decay $\rho\to\pi\gamma$ has not been measured yet (only upper limits are available); therefore we shall rely on indirect estimates for $g_{\rho\pi\gamma}$. This coupling can be related to $g_{\omega\pi\gamma}$ (evaluated from the

TABLE IV. Value of $10^2S_0{}^0$ [a] in the three-parameter resonance approximation.[b] This parametrization, with $\Gamma_R=\mu$, $M_R=750$ MeV ($\nu_R=6$), and $\delta_{\pi\pi}{}^0(\epsilon=420\text{ MeV})=25°$, corresponds to solution b of Marateck *et al.*[c] In the first three columns we give results for Γ_R, M_R varying, and the last parameter fixed as above. The last four columns correspond to values allowed by the Veneziano-type model, Eqs. (5.13) and (5.16).

Γ/μ / ν_R	0.5	1	Γ/μ / ν_R	0.10	0.31	0.41
6	3.40	3.82	16.5	2.55		
7	3.04	3.38	17		3.05	
8	2.83	3.11	17.5			3.55
10	2.61	2.81	18			1.96
12	2.50	2.65	18.5		0.78	
14	2.44	2.56				
16	2.40	2.50				
18	2.38	2.46				

[a] $S_0{}^0$ is related to the decay rate through Eq. (6.5).
[b] In this approximation, the π-π scattering amplitude is defined in Eqs. (4.20)–(4.23).
[c] S. Marateck *et al.*, Phys. Rev. Letters **21**, 1613 (1968).

[25] F. Berends and P. Singer, Phys. Letters **19**, 249 (1965); **19**, 616(E), (1965).

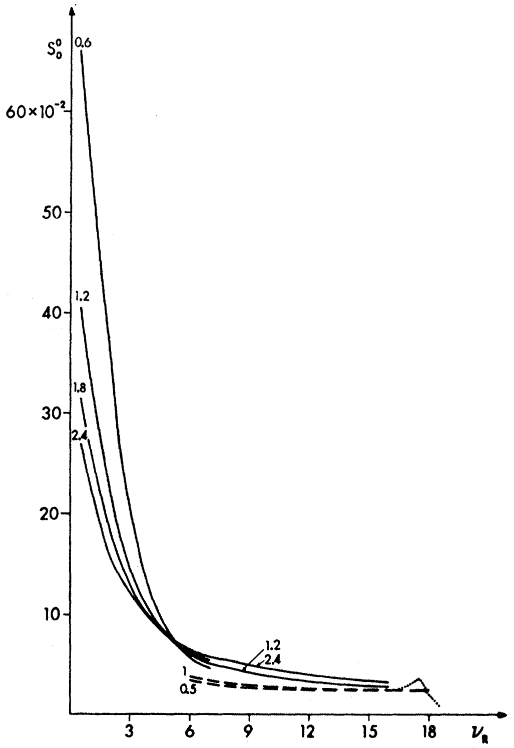

FIG. 5. S_0^0 in the resonance approximation. The solid curves are for the two-parameter case, with widths $\Gamma/\mu=0.6$, 1.2, 1.8, 2.4 and resonance location $0.5\leqslant\nu_R\leqslant16$ (see also Table III); the dashed lines are for the three-parameter case, with $\Gamma/\mu=0.5$, 1 and $6\leqslant\nu_R\leqslant18$ (see also Table IV, first three columns); the dotted line gives the values obtained with the Veneziano-type model (see also Table IV, last four columns).

decay $\omega\rightarrow\pi\gamma$ to be $g_{\omega\pi\gamma}^2/4\pi=0.17\alpha$, where α is the fine-structure constant) by (1) $SU(3)$ symmetry and ω-ϕ mixing by an angle θ giving $g_{\rho\pi\gamma}=g_{\omega\pi\gamma}(\sin\theta)/\sqrt{3})$, and (2) vector dominance which gives $g_{\rho\pi\gamma}=g_{\omega\pi\gamma}(g_\rho/g_\omega)$, where g_ρ, g_ω are determined from the appropriate decays to lepton pairs. The available experimental information[26] on θ, g_ρ, and g_ω allows the range

$$0.7g_{\omega\pi\gamma}/3\leqslant g_{\rho\pi\gamma}\leqslant 1.3g_{\omega\pi\gamma}/3\,.$$

In the following we shall then use $g_{\rho\pi\gamma}=\frac{1}{3}g_{\omega\pi\gamma}$ [the $SU(6)$ value]; the results will be trivially readjustable when a firm experimental value is at hand.

