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APPENDIX: NOTATION FOR VECTOR-MESON DOMINANCE

Operator relations like $\mathbf{j}_\mu = RS\mathbf{\Phi}_\mu$ used in the text are rigorously valid in the sense

$$\langle 0 | j^\alpha{}_\mu(x) | \text{v.m.}\rangle = (RS)_{\alpha\beta} \langle 0 | \Phi^\beta{}_\mu(x) | \text{v.m.}\rangle ,$$

where $|\text{v.m.}\rangle$ is a vector-meson state. For the elements of $RS = g_D{}^{-1}TM^2$ [Eq. (3.1)], we employ the notation

$$\langle 0 | j^3{}_\mu(0) | \rho^0\rangle = m_\rho{}^2 f_\rho{}^{-1} \epsilon_\mu ,$$

$$\langle 0 | j^6{}_\mu(0) | (\surd\tfrac{1}{2})(K^{*0} + \bar{K}^{*0})\rangle = m_{K^*}{}^2 f_{K^*}{}^{-1} \epsilon_\mu ,$$

$$\langle 0 | j^8{}_\mu(0) | \varphi\rangle = \tfrac{1}{2}\sqrt{3} \langle 0 | j^Y{}_\mu(0) | \varphi\rangle = \tfrac{1}{2}\sqrt{3} m_\varphi{}^2 f_Y{}^{-1} \cos\theta_Y \; \epsilon_\mu ,$$

$$\langle 0 | j^8{}_\mu(0) | \omega\rangle = \tfrac{1}{2}\sqrt{3} \langle 0 | j^Y{}_\mu(0) | \omega\rangle = -\tfrac{1}{2}\sqrt{3} m_\omega{}^2 f_Y{}^{-1} \sin\theta_Y \; \epsilon_\mu ,$$

$$\langle 0 | j^0{}_\mu(0) | \varphi\rangle = \surd\tfrac{3}{2} \langle 0 | j^B{}_\mu(0) | \varphi\rangle = \surd\tfrac{3}{2} m_\varphi{}^2 f_B{}^{-1} \sin\theta_B \; \epsilon_\mu ,$$

$$\langle 0 | j^0{}_\mu(0) | \omega\rangle = (\surd\tfrac{3}{2}) \langle 0 | j^B{}_\mu(0) | \omega\rangle = (\surd\tfrac{3}{2}) m_\omega{}^2 f_B{}^{-1} \cos\theta_B \; \epsilon_\mu ,$$

where ϵ_μ is the polarization vector of the corresponding vector meson; $j^Y{}_\mu$ and $j^B{}_\mu$ are the hypercharge and baryon-number currents.

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Consistency of Hard-Pion Theorems in K_{l3} Decays*

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The K_{l3} scalar form factor is studied by the hard-pion method in the framework of a $(3,3^*)+(3^*,3)$-symmetry-breaking model using a modified pole-dominance approximation. A set of consistency relations is found which provides a test of the reliability of the quadratic-smoothness assumption as well as of the symmetry-breaking model. In particular, we find that the solution for the symmetry-breaking parameters which fit the data is inconsistent with the quadratic-smoothness assumption. In addition, the status of other theoretical models is briefly reviewed.

I. INTRODUCTION

THE semileptonic decays of the K meson have been the subject of much attention[1] both because of their accessibility to experiment and because they provide a simple process for testing the ideas of current algebra, pole dominance, and symmetry breaking. Owing to some experimental uncertainties concerning the determination of the parameter $\xi(0)$ as well as uncertainty about the existence of the κ meson,[2,3] an adequate theoretical understanding of these decays has not yet been achieved.

In a recent survey of the experimental situation, Gaillard and Chounet[1] have found that a world average on K^+ decays gives $\xi(0) = -0.85 \pm 0.20$. Should this value for $\xi(0)$ survive additional measurements, a discrepancy will exist between it and the predictions of the conventional theoretical models.[4]

In Sec. II, the results of these models are briefly reviewed. It is argued that within a dispersion-theory approach the experimental result for $\xi(0)$ implies that the idea of κ dominance must be modified. The predictions of current-algebra calculations are dependent on the pole-dominance assumptions for current divergences as well as the type of $SU(3)\times SU(3)$-symmetry-breaking interactions that are assumed.[5] Again the most straightforward approach based on pole-dominance approximations gives results in contradiction with the experimental value of $\xi(0)$.

Accordingly in Sec. III, we carry out a study of the hard-pion method[6] which provides a means for including

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[1] See, e.g., M. K. Gaillard and L. M. Chounet, CERN Report No. 70-14 (unpublished); M. K. Gaillard and L. M. Chounet, Phys. Letters **32B**, 505 (1970).

