one would expect on the basis of the Ademollo-Gatto theorem.[21]

On the other hand, the Gell-Mann, Oakes, and Renner type of solution with $\epsilon_\pi/\epsilon_\kappa \cong -\mu^2/m^2$ corresponds to values of $(f_\pi/f_K)^2$ in the first interval of (35) and one gets $(m^2-\mu^2)\lambda_0 \cong 0.27$. The slope is the same as that predicted by the simple treatments discussed in Sec. II, but again not in good agreement with the data. The value of $f_+(0)$ is close to unity while one has $\lambda_K \cong 10\lambda_\pi$ and $\lambda_K \cong 40\lambda_\pi$. If one sets an upper limit $\lambda_\pi \cong 0.1$ on the basis of the Goldberger-Treiman relation, then these may not be unreasonable order-of-magnitude estimates of the extent to which the hypothesis of K and κ dominance are violated in the propagators. Hence the quadratic-smoothness assumption could be reliable if $\epsilon_\pi/\epsilon_\kappa \cong -\mu^2/m^2$.

Since neither of these two types of solutions is acceptable, one must consider other possibilities. If one is partial to the Hamiltonian (14) with no additional terms, the results of this work show that the quadratic-smoothness approximation is not adequate and higher terms must be kept in the power-series expansion of $G(k^2,p^2,q^2)$. The alternative to this is to consider the inclusion of other terms in the symmetry-breaking Hamiltonian, which would then introduce more parameters to the problem.

[21] M. Ademollo and R. Gatto, Phys. Rev. Letters **13**, 264 (1965).

Finally, we note that none of our conclusions is significantly changed if the κ mass is varied as much as 100 MeV. While there are no indications for a low κ mass, the existence of such a particle would cause a larger discrepancy between (31) and (6b) as was also found in Sec. II. If the κ meson does not exist, this situation can be described by taking the limits $m_\kappa^2 \to \infty$ and $f_\kappa \to 0$ in our results. One then recovers the Dashen-Weinstein theorem[22] from (31).

Note added in manuscript. After submission of this paper for publication, we received an experimental report by C.-Y. Chien *et al.*[23] in which a new value of $\lambda_+ = 0.08 \pm 0.01$ is suggested. If this is indeed the case, then the use of Eqs. (11) and (12) can easily obtain $\xi = -0.74 \pm 0.13$ so that, from Eq. (5), $(m^2-\mu^2)\lambda_0 = 0.2 \pm 0.2$. Furthermore, all of the consistency conditions are then satisfied with the Gell-Mann, Oakes, and Renner type of solutions with $\epsilon_\pi/\epsilon_K = -\mu^2/m^2$.

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[22] R. Dashen and M. Weinstein, Phys. Rev. Letters **22**, 1337 (1969).

[23] C.-Y. Chien *et al.*, Phys. Letters **33B**, 627 (1970).

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Resonances of Arbitrary Multiplicity*

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The constraints of unitarity on an amplitude containing J degenerate resonances are solved, and several parametrizations of the solution are given. In one solution, the unitary amplitude is obtained from the narrow-width limit. Forms suitable for other theoretical applications and for phenomenology are included. The wide variety of possible cross sections and Argand diagrams for the general case are discussed, and examples of the tripole are shown. The well-known results for the dipole are rederived without effort.

I. INTRODUCTION

THE possibility of multiple resonances in the meson spectrum has recently received both experimental and theoretical support. The doubling of the A_2 peak in the $K\bar{K}$ channel as seen by the CERN boson spectrometer group,[1] and recent spin-parity analyses,[2] strongly support the hypothesis that both peaks of the A_2 have a spin and parity of 2^+. One of the simplest models that accounts for the phenomena of the A_2 consists of two interfering resonances with the same quantum numbers, which may or may not be degenerate. Although the A_2 has been studied experimentally in some detail,[1,3] not all aspects of its phenomenology are resolved.

The R region, with its larger mass and background, is not nearly so well explored.[4] Even so, multiple-resonance behavior is consistent with the data, and

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[1] R. Baud *et al.* Phys. Letters **31B**, 401(1970); P. Schübelin, Physics Today **23**, No. 11, 32 (1970).

[2] See, for example, G. Ascoli *et al.*, University of Illinois Report No. COO-1195-193 (unpublished).

[3] M. Alston-Garnjost *et al.*, Phys. Letters **33B**, 607 (1970).

[4] J. Bartsch *et al.*, Nucl. Phys. **B22**, 109 (1970); B. Levrat *et al.*, Phys. Letters **22**, 714 (1966); M. N. Focacci *et al.*, Phys. Rev. Letters **17**, 890 (1966); L. Dubal *et al.*, Nucl. Phys. **B3**, 435 (1967).

there is a definite possibility that the R is a spin-3 tripole.[5] If one dares to extrapolate from the (ρ,A_2,R) sequence, this pattern suggests that the mesons along the leading trajectory have multiplicities equal to their spins.[5]

On the theoretical front, two dynamical models have been suggested in which the multiplicity structure along the leading Regge trajectory increases with spin. The first was suggested in the context of the dual resonance models, where the difficulty in obtaining the physical particle spectrum is well known. If reasonable restrictions are imposed on the $\pi\rho$ amplitude, including the requirement that the 2^- recurrence of the π does not have a parity doublet, then it appears that the residue of the 2^- particle does not factor in the $\pi\rho$ helicity space.[6] Whether this increase in multiplicity is a general feature of dual models that incorporate the physical particle spectrum remains to be seen. Although the mechanism of Ref. 6 does not apply to the A_2, it is still possible that there is a loss of factorization between the $\pi\rho$ and pseudoscalar channels. The second example is a field-theory model. Gürsey and Koca have constructed a nonlocal, infinite-component field theory that produces a doubled A_2 in a natural fashon.[7] This theory has the additional feature that the multiplicity of each resonance along the leading trajectory is equal to its spin. Thus, their theory predicts a tripole R meson, a fourfold S meson, and so on.

The purpose of this paper is to investigate further the amplitudes, mass spectra, and cross sections resulting from a multiple resonance. We present the general solution to the constraints of unitarity on a degenerate J-fold resonance.[8] The solution is given in several different forms, suitable for both theoretical and phenomenological applications. In one form of the solution, the cross sections and Argand diagrams for a J-fold resonance can be predicted from narrow-width models, since unitarity is sufficient to construct a unitary amplitude from the level couplings (the term level coupling will be given a precise definition). Unitarity determines the relative magnitude of all the terms in the amplitude that are higher order in the width scale.

A "degenerate" J-fold resonance amplitude is one in which J resonance poles are located at $E=M-i\frac{1}{2}\Gamma$.[9] This is the case of most interest, since it depends on fewer parameters. Moreover, from the experience with doubled resonances, cross sections will not be sensitive to small changes of the spacings between the resonance poles.[10] The shapes of the cross sections vary smoothly as all the poles approach $E=M-i\frac{1}{2}\Gamma$.

For phenomenological applications, we derive the most general form of the production or formation cross section obtainable from a J-fold resonance. These formulas are suitable for phenomenology and data fitting. In order to give the reader some feeling for the variety of possible shapes, we have included figures showing the cross sections and Argand diagrams resulting from a tripole. It takes only 7 parameters to fit a tripole in any reaction, and in general, it takes $2J+1$ parameters for a J-fold resonance. We work in the framework of the mass-matrix formalism, where both unitarity and the degeneracy condition are easily imposed. The main device for finding the solution so simply is the use of nilpotent matrices. The resulting solution depends on $(n-1)J$ parameters (n is the number of channels, and J is the multiplicity of the resonance) plus M and $\frac{1}{2}\Gamma$, the real and imaginary parts of the location of the J-fold resonance in the complex energy plane. The solution is given by Eqs. (5) and (7)–(9).