We are now in the position to calculate η in (6.5), obtaining

$$\eta=0.89\text{ MeV}. \qquad (6.7)$$

From here on, results are given for the charged partial decay $\omega\rightarrow\pi^+\pi^-\gamma$, the neutral mode then being given by $\Gamma(\omega\rightarrow2\pi^0\gamma)=\frac{1}{2}\Gamma(\omega\rightarrow\pi^+\pi^-\gamma)$.

With no π-π interaction present, the decay rate is given by the Born term. Calculating this rate from

[26] S. C. C. Ting, in *Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 43.

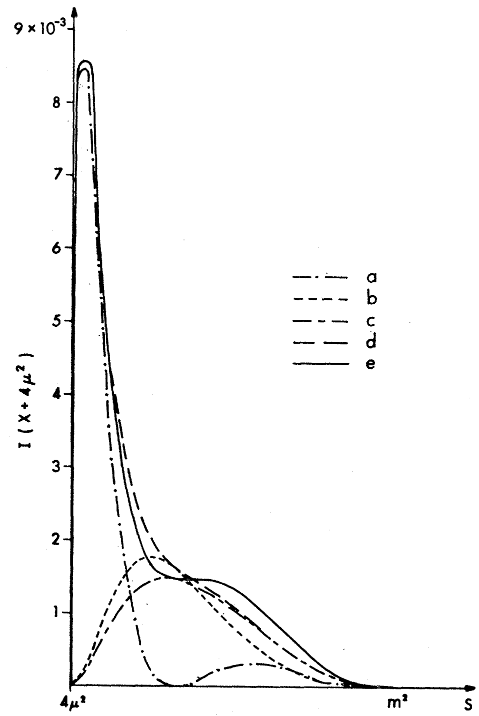

FIG. 6. Decay spectrum in the Born approximation. (a) The ρ contribution to the $J=0$ wave; (b) the ρ contribution to the $J=2$ wave; (c) the ρ and f^0 contribution to the $J=2$ wave; (d) the ρ contribution to the $J=0$, 2 waves [(a)+(b)]; (e) the ρ and f contribution to the $J=0$, 2 waves [(a)+(c)].

(6.3) with the expressions (3.6) and (3.9), one gets

$$\Gamma_B=\eta\times16.9\times10^{-4}=15.0\times10^{-4}\text{ MeV}. \qquad (6.8)$$

In all subsequent calculations we use the amplitudes (3.17), which were derived with the approximated Born amplitudes (3.12) in the integral equation. The approximation consists of retaining only the first term in the expansion of Q_J (Eq. 3.11) and neglecting partial waves higher than $J=2$. Calculating Γ_B with the approximated amplitudes (3.12), we get

$$\Gamma_{B'}=\eta(S_0^0+\sum_{\lambda=0}^{2}S_\lambda^2)=\eta(9.16+6.22)\times10^{-4}$$
$$=13.6\times10^{-4}\text{ MeV}. \qquad (6.9)$$

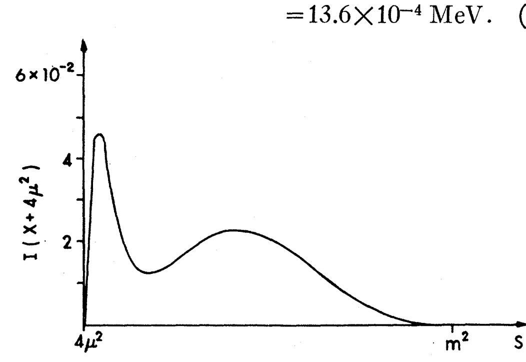

FIG. 7. Decay spectrum for the Veneziano-type model [corresponding to $P=-5$ and $a_0(\mu^{-1})=1/12.5$].

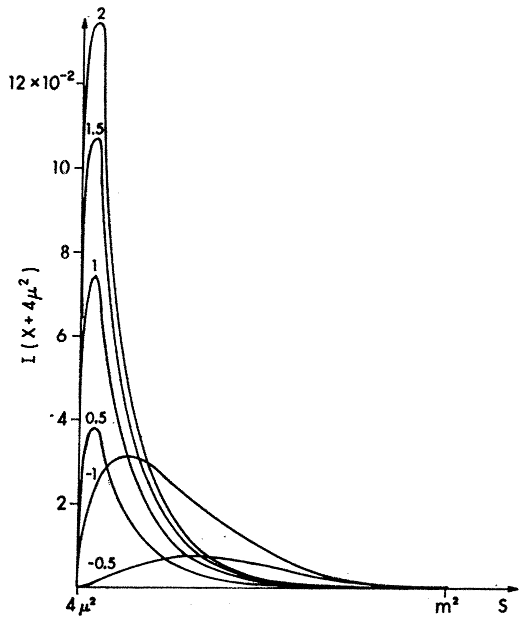

FIG. 8. Decay spectrum for the scattering-length approximation, with $a_0(\mu^{-1})=-1$, -0.5, 0.5, 1, 1.5, 2 and subtraction constant $P=-4$.

