Relativistic Calculation of Nucleon-Nucleon Phase Parameters*

ALEXANDER GERSTEN†
University of Florida, Gainesville, Florida 32601
and
Texas A&M University, College Station, Texas 77840

AND

RICHARD H. THOMPSON‡ AND A. E. S. GREEN
University of Florida, Gainesville, Florida 32601

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An approximate Bethe-Salpeter equation is used to calculate the N-N phase parameters. The one-boson-exchange (OBE) hypothesis is invoked to approximate the left-hand singularity. A good fit to the existing data is obtained, and certain phase parameters ($^3D_2$, $^1D_2$, and $^3P_2$), which are not fitted well by existing OBE-potential (OBEP) models, are much improved. The observables are not fitted directly, but rather an approximate procedure suggested by MacGregor *et al.* is used to minimize the experimental χ^2. Comparisons are made with the nonrelativistic OBEP used in conjunction with the Schrödinger equation.

I. INTRODUCTION

DURING the past decade, generalized forms of the one-boson-exchange potential (OBEP) have been partially successful in describing a rather large accumulation of N-N scattering data up to the 300-MeV region.[1,2] The N-N potential, according to these models, arises from the exchange of pseudoscalar, scalar, and vector mesons. The present work is an attempt to extend the OBEP model by using a relativisitic formalism which has been presented by one of the authors[3] and is based on the Blankenbecler-Sugar[4] reduction of the Bethe-Salpeter[5] equation. It is hoped that it will be possible to extend the model into the 400-MeV region. An attempt to include relativistic effects using the two-particle Dirac equation was initiated by one of the authors, but this study is not complete.[6]

The linear integral equation used to calculate the scattering amplitude in this work is known to introduce a spurious left-hand singularity[4] at $s=4m^2-4(m+m_B)^2$, where m_B is the exchanged boson mass. However, for the N-N system, this singularity is very far removed from the physical region and should not therefore play an important role in the calculation.

In the previous derivations of OBEP, the N-N potential is expanded in terms of ∇/m, and terms of order higher than $(\nabla/m)^2$ are neglected.[1,2] In the preceding paper[7] we examined this question, and found the approximation to be unreliable for the case of the one-pion-exchange potential (OPEP). Therefore, the ∇/m expansion should be circumvented. Hence in this work the calculations are carried out directly in momentum space.

The short-range part of the interaction is parametrized using a phenomenological form factor[1,8] which cuts off the interaction at small N-N separations or, alternatively, at large momenta. This particular type of form factor has been used in the past in fitting to N-N data.[1,2] The calculation is insensitive to the cutoff mass Λ as long as it is taken to be relatively large ($\Lambda\sim1500$ MeV); however, the S waves do depend on this parameter and hence must be thought of as essentially phenomenological.

The exchange mesons used are the π, ρ, ω, η, ϵ, and δ. With the exception of the ϵ, the masses are taken from the Rosenfeld tables.[9] This ϵ is the broad mass scalar in the $J=T=0$ π-π system located at $\sim$740 MeV. Since its mass is not well known, it was treated as a parameter in this work and is searched on to obtain a best fit to the data. The coupling constants are not well known and are all treated as adjustable. One of the more encouraging things to come out of this work is that the coupling constants and masses used all seem to be compatible with other estimates of these parameters.

II. EQUATIONS

The details of the reduction of the Bethe-Salpeter equation to its three-dimensional form are given in I.

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† Present address: University of Nijmegen, Nijmegen, The Netherlands.

‡ Present address: Department of Physics, Drexel University, Philadelphia, Pa. 19104.

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Hence, only the relevant equations are given in this work for completeness.

The three-dimensional equation for the scattering amplitude M derived in I is given by

$$M(\mathbf{p},\mathbf{q})=W(\mathbf{p},\mathbf{q})+\int d\mathbf{k}\, W(\mathbf{p},\mathbf{k})E_2^{(+)}(\mathbf{k},s)M(\mathbf{k},\mathbf{q}), \quad (1)$$

where $E_2^{(+)}$ is the two-particle Green's function given by

$$E_2^{(+)}(\mathbf{k},s)=\frac{4\pi^3m^2\Lambda_{(1)}{}^{(+)}(\mathbf{k})\Lambda_{(2)}{}^{(+)}(-\mathbf{k})}{E^2(\mathbf{k})[E(\mathbf{k})-\frac{1}{2}\sqrt{s}-i\epsilon]}, \quad (2)$$

$$E(\mathbf{k})=(\mathbf{k}^2+m^2)^{1/2}, \quad s=4(\mathbf{q}^2+m^2), \quad (3)$$

$\Lambda_{(i)}{}^{(+)}(\mathbf{k})$ is the positive-energy projection operator for the ith particle, and $W(\mathbf{p},\mathbf{k})$ is the interaction kernel. In this work, $W(\mathbf{p},\mathbf{k})$ is approximated to order $g_B{}^2$, where g_B is the nucleon-boson coupling strength.