[2] D. J. Crennel, U. Karshon, K. W. Lai, J. S. O'Neall, and J. M. Scarr, Phys. Rev. Letters **22**, 483 (1969).

[3] T. G. Trippe, C. Y. Chien, E. Malamud, J. Mellema, P. E. Schlein, W. E. Slater, D. H. Stork, and H. K. Ticho, Phys. Letters **28B**, 203 (1968).

[4] For conventional theoretical models, we refer to those papers quoted in Ref. 1 as well as C. G. Callan, in *Proceedings of Topical Conference on Weak Interactions*, edited by J. S. Bell (CERN, Geneva, 1969).

[5] S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968); M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

[6] For earlier references, we refer to S. Weinberg, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).

corrections to the pole dominance for the current divergences. In doing so, a new consistency condition is found which must be satisfied in order for the previous hard-pion results to follow.

Finally, the results are discussed in Sec. IV. In particular, it is found that if the $(3,3^*)+(3^*,3)$-symmetry-breaking model is forced to fit the experiments, then the consistency condition is badly violated.

II. SCALAR FORM FACTOR FROM DISPERSIVE AND CURRENT-ALGEBRAIC APPROACHES

The hadronic matrix element for the decay $K^+\to\pi^0+\bar{l}+\nu_l$ is conventionally written in terms of two invariant form factors $f_\pm(q^2)$ defined by[7]

$$\langle\pi^0(k)|V_\mu{}^{K^-}(0)|K^+(p)\rangle =\tfrac{1}{2}[(p+k)_\mu f_+(q^2)+(p-k)_\mu f_-(q^2)], \quad (1)$$

where $q=p-k$ and $V_\mu{}^{K^-}=(V_\mu{}^4-iV_\mu{}^5)/\sqrt{2}$. If $f_\pm(q^2)$ are parametrized in the decay region by

$$f_\pm(q^2)=f_\pm(0)(1+q^2\lambda_\pm+\cdots), \quad (2)$$

then the quantities presently accessible to measurement are λ_+, $\xi(0)=f_-(0)/f_+(0)$, and $f_+(0)\tan\theta$, where θ is the Cabibbo angle.

However, as emphasized by Gaillard and Chounet,[1] these two form factors are not dynamically independent and, as a consequence, the experimental values for $\xi(0)$ and λ_+ are highly correlated. $f_+(q^2)$ governs the decay into a p-wave pion while the matrix element of the divergence of the current,

$$\langle\pi^0(k)|\partial_\mu V_\mu{}^{K^-}(0)|K^+(p)\rangle=-\tfrac{1}{2}id(q^2), \quad (3)$$

governs the s-wave decay. $f_+(q^2)$ and $d(q^2)$ are the dynamically independent form factors. If the scalar form factor $d(q^2)$ is parametrized by

$$d(q^2)=(m^2-\mu^2)f_+(0)(1+q^2\lambda_0+\cdots), \quad (4)$$

then the parameter λ_0 is preferable to $\xi(0)$ from both theoretical and experimental standpoints. The relation between these two parameters is easily found to be

$$\xi(0)=(m^2-\mu^2)(\lambda_0-\lambda_+), \quad (5)$$

where m and μ are the K- and π-meson masses, respectively.

The experimental situation for K^+ decays according to the survey by Gaillard and Chounet is that

$$(m^2-\mu^2)\lambda_+=0.53\pm0.21 \quad (6a)$$

and

$$(m^2-\mu^2)\lambda_0=-0.30\pm0.25. \quad (6b)$$

Now we show that the predictions for the slope $(m^2-\mu^2)\lambda_0$ of the scalar form factor based either on dispersion-theory or current-algebra methods are not consistent with the experimental value (6b). No attempt is made to review all the relevant calculations,[4] but rather typical calculations are sketched in order to point out the salient characteristics of each method. It should also be noted that since attention is focused on the scalar form factor $d(q^2)$, rather than the usual form factors $f_\pm(q^2)$, our discussion is independent of the dynamical assumptions used in computing $f_+(q^2)$ such as K^* dominance of the p-wave π-K state.

[7] $f_+(0)=1$ in the $SU(3)$ limit.