Comparison of Γ_B and $\Gamma_{B'}$ shows that the approximation discussed above is accurate within $\sim 10\%$.

The inclusion of the f resonance [Eq. (4.4)] in the expression for t_λ^2, (3.17a)–(3.17c), causes merely a slight change in $\sum_\lambda S_\lambda^2$, the new value being 6.3×10^{-4}.

In Tables I–IV and Figs. 3–5 we give the values of S_0^0 [which is related to the decay rate by (6.5) and (6.7)] for the various parametrizations of the π-π S wave discussed in Sec. IV. The appropriate decay rate is then given by $\Gamma=\eta(S_0^0+6.3\times 10^{-4})$. Figures 6–13 show the decay spectrum for several of the above-mentioned parametrizations. For the resonance approximations we confined ourselves to the value $P=0$ for the subtraction constant, as presumably a two- or three-parameter representation suffices to describe the amplitude in the decay region.

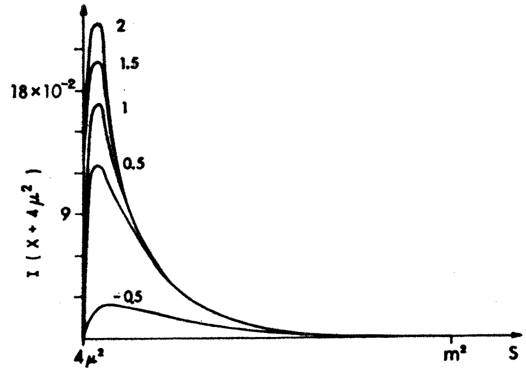

FIG. 9. Decay spectrum for the scattering-length approximation, with $a_0(\mu^{-1})=-1$, -0.5, 0.5, 1, 1.5, 2 and subtraction constant $P=0$.

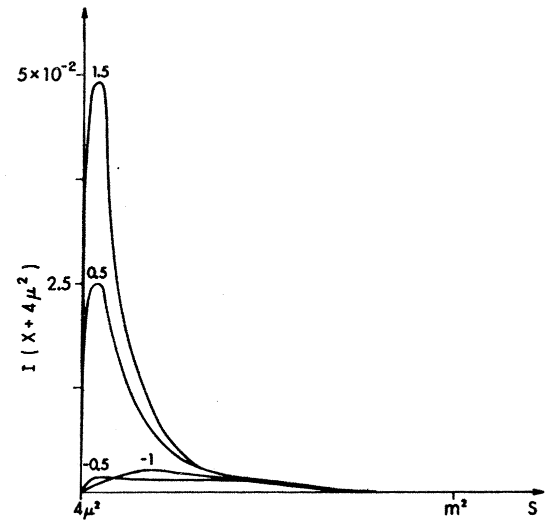

FIG. 10. Decay spectrum for the effective-pole approach, with $a_0(\mu^{-1})=-1$, -0.5, 0.5, 1.5 and pole location $\Lambda=2x$.

VII. DISCUSSION

In our analysis we have allowed for a wide spectrum of alternatives for the π-π interaction. As a result, we have obtained a wide range of predictions for the decay rate. Nevertheless, we can generally classify our results into three main groups. (a) If the decay rate is of the same order of magnitude as the one calculated from the Born term, the parametrization able to account for it is the scattering-length approximation with either $-1\leqslant\mu a_0\leqslant -0.5$ and $P\approx 0$, or with the effective-pole approach for $-0.5\leqslant\mu a_0\leqslant 2$, or normalized to the Veneziano-type amplitude with $\mu a_0=1/12.5$ and $P=-5$. (b) An enhancement of the order of 15–50, i.e., $2\times 10^{-2}\leqslant\Gamma\leqslant 7\times 10^{-2}$ MeV, results from the scattering-length approximation with $P=0$, $0.5\leqslant\mu a_0\leqslant 2$, as well as from the resonance model with 730 MeV$<M_R<$1220

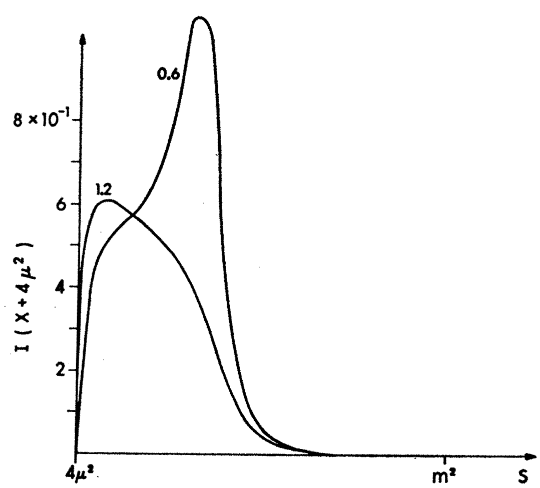

FIG. 11. Decay spectrum for the resonance approximation (two parameters), with $\nu_R=2.5$ ($M_R=525$ MeV), $\Gamma/\mu=0.6$, 1.2.