In practice one wishes to calculate phase parameters rather than directly solve Eq. (1). This is accomplished by projecting the partial-wave amplitudes from Eq. (1), resulting in

$$T^{J,S}(p,l_f;q,l_i)=W^{J,S}(p,l_f;q,l_i)+\sum_{l=|J-S|}^{l=|J+S|}\int_0^\infty dk\, k^2W^{J,S}(p,l_f;k,l)\times g(+,\mathbf{k},s)T^{J,S}(k,l;q,l_i), \quad (4)$$

where J is the total angular momentum, S is the spin angular momentum, l_i and l_f are the initial and final orbital angular momenta, respectively, $T^{J,S}(p,l_f;q,l_i)$ is the off-energy-shell partial-wave scattering amplitude, $W^{J,S}(p,l_f;q,l_i)$ is the partial-wave projection of the interaction kernel $W(\mathbf{p},\mathbf{q})$, and

$$g(+,\mathbf{k},s)=4\pi^3m^2/E^2(\mathbf{k})[E(\mathbf{k})-\tfrac{1}{2}\sqrt{s}-i\epsilon]. \quad (5)$$

For calculational reasons it is more convenient to work with the real K matrix which is related to $T^{J,S}(p,l_f;q,l_i)$ algebraically. This relation is given in II and also K is related to the bar phase parameters in II. The K matrix satisfies Eq. (4) except that $i\epsilon$ must be replaced by a principal-value singularity, i.e.,

$$K^{J,S}(p,l_f;q,l_i)=W^{J,S}(p,l_f;q,l_i)+\sum_{l=|J-S|}^{l=|J+S|}\int_0^\infty dk\, k^2W^{J,S}(p,l_f;k,l)\times g'(+,\mathbf{k},s)K^{J,S}(k,l;q,l_i), \quad (6)$$

where

$$g'(+,\mathbf{k},s)=P\{4\pi^3m^2/E^2(\mathbf{k})[E(\mathbf{k})-\tfrac{1}{2}\sqrt{s}]\} \quad (7)$$

and P denotes that the principal value is to be taken.

III. PARTIAL-WAVE PROJECTIONS

In this section most of the details of projecting the partial-wave amplitudes from the interaction kernel are presented. More specifically, the amplitudes due to the exchange of vector, scalar, and pseudoscalar mesons are treated.

The helicity formalism of Jacob and Wick[10] is used as an intermediate step in going from matrix elements between plane-wave states to matrix elements between states of good J, l, and S. Basically our method is very similar to that used by Goldberger *et al.*[11] in their fundamental paper on the application of dispersion theory to the N-N interaction. However, because the matrix elements involved in the present work are off the energy shell, the helicity formalism application must undergo modifications. Therefore, the authors of this article felt that it would be worthwhile to present some of the details of this aspect of the work.

The matrix element relating helicity states with a definite total J to plane-wave states of good $\mathbf{p}$ is given by[10]

$$\langle pJ\lambda_2'\lambda_1'|\,\mathbf{p}\lambda_2\lambda_1\rangle=[(2J+1)/4\pi]^{1/2}\delta_{\lambda_2\lambda_2'}\delta_{\lambda_1\lambda_1'}\times D_{M\lambda}{}^J(\phi,\theta,-\phi), \quad (8)$$

where M is the projection of J onto a fixed z axis, $\lambda=\lambda_2-\lambda_1$, θ and ϕ are the polar and azimuthal angles defining the direction of $\mathbf{p}$ with respect to the z axis, and $D_{M\lambda}{}^J(\phi,\theta,-\phi)$ is the usual rotation function.[12] Using the completeness relation and Eq. (8), it is straightforward to deduce the relation between the interaction kernel in the $\mathbf{p},\lambda_2,\lambda_1$ representation and in the p,J,λ_2,λ_1 representation. This result is given by

$$W^J(p\lambda_4\lambda_3;q\lambda_2\lambda_1)=2\pi e^{i\phi(\lambda_f-\lambda_i)}\int_{-1}^1 d(\cos\theta)\times d_{\lambda_i\lambda_f}{}^J(\theta)W(\mathbf{p},\lambda_4,\lambda_3;\mathbf{q},\lambda_2,\lambda_1), \quad (9)$$

where $\lambda_f=\lambda_4-\lambda_3$, $\lambda_i=\lambda_2-\lambda_1$, and

$$d_{\lambda_i\lambda_f}{}^J(e)=e^{i\lambda_i\phi}D_{\lambda_i\lambda_f}{}^J(\phi,\theta,-\phi)e^{-i\lambda_f\phi}. \quad (10)$$

In general there are 16 helicity amplitudes $W^J(p\lambda_4\lambda_3;q\lambda_2\lambda_1)$ necessary to describe the N-N interaction. If we denote $\lambda=\pm\frac{1}{2}$ by $\pm$, then the helicity amplitudes are given by the following matrix for a given value of J, p,