A. Dispersive Approaches

Because of the simple unitarity relation for $d(q^2)$ in the elastic πK scattering region, one expects an approximate solution of the form

$$d(q^2)=(m^2-\mu^2)f_+(0)\times\exp\left(\frac{q^2}{\pi}\int_{(m+\mu)^2}^{\infty}dq'^2\frac{\delta_0(q'^2)}{q'^2(q'^2-q^2)}\right), \quad (7)$$

where $\delta_0(q^2)$ is the s-wave $I=\frac{1}{2}$ $K\pi$ phase shift. To estimate (7), one must make further dynamical assumptions, such as that s-wave $K\pi$ scattering is dominated by a κ meson or that it is given by scattering length or effective-range approximations.

1. Pole Dominance

If one assumes that s-wave $K\pi$ scattering is dominated by a κ meson with mass m_κ, then (7) can be approximated by[8]

$$d(q^2)=(m^2-\mu^2)f_+(0)m_\kappa{}^2/(m_\kappa{}^2-q^2), \quad (8)$$

which gives

$$(m^2-\mu^2)\lambda_0=(m^2-\mu^2)/m_\kappa{}^2. \quad (9)$$

If one uses $m_\kappa\cong1$ BeV, then (9) gives $(m^2-\mu^2)\lambda_0\cong+0.23$. While there seems to be no experimental evidence at the moment for a low-mass κ, the existence of such a meson would cause even greater difficulties. On the one hand, one would expect the pole-dominance approximation to be more reliable if the κ had a low mass; on the other hand, there would be even larger discrepancy between the predicted value and the experimental value of the slope (6b).

2. Scattering-Length Approximation

If the κ does not exist, then s-wave $K\pi$ scattering may be represented by a scattering-length approximation.[9] One can estimate (7) by using the expression

$$\delta_0(q^2)=a_0\mu k(q^2)/2q,$$

[8] P. Dennery and H. Primakoff, Phys. Rev. **131**, 1334 (1963); K. Kang, Phys. Rev. Letters **21**, 857 (1968); H. T. Nieh, Phys. Rev. **164**, 1780 (1968); J. E. Mackey, J. M. McKisic, D. M. Scott, and W. W. Wada, *ibid.* **172**, 1590 (1968); **183**, 1520 (1969) (E); J. C. Pati and K. J. Sebastian, *ibid.* **174**, 2033 (1968).

[9] N. Cabibbo and R. Gatto, Nuovo Cimento **13**, 1086 (1959); S. W. MacDowell, Phys. Rev. **116**, 1047 (1959); N. H. Fuchs, *ibid.* **172**, 1532 (1968).

where

$$k(q^2)=\{[q^2-(m+\mu)^2][q^2-(m-\mu)^2]/4q^2\}^{1/2}$$

and a_0 is the $I=\frac{1}{2}$ s-wave $K\pi$ scattering length. Then it follows that

$$(m^2-\mu^2)\lambda_0=(m^2-\mu^2)\frac{a_0\mu}{2\pi}\int_{(m+\mu)^2}^{\infty}dq^2\frac{k(q^2)}{q^5}, \quad (10)$$

which gives $(m^2-\mu^2)\lambda_0\cong 0.5a_0\mu$. If one uses the current-algebra prediction $a_0\cong 0.2\mu^{-1}$, then the result is $(m^2-\mu^2)\lambda_0\cong 0.1$.[10] The integrand on the right-hand side of (10) is positive definite, so that any positive scattering length corresponding to attractive s-wave $K\pi$ scattering cannot explain the experimental value (6b). Modifications to this approach such as the use of the Chew-Mandelstam effective-range approximation do not significantly alter the results.

From the viewpoint of dispersion theory, a negative λ_0 means that high-mass states must dominate over the low-energy π-K effects which are accounted for in these calculations. In order to obtain a negative λ_0, one must therefore develop a dynamical scheme which takes into account high-mass intermediate states or make additional assumptions.[11]

B. Soft-Meson Approaches

The most direct current-algebra prediction comes from the Callan-Treiman relation.[6,12] If the soft-pion theorem derived from the matrix element (1) is assumed to be approximately valid for the physical pion process, this relation gives

$$f_+(m^2)+f_-(m^2)\cong f_K/f_\pi, \quad (11)$$

where f_K and f_π are the K- and π-meson decay constants.[13] If one uses the phenomenological result[14]

$$f_K/f_\pi f_+(0)\cong 1.28, \quad (12)$$

which is based on the Cabibbo theory and the measured amplitudes for K_{e3}, π_{e3}, $K_{\mu 2}$, and $\pi_{\mu 2}$ decays, one gets $d(m^2)\cong 1.28f_+(0)$ or

$$(m^2-\mu^2)\lambda_0\cong 0.28. \quad (13)$$

The corrections to (11) can be estimated by using the Fubini-Furlan technique,[15] according to which the corrections are small as long as the pion pole dominates a certain dispersion integral. Because of the success of pion-pole dominance in the Goldberger-Treiman relation,[16] the Adler-Weisberger relation,[17] and the scattering-length calculation of Weinberg,[10] one might also expect the approximate validity of (11) and therefore (13). Furthermore, reasonable estimates show that the corrections to the Fubini-Furlan dispersion integral coming from nearby singularities are indeed small in comparison to the pion pole term.[18] Therefore, in order to understand a negative λ_0, one must postulate significant effects coming from high-mass states, presumably with the quantum numbers of the κ meson.