We give several different choices of independent parameters. The factorization properties, narrow-width limit, the structure of production amplitudes, cross-section formulas, and general phenomenology of the J-fold resonance formula are also discussed. Some of the interesting features include the large variety of shapes possible for a general J pole, including the variation of the number of peaks, and in those cases in which there is only one peak, the tremendous variation is the possible width of the single peak for a given value of Γ. In particular, the narrow structures observed in the high-mass boson spectrum are not inconsistent with the J-fold resonance structure.

In Sec. III, we apply the results to the dipole, to obtain without effort the results already known for the dipole.[10,11] We discuss both the degenerate and nondegenerate cases.

Section IV is concerned with the cross sections and Argand diagrams of the tripole amplitude. A number of figures are plotted there, and should give some feeling for the tremendous variety of cross sections possible for the tripole. Some examples which look like the data are shown, but the difficulties of making a serious fit are also mentioned.

The paper also serves as a sequel to Ref. 5, and a number of propositions stated in that paper without proof are proved here. In particular, we have succeeded in generalizing the single-channel J-pole solution to an arbitrary number of channels, have proved the fac-

[5] L. H. Chan, R. Slansky, and D. Sutherland, Phys. Rev. Letters **25**, 482 (1970).

[6] V. Rittenberg and H. R. Rubinstein, Phys. Rev. Letters **25**, 191 (1970); J. D. Dorren, V. Rittenberg, and H. R. Rubinstein (unpublished).

[7] F. Gürsey and M. Koca, Nuovo Cimento **1A**, 429 (1971).

[8] In this paper, J refers to the multiplicity of the resonance, not its spin. However, in some cases these may be equal, as suggested in Ref. 5.

[9] For those who prefer to analyze the amplitudes or cross sections in the variable $s=E^2$, the following replacements should be made everywhere: $E\to s$, $M\to M^2$, and $\frac{1}{2}\Gamma\to M\Gamma$. For a degenerate pole, all resonance poles are located at $s=M^2-iM\Gamma$.

[10] C. Rebbi and R. Slansky, Phys. Rev. **185**, 1838 (1969).

[11] S. Coleman, in *Theory and Phenomenology in Particle Physics*, edited by A. Zichichi (Academic, New York, 1969); Y. Dothan and D. Horn, Phys. Rev. D **1**, 916 (1970); Y. Fujii and M. Kato, Phys. Rev. **188**, 2319 (1969); K. W. McVoy, Ann. Phys. (N. Y.) **54**, 552 (1969).

torization properties, and have shown that the narrow-width limit is, indeed, sufficient to determine the shapes of the cross sections and the Argand diagrams.

II. UNITARY DEGENERATE J-FOLD RESONANCES

In this section we construct a unitary partial-wave amplitude in which J resonance poles[8] have coalesced to the same point, $E=M-i\frac{1}{2}\Gamma$,[9] and the background is neglected. This amplitude can be used to construct a unitary amplitude which includes the background, as shown in the Appendix. Suppose there are n channels. The number of arbitrary parameters needed to specify this amplitude is easily counted by first considering J widely separated resonances. There are J couplings to each of the n channels, J masses, and J widths. Unitarity imposes J conditions (which relate the widths to sums of squares of coupling constants in the limit where the poles are widely separated), which leaves $(n+1)J$ parameters. Now let the poles all approach $E=M-i\frac{1}{2}\Gamma$: In this limit, the number of constraints due to unitarity does not change. This imposes $2J-2$ more conditions, since all J widths become equal to Γ, and all J masses become equal to M. Thus it takes $(n-1)J+2$ parameters, including M and Γ, to specify a degenerate J-pole that couples to n open channels. The mass-matrix formalism is convenient for our purposes, since unitarity is easily imposed on this form of the amplitude. The general time-reversal-invariant mass-matrix form of the amplitude, which allows for both degenerate and nondegenerate configurations of J resonances, is

$$A_{ij}(E)=-\tfrac{1}{2}\Gamma X_{i\alpha}(E-\mathbf{M}+i\tfrac{1}{2}\mathbf{\Gamma})^{-1}{}_{\alpha\beta}X_{j\beta}\,, \qquad (1)$$

where $\mathbf{M}$ and $\mathbf{\Gamma}$ are J-by-J real symmetric matrices, the $X_{i\alpha}$ are real, and the J-by-J identity matrix multiplying the energy E is implicit. The parameter Γ is inserted for dimensional convenience, and we define it to be the average of the Γ_i in the case that the J resonance poles are not degenerate. The implied sums on α and β run from 1 to J, and the channel indices i and j take on the values $1,\ldots,n$. Thus the matrix X is a real n-by-J matrix whose first index operates on channel space, and whose second index acts on an internal space used to describe the J resonances or levels. The operator X transforms a vector in the internal or level space into a vector in channel space; X^T, the transpose of X, does the opposite. It is important to note that X and X^T operate on different spaces. The $X_{i\alpha}$ are called coupling constants, or level couplings.

The amplitude in Eq. (1) must satisfy partial-wave unitarity. If, for example, we assume that the quasi-two-particle channels dominate, then the many-channel unitarity relation is

$$\mathrm{Im}A_{ij}(E)=\sum_k A_{ik}(E)\rho_k(E)A_{jk}{}^*(E)=(A\rho A^\dagger)_{ij}\,, \qquad (2)$$

where $\rho_{kl}=\rho_k\delta_{kl}$ and ρ_k, the phase-space factor, is nonzero only for open channels. The constraints of unitarity on $A(E)$ follow trivially, if one notes that an invertible matrix, such as the matrix propagator, satisfies

$$2i\,\mathrm{Im}(B^{-1})=B^{-1}-B^{-1*}=-B^{-1}(B-B^*)B^{-1*}$$
$$=-2iB^{-1}(\mathrm{Im}B)B^{-1*}.$$

Substituting Eq. (1) into Eq. (2), and recalling that X is real, we find

$$\mathbf{\Gamma}=X^T\rho X\Gamma\,. \qquad (3)$$

Thus X completely determines the matrix elements of $\mathbf{\Gamma}$.

We now require that all the poles coincide at $E=M-i\frac{1}{2}\Gamma$. To accomplish this, we write the amplitude (in matrix notation) as

$$A(E)=-X\left(\frac{\frac{1}{2}\Gamma}{E-M+i\frac{1}{2}\Gamma+\frac{1}{2}\Gamma N}\right)X^T, \qquad (4)$$

where N, which is defined by Eq. (4), is a J-by-J symmetric matrix (time-reversal invariant) whose elements are in general complex. The requirement that all the poles of $A(E)$ are located at $E=M-i\frac{1}{2}\Gamma$ implies that the eigenvalues of N must all be 0. Since N satisfies its eigenvalue equation $N^J=0$, N must be a J-by-J complex nilpotent matrix. We shall assume that $N^{J-1}\neq 0$. The case where $N^{J-1}=0$ with $N^{J-2}\neq 0$ is just the problem of a $(J-1)$-fold resonance.

The mass propagator of Eq. (4) is easily inverted to give

$$A(E)=\sum_{k=0}^{J-1}\left(\frac{-\frac{1}{2}\Gamma}{E-M+i\frac{1}{2}\Gamma}\right)^{k+1}XN^kX^T. \qquad (5)$$

Equation (4) satisfies unitarity if

$$I+\mathrm{Im}(N)=X^T\rho X\,, \qquad (6)$$

where I is the J-by-J identity matrix. Equation (6) gives $\frac{1}{2}J(J+1)$ independent relations among the elements of X and N.