MeV. (c) A rate of the order of 0.1 MeV or higher can be accounted for by a resonance model with a low-lying resonance 350 MeV$\leqslant M\leqslant$650 MeV.

As we stressed in the Introduction and as also emerges from the numerical results, our ignorance concerning the high-energy behavior of the S-wave amplitude prevents us from unambiguously relating a particular decay-rate value to a specific form of the low-energy π-π interaction. In the scattering-length approximation, for the same value of a_0, we get very different results for the decay rate when using the two approaches for treating the high-energy contribution, namely, the subtraction procedure and the effective pole. This holds even when the two procedures describe the same asymptotic behavior of the amplitude, namely, for $P=0$. This can be seen by comparing Tables I and II and the respective figures. Although the results are fairly stable for P around zero and positive scattering length, one sees from Table I and Fig. 3 that the decay rate varies sharply with P for constant a_0 when $|P|\gtrsim 2$. This fact is not disturbing, large values of P being unphysical since they overshadow the low-energy behavior of the π-π amplitude. The value $P=0$ gives also what is usually considered the most desirable high-energy behavior. The effective-pole approach summarized in Table II and Fig. 4 is only slightly dependent on the location of the second pole, for $\Lambda\gtrsim 5x$.

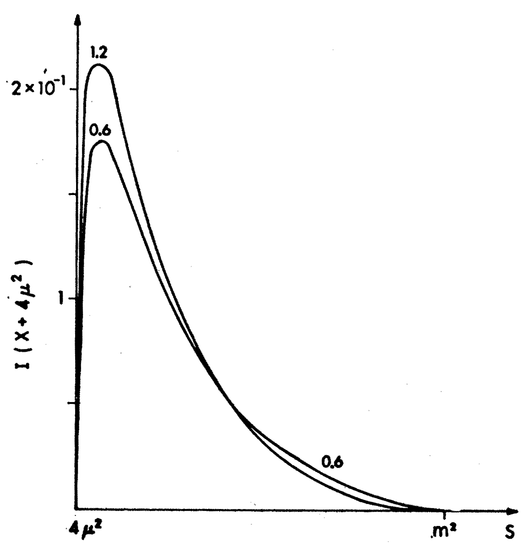

FIG. 12. Decay spectrum for the resonance approximation (two parameters), with $\nu_R=6.5$ ($M_R=765$ MeV), $\Gamma/\mu=0.6$, 1.2.

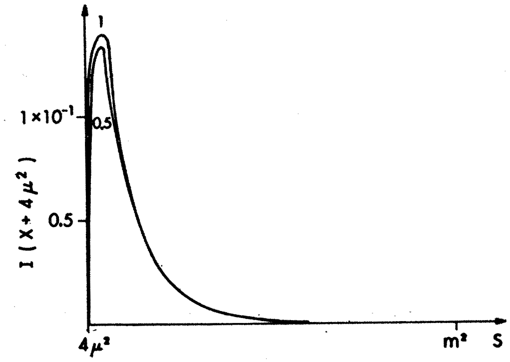

FIG. 13. Decay spectrum for the resonance approximation (three parameters), with $\nu_R=18$ ($M_R=1220$ MeV) and $\Gamma/\mu=0.5$, 1. The third parameter is fixed from experiment to give $\delta_{\pi\pi}{}^0(E=420$ MeV$)=25°$ (see also caption for Table IV).

The strong dependence of the results on the high-energy behavior of the amplitude has also an advantage, since it allows the experiment to pick out the correct alternative.

An additional tool for distinguishing among the various models is the decay spectrum. From the inspection of Figs. 6–13 it can be seen that most models have the common feature of the spectrum peaking at low energy, except for the scattering-length model fitted to Veneziano-type amplitude and the low-lying ($M_R\lesssim 700$ MeV) resonance amplitudes.

Our conclusions concerning the effect of the high-energy behavior of the $\omega+\gamma\to\pi+\pi$ amplitude on the decay rate, as well as the effect of the low-lying resonances, agree in general, though not in all details, with those obtained in the finite-energy sum rules approach to this problem.[6]