In addition to the approaches mentioned above, there have been further attempts[19] to calculate K_{l3} form factors by making use of the Veneziano model (usually in its simplest form) when supplemented by the Adler condition. Without reproducing the details of this type of calculation, we state that again one does not get a negative λ_0 as long as smooth extrapolation away from the physical masses is assumed.

From this review of theoretical models it is clear that one must proceed beyond the pole-dominance approximations if one is to understand the measured slope of the scalar form factor. The hard-pion method provides the framework which seems best suited for such a calculation.

III. HARD-PION METHOD

In order to apply the hard-pion method[20] to K_{l3} decay, it is convenient to work within the framework of a specific model of symmetry breaking. The most simple conjecture[5] is that the symmetry-breaking Hamiltonian is given by

$$H'=\epsilon_0 u_0+\epsilon_8 u_8, \quad (14)$$

where u_0 and u_8 are members of a nonet of scalar operators u_i $(i=0,\ldots,8)$ which together with a nonet of pseudoscalar operators v_i $(i=0,\ldots,8)$ transform according to the representation $(3,3^*)+(3^*,3)$ of $SU(3)\times SU(3)$. The transformation properties of the u_i and v_i are expressed by the relations

$$\begin{aligned}
[Q_i(x_0),u_j(x)]&=if_{ijk}u_k(x),\\
[Q_i(x_0),v_j(x)]&=if_{ijk}v_k(x),\\
[Q_i^5(x_0),u_j(x)]&=-id_{ijk}v_k(x),\\
[Q_i^5(x_0),v_j(x)]&=+id_{ijk}u_k(x),
\end{aligned} \quad (15)$$

where Q_i and Q_i^5 $(i=1,\ldots,8)$ are the generators of $SU(3)\times SU(3)$.

[10] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[11] See, e.g., K. Kang, Phys. Rev. Letters **25**, 414 (1970).

[12] C. G. Callan and S. B. Treiman, Phys. Rev. Letters **16**, 153 (1966); M. Suzuki, *ibid.* **16**, 312 (1966); V. S. Mathur, S. Okubo, and L. K. Pandit, *ibid.* **16**, 371 (1966).

[13] In our notation, $f_\pi=0.69\mu$.

[14] N. Cabibbo, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, Calif., 1966* (California U. P., Berkeley, Calif., 1967); N. Brene, M. Roos, and A. Sirlin, Nucl. Phys. **B6**, 225 (1968).

[15] S. Fubini and G. Furlan, Ann. Phys. (N. Y.) **48**, 332 (1968).

[16] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).

[17] S. L. Adler, Phys. Rev. **143**, 1144 (1966); W. I. Weisberger, *ibid.* **143**, 1302 (1966).

[18] M. Ademollo, G. Denardo, and G. Furlan, Nuovo Cimento **57A**, 1 (1968).

[19] See, e.g., J. A. Cronin and K. Kang, Phys. Rev. Letters **23**, 1004 (1969).

[20] I. S. Gerstein, J. H. Schnitzer, and S. Weinberg, Phys. Rev. **175**, 1873 (1968); I. S. Gerstein and H. J. Schnitzer, *ibid.* **175**, 1876 (1968); N. G. Deshpande, Phys. Rev. D **2**, 569 (1970); R. Arnowitt, M. H. Friedman, and P. Nath, Nucl. Phys. **B10**, 578 (1969).

The commutation relations (15) do not determine a scale for the densities u_i and v_i. Hence the only quantity of physical significance is the ratio ϵ_8/ϵ_0 of the parameters appearing in the Hamiltonian (14). Another quantity of importance in determining the physical content of the model is the ratio ω_8/ω_0, made of the vacuum expectation values

$$\omega_i = \langle 0|u_i|0\rangle \quad (i=0,8). \tag{16}$$

It has been shown by Gell-Mann, Oakes, and Renner[5] that if one assumes pole dominance for axial-vector-current divergences as well as approximate $SU(3)$ symmetry for certain vertices, then $\epsilon_8/\epsilon_0 \cong -\sqrt{2}$ and $\omega_8/\omega_0 \cong 0$. In what follows we modify the pole-dominance assumption. No assumptions of the approximate $SU(3)$ symmetry for vertex functions are made. Consequently, these parameters ϵ_8/ϵ_0 and ω_8/ω_0 are free in the present discussion, so that we are able to study the dependence of the K_{l3} scalar form factor on them.