There must be other linear dependences between X and N. The number of dependences can be found by counting the parameters on which Eq. (4) depends [$nJ+J(J-1)$, excluding M and Γ] and subtracing the $\frac{1}{2}J(J+1)$ constraints of Eq. (6). [The total number of real parameters specifying a J-by-J complex nilpotent matrix is $J(J-1)$.] Thus, there must be $\frac{1}{2}J(J-1)$ other linear dependences.

The source of these other dependences is the arbitrariness in defining the orientation of the J-dimensional internal space: A redefinition of the internal basis leaves the amplitude invariant, while transforming X and N. We restrict this set of transformations to those which leave the form of the unitarity constraints of Eq. (6) invariant. The group satisfying this condition is the group of orthogonal transformations in J dimensions, which is a $\frac{1}{2}J(J-1)$-parameter group. Thus, N need be specified only up to an equivalence of J-dimensional orthogonal transformations.

This freedom may be used in making a convenient construction of N. In the form we find most useful (which is given below), N depends only on $\frac{1}{2}J(J-1)$ parameters, *all* of them being determined by the off-diagonal elements of $X^T\rho X$. The diagonal elements of N are real. We have satisfied the linear dependences by our choice of the other $\frac{1}{2}J(J-1)$ extra parameters needed to specify N. Of course, other solutions of the linear dependences may also be useful for some applications. With our choice, N is completely determined by the $X_{i\alpha}$, and the only constraint on the $X_{i\alpha}$ is that the diagonal elements of $X^T\rho X$ equal 1. Thus, $A(E)$ depends on $nJ-J$ independent parameters, as it should. This form for the unitarized J pole is clearly useful for narrow-width models, such as the dual models or nonlocal field theories, that should be expected to furnish the level couplings, i.e., the elements of X. It is necessary for a degenerate-unitarized J pole that the elements $(X^T\rho X)_{ii}=1$ (no sum on i). The other elements of $X^T\rho X$ determine N and, consequently, all the cross sections and phase shifts.

It remains to find explicitly this solution for N. In the case that J is odd, N has the form

$$N=\begin{bmatrix} \alpha_1 & i\alpha_1 & \beta_{13} & \beta_{14} & \cdots & \beta_{1J} \\ i\alpha_1 & -\alpha_1 & i\beta_{13} & i\beta_{14} & \cdots & i\beta_{1J} \\ \beta_{13} & i\beta_{13} & \alpha_3 & i\alpha_3 & \cdots & \beta_{3J} \\ \beta_{14} & i\beta_{14} & i\alpha_3 & -\alpha_3 & \cdots & i\beta_{3J} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{1J} & i\beta_{1J} & \beta_{3J} & i\beta_{3J} & \cdots & 0 \end{bmatrix}, \tag{7}$$

where the α_i are real and the β_{ij} are complex. In the case that J is even, simply remove the last row and column from Eq. (7). A moment's contemplation reveals that N is, indeed, the most general symmetric representation of a complex nilpotent matrix of order J, up to a J-dimensional rotation. It depends on $\frac{1}{2}J(J-1)$ parameters, as required, and we can directly relate these parameters to the elements of $X^T\rho X$. We discuss the derivation of Eq. (7) in Sec. IV for the tripole. We define a real symmetric matrix γ by

$$\gamma=\mathrm{Im}N=\begin{bmatrix} 0 & \alpha_1 & \mathrm{Im}\beta_{13} & \mathrm{Im}\beta_{14} & \cdots & \mathrm{Im}\beta_{1J} \\ \alpha_1 & 0 & \mathrm{Re}\beta_{13} & \mathrm{Re}\beta_{14} & \cdots & \mathrm{Re}\beta_{1J} \\ \mathrm{Im}\beta_{13} & \mathrm{Re}\beta_{13} & 0 & \alpha_3 & \cdots & \mathrm{Im}\beta_{3J} \\ \mathrm{Im}\beta_{14} & \mathrm{Re}\beta_{14} & \alpha_3 & 0 & \cdots & \mathrm{Re}\beta_{3J} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \mathrm{Im}\beta_{1J} & \mathrm{Re}\beta_{1J} & \mathrm{Im}\beta_{3J} & \mathrm{Re}\beta_{3J} & \cdots & 0 \end{bmatrix}. \tag{8}$$

The unitarity constraints, Eq. (6), are then given by

$$I+\gamma=X^T\rho X. \tag{9}$$

Thus, N is completely determined by the elements of X, and the relations between different elements of N and $X^T\rho X$ are completely uncoupled. It is easy to see that the shapes of the cross sections can be determined from the narrow-width limit. From Eq. (5), the narrow-width limit is

$$A_{NW}(E)=-\tfrac{1}{2}\Gamma[XX^T/(E-M)]. \tag{10}$$

In principle, the scale of X is not determined in narrow-width models. However, if some independent principle, such as that suggested in Ref. 5, determines that the multiplicity of the resonance is J, then unitarity can be used to set this scale, since Eq. (9) implies that $\mathrm{Tr}(X^T\rho X)=J$. Thus, X is calculable from the narrow-width model, and this provides adequate information to compute N through Eqs. (8) and (9). Then the unitarized amplitude follows from Eqs. (4) or (5).

The factorization properties of each term in Eq. (5) are obtained from the properties of the nilpotent matrices. Let us consider the case where $N^{J-1}\neq 0$, and the number of channels $n>J$. Other cases are similarly treated. The n-by-n matrix XN^kX^T is a matrix in channel space. We prove that XN^kX^T can be written as a matrix product of a n-by-$(J-k)$ matrix with its transpose. The order of the factorization is defined to be $J-k$. Thus XX^T is the Jth order factorizable, and $XN^{J-1}X^T$ is the first-order factorizable, or completely factorizable, since it is given by a column vector times a row vector.

The proof of this statement uses the fact that $\det(N^k)=0$ for $k=1, \ldots, J-1$. From $\det(N)=0$, N can be written as $\psi\psi^T$, where ψ is a J-by-$(J-1)$ matrix. Now, N^2 is also a nilpotent matrix, equal to $\psi\psi^T\psi\psi^T$. Since $\det(N^2)=0$ implies $\det(\psi^T\psi)=0$, the $(J-1)$-by-$(J-1)$ matrix $\psi^T\psi$ can be written as $\phi\phi^T$, where ϕ is a $(J-1)$-by-$(J-2)$ matrix. Thus, we obtain $N^2=\psi\phi\phi^T\psi^T$, where $\psi\phi$ is a J-by-$(J-2)$ matrix. This procedure can be applied until $k=J-1$, since $\det(N^{J-1})=0$. Thus, N^{J-1} can be written as the matrix product of a J-by-1 matrix with its transpose.

These factorization properties were mentioned in Ref. 5. This procedure also suggests a method for explicitly constructing N, as given in Eq. (7). We will discuss this construction only for the tripole (see Sec. IV).