From (14) and (15) together with the Heisenberg equation of motion, one arrives at the following expressions for the divergences of certain currents:

$$\begin{aligned} \partial_\mu A_\mu{}^i(0) &= -\epsilon_\pi v_i(0) \qquad (i=1,2,3), \\ \partial_\mu A_\mu{}^i(0) &= -(\epsilon_\pi - \epsilon_\kappa) v_i(0) \qquad (i=4,5,6,7), \\ \partial_\mu V_\mu{}^{K^+}(0) &= i\epsilon_\kappa u_{K^+}(0), \end{aligned} \tag{17}$$

where

$$\epsilon_\pi = (\sqrt{2}\epsilon_0 + \epsilon_8)/\sqrt{3}, \quad \epsilon_\kappa = \tfrac{1}{2}\sqrt{3}\epsilon_8. \tag{18}$$

To begin the study of the K_{l3} scalar form factor, let us define a three-point function $G(k^2,p^2,q^2)$ with π, K, and κ meson poles removed by

$$G(k^2,p^2,q^2) = \frac{\mu^2-k^2}{\mu^2 f_\pi}\frac{m^2-p^2}{m^2 f_K}\frac{m_\kappa{}^2-q^2}{m_\kappa{}^2 f_\kappa}\int\int d^4x d^4y\, e^{ikx}e^{iqy} \times\langle 0|T(\partial_\mu A_\mu{}^{\pi^0}(x)\partial_\nu V_\nu{}^{K^-}(y)\partial_\lambda A_\lambda{}^{K^+}(0))|0\rangle, \tag{19}$$

where f_κ is the decay constant for the κ meson.

From the commutation relations (15) and the specific model of the symmetry-breaking Hamiltonian (14), one can derive the following identities for $G(k^2,p^2,q^2)$:

$$G(0,q^2,q^2) = -\frac{i}{2f_\pi f_K f_\kappa}\frac{m^2-q^2}{m^2}\frac{m_\kappa{}^2-q^2}{m_\kappa{}^2} \times\left[\frac{\epsilon_\kappa}{\epsilon_\pi-\epsilon_\kappa}\Delta_K(q^2) - \frac{\epsilon_\pi-\epsilon_\kappa}{\epsilon_\kappa}\Delta_\kappa(q^2)\right], \tag{20a}$$

$$G(q^2,0,q^2) = \frac{-i}{2f_\pi f_K f_\kappa}\frac{\mu^2-q^2}{\mu^2}\frac{m_\kappa{}^2-q^2}{m_\kappa{}^2} \times\left[\frac{\epsilon_\kappa}{\epsilon_\pi}\Delta_\pi(q^2) - \frac{\epsilon_\pi}{\epsilon_\kappa}\Delta_\kappa(q^2)\right], \tag{20b}$$

$$G(q^2,q^2,0) = \frac{-i}{2f_\pi f_K f_\kappa}\frac{\mu^2-q^2}{\mu^2}\frac{m^2-q^2}{m^2} \times\left[\frac{\epsilon_\pi}{\epsilon_\pi-\epsilon_\kappa}\Delta_K(q^2) - \frac{\epsilon_\pi-\epsilon_\kappa}{\epsilon_\pi}\Delta_\pi(q^2)\right], \tag{20c}$$

where $\Delta_\pi(q^2)$, $\Delta_K(q^2)$, and $\Delta_\kappa(q^2)$ refer to the two-point functions

$$\Delta(q^2) = -i\int d^4x\, e^{iqx} \times\langle 0|T(\partial_\mu A_\mu{}^{\pi^0}(x)\partial_\mu A_\mu{}^{\pi^0}(0))|0\rangle, \tag{21a}$$

$$\Delta_K(q^2) = -i\int d^4x\, e^{iqx} \times\langle 0|T(\partial_\mu A_\mu{}^{K^+}(x)\partial_\mu A_\mu{}^{K^-}(0))|0\rangle, \tag{21b}$$

$$\Delta_\kappa(q^2) = -i\int d^4x\, e^{iqx} \times\langle 0|T(\partial_\mu V_\mu{}^{K^+}(x)\partial_\mu V_\mu{}^{K^-}(0))|0\rangle. \tag{21c}$$

One can also derive the following theorems for the two-point functions:

$$\Delta_\pi(0) = \epsilon_\pi\omega_\pi, \tag{22a}$$

$$\Delta_K(0) = (\epsilon_\pi-\epsilon_\kappa)(\omega_\pi-\omega_\kappa), \tag{22b}$$

$$\Delta_\kappa(0) = \epsilon_\kappa\omega_\kappa, \tag{22c}$$

where $\omega_\pi = (\sqrt{2}\omega_0+\omega_8)/\sqrt{3}$ and $\omega_\kappa = \frac{1}{2}\sqrt{3}\omega_8$.