The solution just presented is particularly useful for narrow-width models, since all $n(J-1)$ of the arbitrary parameters are elements of X. In another simple solution, which has a somewhat more geometrical interpretation, the $\frac{1}{2}J(J-1)$ independent matrix elements γ_{ij} of Eq. (8) are chosen to be the independent parameters. The advantage of this choice is that the parameters γ_{ij} provide a measure of the mixing of a J-fold resonance. To see this, consider the vectors $\mathbf{x}_\alpha\equiv(X_{1\alpha},\ldots,X_{n\alpha})$,

$\alpha=1,\ldots,J$, which form a set of J vectors in channel space. From Eq. (9), the $\mathbf{x}_\alpha$ are normalized, and the projection of $\mathbf{x}_\alpha$ onto $\mathbf{x}_\beta$ (which is less than or equal to 1) is just the mixing parameter $\gamma_{\alpha\beta}$. This projection is defined by Eq. (9). The mixing between two components of the J pole is a maximum if $\gamma_{\alpha\beta}=1$, and is a minimum when $\gamma_{\alpha\beta}=0$. For example, consider the "maximally mixed" case, where $\gamma_{\alpha\beta}=1$ for all α and β. Then, we recover the trivial case discussed in Ref. 5, where the $|A|^2$ for the J-fold resonance has J peaks of unit height in all channels, with $J-1$ dips to zero between each pair of adjacent peaks. The opposite extreme is $\gamma_{\alpha\beta}=0$, where all channels contain a simple resonance. (This extreme can occur if a selection rule forbids the resonances to mix.) When some of the $\gamma_{\alpha\beta}$ are 0, then N^k can be zero for $k<J-1$. The construction of N in this case presents no difficulties, as it is equivalent to the N constructed for a k-fold resonance, with $k<J$.

If the $\gamma_{\alpha\beta}$ are chosen as free parameters, then there are an additional $\frac{1}{2}J(J-1)$ linear dependences among the X_{ij}. We first solve these constraints for the case where the number of channel $n=J$, so that X is a J-by-J matrix. Let us choose a reference form for X, call it $X^{(0)}$, to be in the Jordan canonical form. Thus, all the elements of $X^{(0)}$ below the major diagonal are zero. This is the crucial step in our solution since $X^{(0)}$ specifies a reference orientation of channel space with respect to the internal space. However, all other possible "orientations" of the level couplings $X_{i\alpha}$ can be reached by rotating $X_{i\alpha}{}^{(0)}$ by a real orthogonal transformation defined in channel space. This rotation depends on $\frac{1}{2}J(J-1)$ rotation angles. (The metric of these transformations is the phase space, so that $R\rho R^T=\rho$.) Thus, in the case that $n=J$, the set of $\frac{1}{2}J(J-1)$ mixing parameters γ_{ij} and the set of $\frac{1}{2}J(J-1)$ "channel orientation angles" specify the J pole completely. The elements of $X^{(0)}=R^TX$ can be obtained from Eq. (9), since $X^{(0)}$ must satisfy the same normalization conditions as X. Thus, we find the equations

$$\begin{aligned}\rho_\alpha X_{\alpha\alpha}{}^{(0)2}&=1-\sum_{i=1}^{\alpha-1}X_{i\alpha}{}^{(0)2}\rho_i\,,\quad \alpha=2,\ldots,J\\ \sum_{i=1}^{\alpha}X_{i\alpha}{}^{(0)}\rho_iX_{i\beta}{}^{(0)}&=\gamma_{\alpha\beta}\,,\qquad \beta=\alpha+1,\ldots,J.\end{aligned}\tag{11}$$

These equations can be solved in a stepwise manner. Absorbing the ρ_i factors into the $X_{i\alpha}$, we find the first few rows of $X_{i\alpha}{}^{(0)}$ to be

$$\begin{aligned}&X_{11}{}^{(0)}=1\,,\quad X_{1\alpha}{}^{(0)}=\gamma_{1\alpha}\,,\quad \alpha=2,\ldots,J\\ &X_{22}{}^{(0)}=(1-\gamma_{12}{}^2)^{1/2}\,,\quad X_{2\alpha}{}^{(0)}X_{22}{}^{(0)}=(\gamma_{2\alpha}-\gamma_{12}\gamma_{1\alpha})\,,\\ &\qquad\qquad \alpha=3,\ldots,J\\ &X_{33}{}^{(0)}=(1-\gamma_{13}{}^2-\gamma_{12}{}^2-\gamma_{23}{}^2+2\gamma_{13}\gamma_{23}\gamma_{13})^{1/2}/(1-\gamma_{12}{}^2)^{1/2}\,,\\ &X_{3\alpha}{}^{(0)}X_{33}{}^{(0)}X_{22}{}^{(0)2}=\gamma_{3\alpha}-\gamma_{13}\gamma_{1\alpha}-\gamma_{23}\gamma_{2\alpha}+\gamma_{23}\gamma_{12}\gamma_{1\alpha}\\ &\qquad -\gamma_{12}{}^2\gamma_{3\alpha}+\gamma_{12}\gamma_{13}\gamma_{2\alpha}\,,\quad \alpha=4,\ldots,J.\end{aligned}\tag{12}$$

This solution is correct if none of the γ_{ij} are equal to 1. The modification if, for example, one of the γ_{ij} is 1, is simple. Let us choose this to be $\gamma_{J-1,J}=1$. Then we may set $X_{JJ}{}^{(0)}=0$, and the remainder of the calculation of $X_{ij}{}^{(0)}$ goes through unchanged. Other solutions of Eq. (12) follow if the negative square roots are taken. Needless to say, the lower rows of $X^{(0)}$ for high-J multiple resonances become rather complicated, but are obtained in a straightforward way. The γ_{ij} are constrained such that all the expressions inside square roots are positive. The source of this constraint is simply that the γ_{ij} are cosines of angles between a set of J real vectors in channel space.

If the number of channels n is greater than J, then not all the $\frac{1}{2}n(n-1)$ parameters that specify a real orthogonal transformation act independently on X. Since the n-by-n rotation R acts on an n-by-J matrix, $\frac{1}{2}(n-J)(n-J-1)$ of the rotation parameters leave the n-by-J matrix unchanged. Thus, only $\frac{1}{2}n(n-1)-\frac{1}{2}(n-J)(n-J-1)$ of the rotation parameters affect the elements of X, and the n-channel J pole is described by $(n-1)J$ parameters (including the γ_{ij}, but not M and Γ), as demanded.

It takes J amplitudes to describe the *production* of a J-fold resonance, just as it takes J real amplitudes $X_{i\alpha}$ $(\alpha=1,\ldots,J)$ to describe its coupling to channel i. However, the production amplitudes, call them $X_\alpha{}^{(P)}$, need not be real, and are not constrained by Eq. (9). Thus, it takes $2J-1$ extra parameters to describe the production of a J pole, since one phase is unobservable. The amplitude for producing a J pole that subsequently decays into channel i is

$$A_{P,i}(E)=-X_\alpha{}^{(P)}\left(\frac{\frac{1}{2}\Gamma}{E-M+i\frac{1}{2}\Gamma+\frac{1}{2}\Gamma N}\right)_{\alpha\beta}X_{i\beta}\,.\tag{13}$$

Of course, the $X_{i\beta}$ remain real and are still constrained by Eq. (9). If the total cross section is measured, then the shape of the J pole depends on fewer parameters. From Eqs. (2) and (4), we find that

$$\sigma_{\mathrm{tot}}{}^{(P)}=X_\alpha{}^{(P)}\,\mathrm{Im}\left(\frac{\frac{1}{2}\Gamma}{E-M+i\frac{1}{2}\Gamma+\frac{1}{2}\Gamma N}\right)_{\alpha\beta}X_\beta{}^{(P)*}\,,\tag{14}$$

which depends on $\frac{1}{2}J(J-1)$ parameters from N, plus $2J-1$ more from $X^{(P)}$. Along with M and Γ, this is a total of $\frac{1}{2}(J+1)(J+2)$ parameters.

If the matrix elements $X_{i\alpha}$ can be computed from a theory, it is then easy to compare the cross sections predicted by the theory with the experimentally observed ones. However, in general, the constraints are not too restrictive for direct phenomenology, when a complete set of formation experiments is not available. Moreover, most of the data are obtained from production experiments. Therefore, it is useful to extract a parametric form for the cross section (or mass plot) of a J pole that can be used quite generally in phenomenological analysis. From Eq.(5), it is clear that $\sigma_{ij}^{(J)}$, the cross section or mass distribution for a J-fold resonance, is equal to $[(E-M)^2+\frac{1}{4}\Gamma^2]^{-J}$ times a polynomial in $(E-M)$ of order $2J-2$. This implies that $2J+1$ parameters (including M and Γ) are sufficient to specify the shape and normalization of the cross section. The polynomial must be arranged so that it can have no more than $J-1$ real zeros and is never negative, or else the polynomial is not obtained from the absolute square of an amplitude. An example of a useful form for data fitting for a particular channel, j going to i, is

$$\sigma_{ij}^{(J)}=[(E-M)^2+\tfrac{1}{4}\Gamma^2]^{-J}\left[\left(\sum_{k=0}^{J-1}a_k^{ij}(E-M)^k\right)^2+\left(\sum_{k=0}^{J-2}b_k^{ij}(E-M)^k\right)^2\right]. \quad (15)$$

This formula depends on $2J+1$ real parameters, as it should. If the experimentalist is able to measure enough formation cross sections, or enough cross sections using the same production reaction, then eventually the a_k^{ij} and b_k^{ij} from the different experiments will have to satisfy constraints. The connection between these parameters and the $X_{i\alpha}$ is rather complicated. Of course, in a single production experiment, the parameters of Eq. (13) are essentially unconstrained. This formula is useful for exploring the shapes possible from a J-fold resonance.

It is of interest to analyze those cases where the J pole manifests itself as a single peak. The variation of the possible widths increases as J increases. From Eq. (15), we can estimate that the narrowest possible full width at half-maximum is

$$\Gamma^J{}_{\min}=\Gamma(2^{1/J}-1)^{1/2}.$$

In this case, only a_0 and b_0 are nonzero. Similarly, the widest possible peak is

$$\Gamma^J{}_{\max}\approx\Gamma(2^{1/J}-1)^{-1/2}.$$

For $J>3$, an adequate estimate of the ratio is

$$\Gamma^J{}_{\max}/\Gamma^J{}_{\min}\approx 1.5J. \quad (16)$$

Thus, the possible variation in the widths of the J poles increases linearly with J. For example, if the S meson is a fourth-order resonance, then the widths of single peaks observed in reactions could vary as much as a factor of 6. This variation is even more dramatic for higher-order resonances.

III. DEGENERATE AND NONDEGENERATE DIPOLE

The solutions of the constraints of the unitarity relations for a partial-wave amplitude containing two resonances, both in the degenerate and nondegenerate cases, have been known for some time.[10,11] The problem has attracted much interest because of its many applications, including the neutral K-meson system,[12] ω-ρ mixing,[13] the doubling of the A_2 meson, and the possible mixing of two 1^+ mesons in the Q region.[14]

An effortless derivation of the degenerate limit follows immediately from the results of Sec. II. We restrict our discussion to two channels; the generalization to more channels causes no difficulty. From Eq. (5), the degenerate dipole amplitude is

$$A(E)=-XX^T\left(\frac{\frac{1}{2}\Gamma}{E-M+i\frac{1}{2}\Gamma}\right)+XNX^T\left(\frac{\frac{1}{2}\Gamma}{E-M+i\frac{1}{2}\Gamma}\right)^2, \quad (17)$$

where N is given by Eq. (7),

$$N=\cos\theta\begin{bmatrix}1 & i\\ i & -1\end{bmatrix}, \quad (18)$$

and θ is the mixing angle. The normalization conditions, Eq. (9), are

$$\sum_i X_{i\alpha}^2=1, \qquad \alpha=1,2$$
$$\sum_i X_{i1}X_{i2}=\cos\theta. \quad (19)$$

(We have absorbed the ρ_i into the X_i for notational convenience.) The sums on i are over the two channels. It follows immediately from Eq. (19) that

$$X=\begin{bmatrix}\cos\phi & \cos(\phi-\theta)\\ \sin\phi & \sin(\phi-\theta)\end{bmatrix}, \quad (20)$$

where ϕ is the orientation angle.

The two-channel dipole amplitude depends on the four parameters M, Γ, θ, and ϕ. Examples of the cross sections and Argand diagrams implied by Eq. (17) can be found in Ref. 10. It is worth emphasizing that $A(E)$ depends on only these four independent parameters; additional parameters are redundant, or are dependent on these parameters.[15]

We recall some of the features of the cross sections for the formation of dipoles.[10] The shapes of the cross sections in a given decay channel depend on the initial channel, and a given initial channel can lead to different shapes, depending on the decay channel. Both single and double peaks are possible, and the phase shift

[12] T. D. Lee and C. S. Wu, Ann. Rev. Nucl. Sci. 16, 511 (1966).
[13] M. Gourdin, Lecture Notes for the Eleventh Scottish Universities' Summer School in Physics, 1970 (unpublished).
[14] Birmingham-Glasgow-Oxford Collaboration, Kiev Conference, 1970 (unpublished).
[15] For example, the angle θ in Ref. 10, which is the orientation of the dipole in the complex energy plane, is redundant, and can be set equal to zero without loss of generality. In this sense, the directional dependence of the dipole is lost in the degenerate limit.

does not always travel through 2π, as it does in the single-channel case. The A_2 meson exhibits this same complicated behavior with both single and double peaks occurring in various reaction channels.

The production amplitude is not as constrained as the formation amplitudes, and all the peculiarities of the formation cross sections are possible with the production cross sections.

We now turn to the nondegenerate dipole, and show that the constraints of unitarity are again easily imposed if we employ the mass-matrix formalism. The unitary mass-matrix form of the amplitude is given by Eqs. (1) and (3),

$$A(E)=-X\left(\frac{\frac{1}{2}\Gamma}{E-\mathbf{M}+i\frac{1}{2}\Gamma X^T\rho X}\right)X^T, \tag{21}$$

which is true for any number of degenerate or nondegenerate resonances. We now examine Eq. (21) for a nondegenerate dipole.

The arbitrariness of the internal basis allows some choice in the form of $\mathbf{M}$ or $X^T\rho X$. In applications this arbitrariness is sometimes used to make $\mathbf{M}$ or $X^T\rho X$ diagonal, at least to leading order in perturbation theory. We may also use this freedom so that the relation between the pole locations (the eigenvalues of $\mathbf{M}-i\frac{1}{2}\mathbf{\Gamma}$) and the elements of X are particularly simple. Although the method is different, this same choice was made in Ref. 10.

The mass matrix $\mathbf{M}-i\frac{1}{2}\mathbf{\Gamma}$ has eigenvalues $M_1-i\frac{1}{2}\Gamma_1$ and $M_2-i\frac{1}{2}\Gamma_2$, with corresponding eigenvectors ϕ_1 and ϕ_2. Thus, we can expand the mass propagator in projection operators formed from ϕ_1 and ϕ_2, and utilize the properties of the projection operators to obtain

$$A_{ij}(E)=-X_{i\alpha}\left[\frac{(P_1)_{\alpha\beta}}{E-M_1+i\frac{1}{2}\Gamma_1}+\frac{(P_2)_{\alpha\beta}}{E-M_2+i\frac{1}{2}\Gamma_2}\right]X_{j\beta}. \tag{22}$$

The factorization of each pole is manifest in Eq. (22).