If the three-point function $G(k^2,p^2,q^2)$ is expanded in a power series in the momenta,

$$G(k^2,p^2,q^2) = g + k^2h_1 + p^2h_2 + q^2h_3 + \cdots, \tag{23}$$

then from (20) the coefficients g, h_1, h_2, and h_3 can be determined provided some assumptions are made about the form of the two-point functions in the region of small momentum transfer.

From the relation

$$G(\mu^2,m^2,q^2) = -\frac{m_\kappa{}^2-q^2}{m_\kappa{}^2 f_\kappa}\langle\pi^0(k)|\partial_\mu V_\mu{}^{K^-}(0)|K^+(p)\rangle \tag{24}$$

it follows that if in the range $|k^2|\le\mu^2$, $|p^2|\le m^2$, $|q^2|\le m^2$ the higher-order terms in the power series (23) are negligible, then

$$d(q^2) = -2i\frac{m_\kappa{}^2 f_\kappa}{m_\kappa{}^2-q^2}(g+\mu^2h_1+m^2h_2+q^2h_3). \tag{25}$$

The basic assumption of our hard-pion approach is that the quartic and higher terms in (23) are indeed negligible. Hereafter, this is referred to as the quadratic-smoothness assumption. An equivalent assumption is made in the previous hard-pion calculations.[20]

For the reasons outlined in Sec. II, the usual assumption of pole dominance in the unsubtracted propagators (21) should be modified. Hence we assume instead pole dominance of the once-subtracted form of these two-point functions and write

$$\Delta_\pi(q^2)=(\epsilon_\pi\omega_\pi-\lambda_\pi q^2 f_\pi^2)\mu^2/(\mu^2-q^2)\,, \tag{26a}$$

$$\Delta_K(q^2)=[(\epsilon_\pi-\epsilon_\kappa)(\omega_\pi-\omega_\kappa)-\lambda_K q^2 f_K^2]\times m^2/(m^2-q^2)\,, \tag{26b}$$

$$\Delta_\kappa(q^2)=(\epsilon_\kappa\omega_\kappa-\lambda_\kappa q^2 f_\kappa^2)m_\kappa^2/(m_\kappa^2-q^2)\,. \tag{26c}$$

By construction these forms satisfy the identities (22). The limit $\lambda_\pi=\lambda_K=\lambda_\kappa=0$ corresponds to the usual pole-dominance case. The requirement that the poles have the correct residues leads to the equations

$$\epsilon_\pi\omega_\pi=\mu^2 f_\pi^2(1+\lambda_\pi)\,, \tag{27a}$$

$$(\epsilon_\pi-\epsilon_\kappa)(\omega_\pi-\omega_\kappa)=m^2 f_K^2(1+\lambda_K)\,, \tag{27b}$$

$$\epsilon_\kappa\omega_\kappa=m_\kappa^2 f_\kappa^2(1+\lambda_\kappa)\,. \tag{27c}$$

The positivity of the absorptive part of the propagators requires that λ_π, λ_K, and λ_κ be non-negative.

By substituting (26) into (20), one finds

$$g=\frac{-i}{2f_\pi f_K f_\kappa}(\epsilon_\kappa\omega_\pi-\epsilon_\pi\omega_\kappa)\,, \tag{28a}$$

$$h_1=\frac{-i}{4f_\pi f_K f_\kappa}\left[f_K^2-f_\pi^2-f_\kappa^2+\frac{2\omega_\kappa}{\omega_\pi}f_\pi^2+\frac{2\lambda_\pi f_\pi^2}{\epsilon_\pi\omega_\pi}(\epsilon_\pi\omega_\kappa-\epsilon_\kappa\omega_\pi)\right], \tag{28b}$$

$$h_2=\frac{-i}{4f_\pi f_K f_\kappa}\left[f_K^2-f_\pi^2+f_\kappa^2+\frac{2\omega_\kappa}{\omega_\pi-\omega_\kappa}f_K^2+\frac{2\lambda_K f_K^2(\epsilon_\pi\omega_\kappa-\epsilon_\kappa\omega_\pi)}{(\epsilon_\pi-\epsilon_\kappa)(\omega_\pi-\omega_\kappa)}\right], \tag{28c}$$

$$h_3=\frac{-i}{4f_\pi f_K f_\kappa}\left[f_\pi^2-f_K^2+f_\kappa^2-\frac{2\omega_\pi}{\omega_\kappa}f_\kappa^2+\frac{2\lambda_\kappa f_\kappa^2}{\epsilon_\kappa\omega_\kappa}(\epsilon_\pi\omega_\kappa-\epsilon_\kappa\omega_\pi)\right]. \tag{28d}$$