If the eigenvalues are not degenerate, then the eigenvectors of $\mathbf{M}-i\frac{1}{2}\mathbf{\Gamma}$, which must be orthogonal and normalized, may always be written in terms of one complex variable as

$$\begin{aligned}\phi_1^T&=(1-c^2)^{-1/2}(1,ic),\\ \phi_1^T&=(1-c^2)^{-1/2}(-ic,1).\end{aligned} \tag{23}$$

We have still not utilized the freedom in choosing the internal basis. The choice which simplifies the relation between the masses, widths, couplings, and c is that $\mathrm{Im}(c)=0$. Other choices of the phase of c can lead to diagonal $\mathbf{M}$ or diagonal $X\rho X^T$, but the relation between X and the pole locations is enormously complicated. Upon setting $\mathrm{Im}(c)=0$, expanding $\mathbf{M}-i\frac{1}{2}\mathbf{\Gamma}$ in projection operators, and equating its imaginary part to $-\frac{1}{2}\Gamma X^T\rho X$, we find the equations

$$\begin{aligned}\mathbf{x}_1\cdot\mathbf{x}_1&\equiv\sum X_{i1}\rho_iX_{i1}=(\Gamma_1-c^2\Gamma_2)[(1-c^2)\Gamma]^{-1},\\ \mathbf{x}_2\cdot\mathbf{x}_2&\equiv\sum X_{i2}\rho_iX_{i2}=(\Gamma_2-c^2\Gamma_1)[(1-c^2)\Gamma]^{-1},\\ \mathbf{x}_1\cdot\mathbf{x}_2&\equiv\sum X_{i1}\rho_iX_{i2}=2c(M_2-M_1)[(1-c^2)\Gamma]^{-1}.\end{aligned} \tag{24}$$

Since the X_i are real and the ρ_i are positive, the Schwarz inequality may be applied to Eq. (24): $(\mathbf{x}_1\cdot\mathbf{x}_1)(\mathbf{x}_2\cdot\mathbf{x}_2)>(\mathbf{x}_1\cdot\mathbf{x}_2)^2$. Along with the positivity of $\mathbf{x}_1\cdot\mathbf{x}_1$ and $\mathbf{x}_2\cdot\mathbf{x}_2$, this condition implies

$$1-c^2>2r/(1+r), \tag{25}$$

where

$$r^2=\frac{\frac{1}{4}(\Gamma_1-\Gamma_2)^2+(M_1-M_2)^2}{\frac{1}{4}(\Gamma_1+\Gamma_2)+(M_1-M_2)^2}.$$

Thus, $1-c^2$ is positive, which explains our choice of the form of the eigenvectors in Eq. (23). The condition, Eq. (25), already implies that Γ_1 and Γ_2 are positive.

This solution is identical (aside from a slightly altered over-all normalization of the $X_{i\alpha}$) to the solution in Ref. 10 found by direct solution of the unitarity relation. The normalization conditions, Eq. (24), are much more complicated if c is taken complex.

IV. TRIPOLE

The variety of the shapes of cross sections, and the behavior of the phase shifts can be fantastically complicated for the tripole. The plan of this section is to apply the results of Sec. II to the tripole, to give some examples of their behavior, and to comment on some possible applications to the data. Our discussion is restricted to three channels and, therefore, there are six arbitrary parameters aside from M and Γ. Larger numbers of channels just introduce more parameters without adding to the wealth of phenomena possible with the tripole. Less than three channels restricts the possibilities.

The tripole amplitude, from Eq. (5), is

$$A(E)=-X\left[\frac{\frac{1}{2}\Gamma}{E-M+i\frac{1}{2}\Gamma}-\left(\frac{\frac{1}{2}\Gamma}{E-M+i\frac{1}{2}\Gamma}\right)^2N+\left(\frac{\frac{1}{2}\Gamma}{E-M+i\frac{1}{2}\Gamma}\right)^3N^2\right]X^T, \tag{26}$$

where N is a 3-by-3 nilpotent matrix of the type given in Eq. (7):

$$N=\begin{pmatrix}\gamma_{12} & i\gamma_{12} & \gamma_{23}+i\gamma_{13}\\ i\gamma_{12} & -\gamma_{12} & i\gamma_{13}-\gamma_{13}\\ \gamma_{23}+i\gamma_{13} & i\gamma_{23}-\gamma_{13} & 0\end{pmatrix}. \tag{27}$$

The mixing parameters $\gamma_{\alpha\beta}$ are real, and are related to the elements of X by the normalization conditions [Eq. (9)]:

$$\sum_i X_{i\alpha}\rho_iX_{i\beta}=\gamma_{\alpha\beta}+\delta_{\alpha\beta}. \tag{28}$$

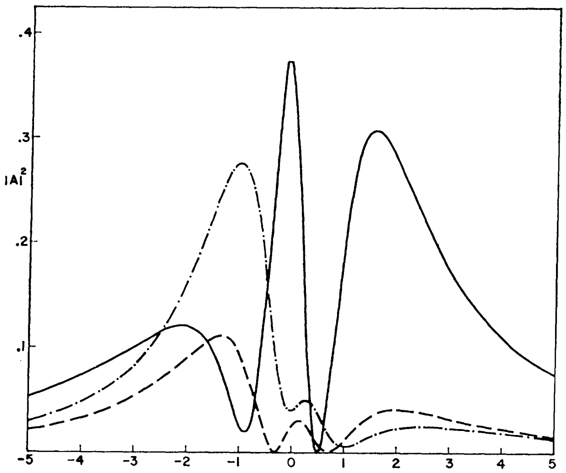

FIG. 1. Absolute squares of the tripole amplitudes for $\gamma_{12}=1$, $\gamma_{13}=0.9$, $\gamma_{23}=0.9$, $\phi=-0.16\pi$, $\theta=0.20\pi$, and $\psi=0.44\pi$. The units along the abscissa are $\epsilon=2(E-M)/\Gamma$. Thus 2 units of ϵ is an energy change of Γ. (The tripole is located at $E=M-i\frac{1}{2}\Gamma$ in the complex energy plane.) The phase-space factor ρ is set equal to 1, which sets the vertical scale. The dash-dot line is $|A_{11}|^2$, the solid line is $|A_{22}|^2$, and the dashed curve is $|A_{13}|^2$. The maximum possible value of the γ_{ij} is 1. The Argand diagram for A_{11} is given in Fig. 6.

Although the above form of N is most easily obtained from the general solution, Eq. (7), it is not difficult to derive this form directly from the properties of the nilpotent matrix. The construction of N follows closely the discussion of factorization of Sec. II. If we assume $N^2\neq 0$, then the matrix N is the product of a 3-by-2 matrix with its transpose and N^2 is the product of a

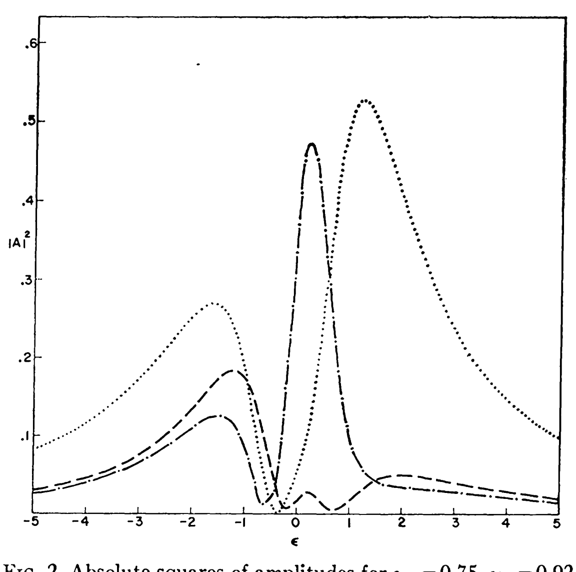

FIG. 2. Absolute squares of amplitudes for $\gamma_{12}=0.75$, $\gamma_{13}=0.92$, $\gamma_{23}=0.91$, $\phi=1.1\pi$, $\theta=1.42\pi$, and $\psi=0.62\pi$. The dash-dot curve is $|A_{11}|^2$, the dashed curve is $|A_{13}|^2$, and the dotted curve is $|A_{33}|^2$. The energy parameter ϵ is defined in the caption to Fig. (1). Note that the narrow peak in $|A_{11}|^2$ has a *full* width at half-maximum of about $\frac{1}{2}\Gamma$, whereas, the width of the peak in $|A_{33}|^2$ is about $\frac{3}{2}\Gamma$.

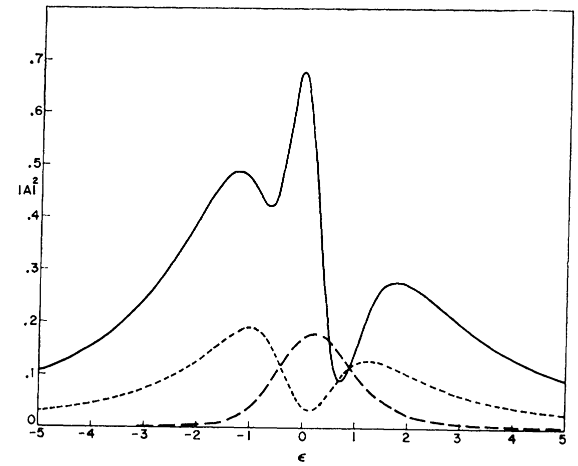

FIG. 3. Absolute squares of amplitudes for $\gamma_{12}=0.75$, $\gamma_{13}=0.55$, $\gamma_{23}=0.75$, $\phi=\frac{1}{6}\pi$, $\theta=0$, and $\psi=0$. The phase shifts for these amplitudes are shown in Fig. 4. The solid line is $|A_{22}|^2$, the short-dashed line is $|A_{12}|^2$, and the long-dashed curve is $|A_{13}|^2$.

3-by-1 matrix with its transpose. Thus, N can be written as $\psi_2\psi_2^T$ and N^2 as $\psi_2\psi_1\psi_1^T\psi_2^T$, where ψ_2 is a 3-by-2 matrix and ψ_1 is a 2-by-1 matrix. Because N^2 is a nilpotent matrix with the property $(N^2)^2=0$, $(\psi_2\psi_1)^T$ is equal to $(1,i,0)$, up to a rotation, an over-all normalization, and the freedom of the choice of basis. Moreover, $\psi_1^T=(1,i)$, since $\psi_1\psi_1^T$ is the 2-by-2 nilpotent matrix. Therefore, we obtain a set of equations for the elements of ψ_2, which are easily solved. Then N is found by simply multiplying $\psi_2\psi_2^T$, giving Eq. (27). The most general 3-by-3 nilpotent matrix can be obtained by making an orthogonal transformation on N, but we choose the internal coordinate basis so that the normalization conditions are uncoupled and simple. This sets the rotation equal to the identity.

The most convenient form for X depends on the problem. If we were dealing with models that give the elements X, such as narrow-width models, then the most simple parametrization is given in terms of six angles,

$$X_{1\alpha}=\sin\theta_\alpha\sin\phi_\alpha,\quad X_{2\alpha}=\sin\theta_\alpha\cos\phi_\alpha,\quad X_{3\alpha}=\cos\phi_\alpha, \tag{29}$$

where we have absorbed the phase-space factors into the $X_{i\alpha}$, and $\alpha=1, 2, 3$.

For displaying examples of the tripole, it is somewhat more convenient to use the second parametrization suggested in Sec. II. In this case, the γ_{ij} are independent, and the channel space indices of X are rotated from the semidiagonal form of Eq. (12) using 3-by-3 rotation matrices:

$$X_{i\alpha}=R_{ij}(\psi,\theta,\phi)X_{j\alpha}^{(0)}. \tag{30}$$

The matrix $X^{(0)}$ is given in Eq. (12). The angles are Euler angles: ϕ is a rotation around the z axis, θ around the y axis, and ψ is around the new z axis.

To give some idea of the wide range of phenomena possible with the tripole, we have plotted a few ex-

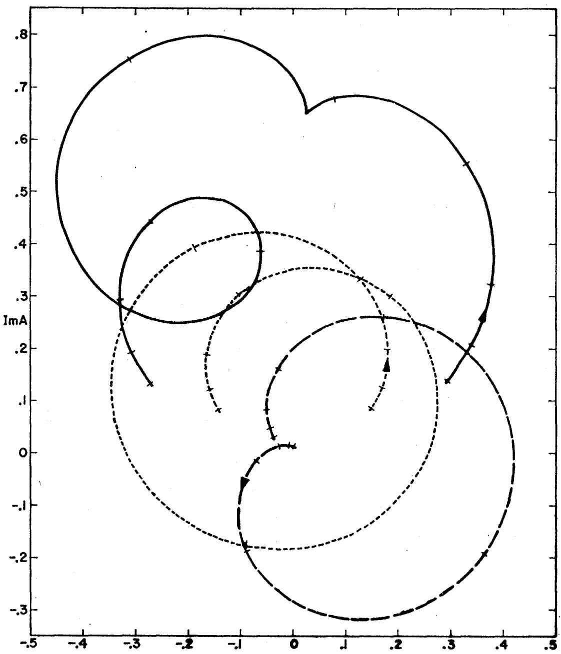

FIG. 4. Argand diagram for tripole with parameters listed in Fig. 3. The arrow denotes increasing energy. The curve begins at $\epsilon=-5$, and each cross bar signifies an increase of one unit of ϵ. Again, the solid line is the Argand diagram for A_{22}, the short-dashed line is A_{12}, and the long-dashed curve is A_{13}.

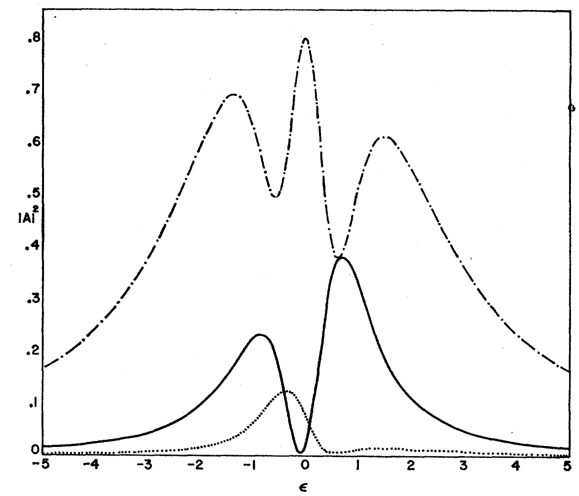

FIG. 5. Absolute square of amplitudes for $\gamma_{12}=0.85$, $\gamma_{13}=0.65$, $\gamma_{23}=0.25$, $\phi=\theta=\psi=0$. The dash-dot curve is $|A_{11}|^2$, the solid curve is $|A_{22}|^2$, and the dotted curve is $|A_{33}|^2$. The Argand diagram for A_{11} is shown in Fig. 6. Note how much this looks like the missing-mass spectrometer data for the R region.

amples. Equation (26), with N defined by Eq. (27) and X defined by Eq. (30), has been used to compute $|A_{ij}(E)|^2$ and the phase shifts for some typical values of the six parameters γ_{12}, γ_{13}, γ_{23}, ϕ, θ, and ψ.[16] We emphasize that these examples are formation experiments, and the structures from production experiments can be even more varied, since the reality properties and normalization of the production amplitudes are not restricted by Eq. (9). The cross sections and the Argand diagrams are shown in the figures.