Imposing the assumption that the quartic terms in the power-series expansion (23) vanish on the right-hand side of (20) leads to new consistency relations

$$\frac{\lambda_\pi\mu^2 f_\pi^2}{\epsilon_\pi^2}=\frac{\lambda_K m^2 f_K^2}{(\epsilon_\pi-\epsilon_\kappa)^2}=\frac{\lambda_\kappa m_\kappa^2 f_\kappa^2}{\epsilon_\kappa^2}\,. \tag{29}$$

In addition, from (25), (28), and (29) come the results

$$f_+(0)=(f_K^2+f_\pi^2-f_\kappa^2)/2f_\pi f_K \tag{30}$$

and

$$(m^2-\mu^2)\lambda_0=\frac{m^2-\mu^2}{m_\kappa^2}+\frac{f_K^2-f_\pi^2+f_\kappa^2+(2m^2/m_\kappa^2)f_K^2[\epsilon_\kappa/(\epsilon_\pi-\epsilon_\kappa)]}{f_K^2+f_\pi^2-f_\kappa^2}\,. \tag{31}$$

Equation (30) was first derived by Glashow and Weinberg and (31) was derived by Gerstein and Schnitzer under the less general assumption that the unsubtracted propagators are dominated by single-particle poles. The consistency relations are of course satisfied in these derivations because $\lambda_\pi=\lambda_K=\lambda_\kappa=0$. Relations (30) and (31), however, are more generally valid provided the corrections to the pole terms satisfy the conditions (29). In Sec. IV, we see that (29) provides an important check on the reliability of the quadratic-smoothness assumption within the context of the symmetry-breaking model we have assumed.

IV. DISCUSSION

With the input $f_K/f_\pi f_+(0)=1.28$, the Glashow-Weinberg relation gives

$$(f_\pi/f_K)^2-(f_\kappa/f_K)^2=0.56\,. \tag{32}$$

Inserting this into (30) gives for the slope of the scalar form factor

$$(m^2-\mu^2)\lambda_0=0.28+\frac{m^2-\mu^2}{m_\kappa^2}+1.28\frac{m^2}{m_\kappa^2}\frac{\epsilon_\kappa}{\epsilon_\pi-\epsilon_\kappa}\,. \tag{33}$$

One can show from (27) and (29) that

$$(\epsilon_\pi/\epsilon_\kappa)^2 m_\kappa^2 f_\kappa^2+(\epsilon_\pi/\epsilon_\kappa)\times(m^2 f_K^2-\mu^2 f_\pi^2-m_\kappa^2 f_\kappa^2)+\mu^2 f_\pi^2=0\,, \tag{34}$$

which is independent of the λ_i. Inserting $m_\kappa\cong 1.1$ BeV and using (32) to eliminate $(f_\kappa/f_K)^2$ from (34) yields a relation between $\epsilon_\pi/\epsilon_\kappa$ and $(f_\pi/f_K)^2$. The requirement that $\epsilon_\pi/\epsilon_\kappa$ be real restricts the allowed values of $(f_\pi/f_K)^2$ to lie in the intervals

$$0.56\leq(f_\pi/f_K)^2\lesssim 0.7 \quad\text{or}\quad 0.9\lesssim(f_\pi/f_K)^2\,. \tag{35}$$

Using $m_\kappa\cong 1.1$ BeV again, from (33) and the experimental value of the slope (6b) one finds

$$\epsilon_\pi/\epsilon_\kappa=+0.68\pm 0.13\,, \tag{36}$$

which corresponds to $(f_\pi/f_K)^2=1.2\pm 0.2$. For the solution (36), however, the consistency relations (29) give rather unsatisfactory predictions for λ_K and λ_κ. One gets $\lambda_K\cong\lambda_\kappa\cong 0.1\lambda_\pi$, indicating that K and κ dominance are much better satisfied than π dominance. Clearly this is an implausible result and one must interpret it as a failure of the quadratic-smoothness assumption.

Another unpleasant feature of this solution is that $f_+(0)\cong 0.7\pm 0.1$, which is somewhat different from what one would expect on the basis of the Ademollo-Gatto theorem.[21]

On the other hand, the Gell-Mann, Oakes, and Renner type of solution with $\epsilon_\pi/\epsilon_\kappa \cong -\mu^2/m^2$ corresponds to values of $(f_\pi/f_K)^2$ in the first interval of (35) and one gets $(m^2-\mu^2)\lambda_0 \cong 0.27$. The slope is the same as that predicted by the simple treatments discussed in Sec. II, but again not in good agreement with the data. The value of $f_+(0)$ is close to unity while one has $\lambda_K \cong 10\lambda_\pi$ and $\lambda_K \cong 40\lambda_\pi$. If one sets an upper limit $\lambda_\pi \cong 0.1$ on the basis of the Goldberger-Treiman relation, then these may not be unreasonable order-of-magnitude estimates of the extent to which the hypothesis of K and κ dominance are violated in the propagators. Hence the quadratic-smoothness assumption could be reliable if $\epsilon_\pi/\epsilon_\kappa \cong -\mu^2/m^2$.

Since neither of these two types of solutions is acceptable, one must consider other possibilities. If one is partial to the Hamiltonian (14) with no additional terms, the results of this work show that the quadratic-smoothness approximation is not adequate and higher terms must be kept in the power-series expansion of $G(k^2,p^2,q^2)$. The alternative to this is to consider the inclusion of other terms in the symmetry-breaking Hamiltonian, which would then introduce more parameters to the problem.

[21] M. Ademollo and R. Gatto, Phys. Rev. Letters **13**, 264 (1965).

Finally, we note that none of our conclusions is significantly changed if the κ mass is varied as much as 100 MeV. While there are no indications for a low κ mass, the existence of such a particle would cause a larger discrepancy between (31) and (6b) as was also found in Sec. II. If the κ meson does not exist, this situation can be described by taking the limits $m_\kappa^2 \to \infty$ and $f_\kappa \to 0$ in our results. One then recovers the Dashen-Weinstein theorem[22] from (31).

Note added in manuscript. After submission of this paper for publication, we received an experimental report by C.-Y. Chien *et al.*[23] in which a new value of $\lambda_+ = 0.08 \pm 0.01$ is suggested. If this is indeed the case, then the use of Eqs. (11) and (12) can easily obtain $\xi = -0.74 \pm 0.13$ so that, from Eq. (5), $(m^2-\mu^2)\lambda_0 = 0.2 \pm 0.2$. Furthermore, all of the consistency conditions are then satisfied with the Gell-Mann, Oakes, and Renner type of solutions with $\epsilon_\pi/\epsilon_K = -\mu^2/m^2$.

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Resonances of Arbitrary Multiplicity*

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The constraints of unitarity on an amplitude containing J degenerate resonances are solved, and several parametrizations of the solution are given. In one solution, the unitary amplitude is obtained from the narrow-width limit. Forms suitable for other theoretical applications and for phenomenology are included. The wide variety of possible cross sections and Argand diagrams for the general case are discussed, and examples of the tripole are shown. The well-known results for the dipole are rederived without effort.

I. INTRODUCTION

The possibility of multiple resonances in the meson spectrum has recently received both experimental and theoretical support. The doubling of the A_2 peak in the $K\bar{K}$ channel as seen by the CERN boson spectrometer group,[1] and recent spin-parity analyses,[2] strongly support the hypothesis that both peaks of the A_2 have a spin and parity of 2^+. One of the simplest models that accounts for the phenomena of the A_2 consists of two interfering resonances with the same quantum numbers, which may or may not be degenerate. Although the A_2 has been studied experimentally in some detail,[1,3] not all aspects of its phenomenology are resolved.

The R region, with its larger mass and background, is not nearly so well explored.[4] Even so, multiple-resonance behavior is consistent with the data, and

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[1] R. Baud *et al.* Phys. Letters **31B**, 401(1970); P. Schübelin, Physics Today **23**, No. 11, 32 (1970).

[2] See, for example, G. Ascoli *et al.*, University of Illinois Report No. COO-1195-193 (unpublished).

[3] M. Alston-Garnjost *et al.*, Phys. Letters **33B**, 607 (1970).

[4] J. Bartsch *et al.*, Nucl. Phys. **B22**, 109 (1970); B. Levrat *et al.*, Phys. Letters **22**, 714 (1966); M. N. Focacci *et al.*, Phys. Rev. Letters **17**, 890 (1966); L. Dubal *et al.*, Nucl. Phys. **B3**, 435 (1967).