Figure 1 is exemplary of the rapid change of the shape structure as a function of γ_{ij}. When all three γ's are 1, the cross section has the triple-peak structure of the maximally mixed case.[5] But as γ_{ij} is decreased, the triple peaking rapidly gives way to single and double peaking. Thus, it is not necessary that a tripole imply a triple peak. In the example of Fig. 1, $\gamma_{12}=1$, $\gamma_{13}=0.9$, $\gamma_{23}=0.9$, and already, some of the cross sections are single-peaked. The phase shift corresponding to A_{11} is shown in Fig. 6.

Figure 2 shows how the widths of peaks can change from cross section to cross section. $|A_{11}|^2$ is sharply peaked, with a full width at half-maximum of about $\frac{1}{2}\Gamma$. Experimentally, $|A_{33}|^2$ might appear as a broadened distorted single peak. The width of the larger peak in $|A_{33}|^2$ is about 1.5Γ.

As the mixing parameters are further reduced, the shapes become even more peculiar, such as those seen in Fig. 3. The sudden, rapid variations in energy are accompanied with sudden rates of change of the phase shift, as can be seen in Fig. 4, where the corresponding phase shifts to the cross sections of Fig. 3 are drawn.

In Fig. 5, we see a tripole cross section which is similar to the missing-mass data on the R region.[4] The phase shift for $|A_{11}|^2$ is shown in Fig. 6. Although it is amusing that the tripole formula can fit the R region,

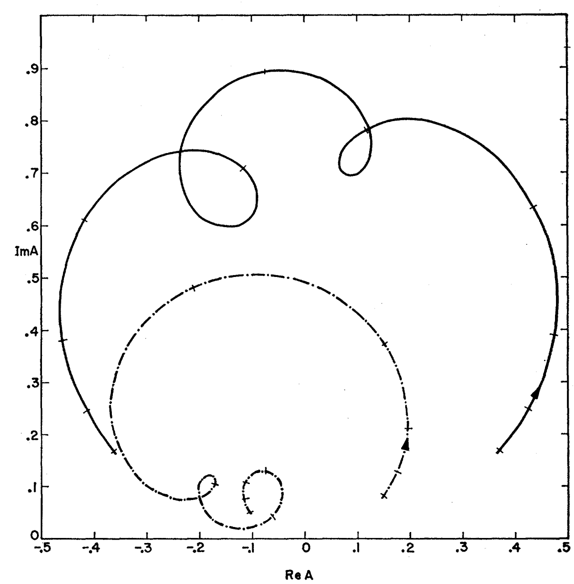

FIG. 6. Argand diagrams. The solid curve is A_{11} for $\gamma_{12}=0.85$, $\gamma_{13}=0.65$, $\gamma_{23}=0.25$ and $\phi=\theta=\psi=0$; the dash-dot curve is A_{11} for $\gamma_{12}=1$, $\gamma_{13}=0.9$, $\gamma_{23}=0.9$, $\phi=-0.16\pi$, $\theta=0.20\pi$, and $\psi=0.44\pi$. The arrows denote increase of energy, beginning at $\epsilon=-5$, and the distance between bars in one unit of ϵ.

[16] We wish to thank J. Vitale for the use of his plotting program.

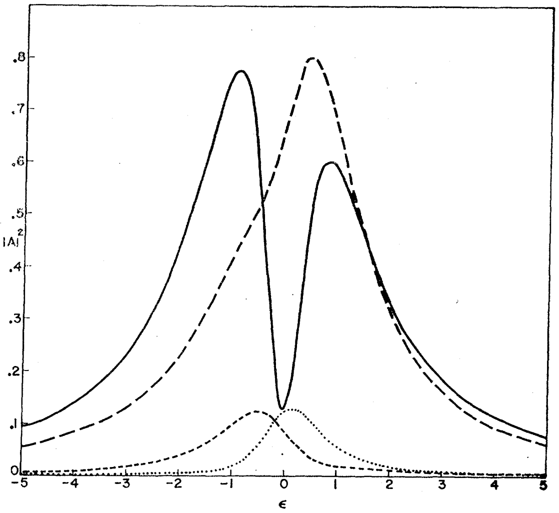

FIG. 7. Absolute square of amplitudes for $\gamma_{12}=0$, $\gamma_{13}=0.5$, $\gamma_{23}=0.71$, $\phi=\theta=\psi=0$. The dash-dot curve is $|A_{11}|^2$, the short-dashed curve is $|A_{12}|^2$, the solid curve is $|A_{22}|^2$, and the dotted curve is $|A_{33}|^2$. The widths of the single peak and the double peaks taken together are comparable, and a feature similar to the A_2 data of CERN and LRL (Refs. 1 and 3).

one should not take this particular result too seriously, because the missing-mass experiment adds together contributions from both G parities. On the other hand, the examples already shown indicate that the tripole could be used to fit the R region after a reliable separation of the negative and positive G parities have been completed, and the background from other angular momenta estimated. According to the conjecture in Ref. 5, the R is a spin-3, positive-G-parity tripole.

Our final example of a tripole, in Fig. 7, shows a double peak in one channel and a single broad peak in another channel. We do not wish to suggest that the A_2 is a tripole. Nevertheless, the *width* of the single peak seen in the LRL data appears too broad to be easily fit with the dipole formula of Sec. III. In Fig. 7 we find a single peak that is as broad as the double-peaked structure, which is much nearer to the comparison of the peak structure seen by LRL and CERN.[1,3]

ACKNOWLEDGMENTS

It is a pleasure to thank Professor Feza Gürsey for his encouragement and careful reading of the manuscript, and members of the Yale Physics Department for stimulating conversations.

APPENDIX

In this appendix, we show how the background may be added to the multiple-resonance amplitude, Eq. (1), in a manner which satisfies unitarity. The general partial-wave S matrix is the matrix in channel space,

$$S=S_B+2i\rho^{1/2}\bar{A}\rho^{1/2}, \tag{A1}$$

where ρ is the (diagonal) phase-space matrix, S_B does not contain the multiple resonance, $\bar{A}$ has the resonance, and $SS^\dagger=I$. S_B also satisfies $S_BS_B{}^\dagger=I$, and is related to a set of "background eigenphase shifts" by

$$S_B=Be^{2i\chi_B}B^T, \tag{A2}$$

where B is the orthogonal matrix which rotates the diagonal matrix $e^{2i\chi_B}$ to channel space. (In general, B is energy dependent, and has kinematic singularities to cancel the kinematic singularities which may appear in χ_B.)

Equation (A1) can then be expressed as

$$S=Be^{i\chi_B}B^T(1+2i\rho^{1/2}A\rho^{1/2})Be^{i\chi_B}B^T. \tag{A3}$$

The most important feature of Eq. (A3) is that $SS^\dagger=I$ and the orthogonality of B implies that

$$\mathrm{Im}A=A\rho A^\dagger. \tag{A4}$$

Of course, we take A to be the unitary multiple-resonance amplitude derived in the text. The full unitary amplitude, including the background, is defined by

$$S=I+2i\rho^{1/2}A_B\rho^{1/2}, \tag{A5}$$

with $SS^\dagger=I$.

Inserting Eq. (A3) for S in Eq. (A5), we find

$$A_B=\rho^{-1/2}Be^{i\chi_B}\sin\chi_B\,B^T\rho^{-1/2} +\rho^{-1/2}Be^{i\chi_B}B^T\rho^{1/2}A\rho^{1/2}Be^{i\chi_B}B^T\rho^{-1/2}. \tag{A6}$$

Thus, A_B depends on the multiple-resonance amplitude A, N background eigenphase shifts χ_B, and $\frac{1}{2}N(N-1)$ rotation parameters, B. This is correct, since the "background" amplitude does not factor, and should depend on $\frac{1}{2}N(N+1)$ parameters or, in general, $\frac{1}{2}N(N+1)$ energy-dependent functions.