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and q:

$$\text{(helicity matrix)}=\begin{pmatrix}\langle++|++\rangle & \langle++|--\rangle & \langle++|+-\rangle & \langle++|-+\rangle\\ \langle--|++\rangle & \langle--|--\rangle & \langle--|+-\rangle & \langle--|-+\rangle\\ \langle+-|++\rangle & \langle+-|--\rangle & \langle+-|+-\rangle & \langle+-|-+\rangle\\ \langle-+|++\rangle & \langle-+|--\rangle & \langle-+|+-\rangle & \langle-+|-+\rangle\end{pmatrix}, \tag{11}$$

where the indices J, p, and q have been suppressed; e.g., $\langle++|++\rangle\equiv W^J(p++;q++)$. Invoking time-reversal invariance and the conservation of parity allows one to relate many of these helicity matrix elements. Because of time-reversal invariance, the matrix of Eq. (11) must be symmetric with respect to interchanges of pairs of helicity indices; i.e., if T denotes the time-reversal operator, then

$$\begin{aligned}\langle pJ\lambda_4\lambda_3|W|qJ\lambda_2\lambda_1\rangle&=\langle pJ\lambda_4\lambda_3|T^\dagger TWT^\dagger T|qJ\lambda_2\lambda_1\rangle\\&=\langle pJ\lambda_4\lambda_3|W^\dagger|qJ\lambda_2\lambda_1\rangle^*\\&=\langle qJ\lambda_2\lambda_1|W|pJ\lambda_4\lambda_3\rangle.\end{aligned}\tag{12}$$

Note that since the matrix element in Eq. (12) is off shell, one must also interchange p and q when interchanging $\lambda_2\lambda_1$ and $\lambda_4\lambda_3$.[13] Application of the conservation of parity leads to the result that

$$\begin{aligned}&\langle pJ\lambda_4\lambda_3|W|qJ\lambda_2\lambda_1\rangle\\&\quad=\langle p\,J\,-\lambda_4\,-\lambda_3|W|q\,J\,-\lambda_2\,-\lambda_1\rangle.\end{aligned}\tag{13}$$

Then using the properties of Eqs. (12) and (13), the following relations hold:

$$a^J(p,q)\equiv\langle++|++\rangle=\langle--|--\rangle,\tag{14a}$$

$$b^J(p,q)\equiv\langle++|--\rangle=\langle--|++\rangle,\tag{14b}$$

$$c^J(p,q)\equiv\langle++|+-\rangle=\langle--|-+\rangle,\tag{14c}$$

$$d^J(p,q)\equiv\langle--|+-\rangle=\langle++|-+\rangle,\tag{14d}$$

$$e^J(p,q)\equiv\langle+-|+-\rangle=\langle-+|-+\rangle,\tag{14e}$$

$$f^J(p,q)\equiv\langle+-|-+\rangle=\langle-+|+-\rangle.\tag{14f}$$

Assuming the conservation of isospin leads to the direct conclusion that there can be no scattering between spin-triplet and spin-singlet states. The usual singlet and triplet states may be formed from the helicity states.[10] These states are also states of the parity operator P and are given in the following expressions:

(triplet)

$$|J,P=J+1\rangle=(|++\rangle+|--\rangle)/\sqrt{2},\tag{15a}$$

$$|J,P=J-1\rangle=(|+-\rangle+|-+\rangle)/\sqrt{2},\tag{15b}$$

$$|J,P=J\rangle_T=(|++\rangle-|--\rangle)|\sqrt{2},\tag{15c}$$

(singlet)
$$|J,P=J\rangle_S=(|+-\rangle-|-+\rangle)/\sqrt{2}.\tag{15d}$$

[13] This is the essential difference between the on-shell and off-shell formalism. Although this is a straightforward result of the formalism, failure to interchange p and q leads to the result that $W^{J,1}(p,J+1;q,J-1)=W^{J,1}(p,J-1;q,J+1)$, which is not true and has led to some confusion in the literature [see, for example, J. Goto and S. Machida, Progr. Theoret. Phys. (Kyoto) 25, 64 (1961)].

Since no scattering can occur between singlet and triplet states, the matrix elements $\langle J,P=J\pm1|J,P=J\rangle_S$ and ${}_T\langle J,P=J|J,P=J\rangle_S$ must be identically zero. These conditions lead to $\langle++|+-\rangle=\langle--|-+\rangle$, which is one of the previous conditions, and $\langle++|+-\rangle=\langle++|-+\rangle$, which means that $c(p,q)=d(p,q)$. Hence the helicity matrix of Eq. (11) reduces to

$$\begin{pmatrix}a^J(p,q) & b^J(p,q) & c^J(p,q) & c^J(p,q)\\ b^J(q,p) & a^J(p,q) & c^J(p,q) & c^J(p,q)\\ c^J(q,p) & c^J(q,p) & e^J(p,q) & f^J(p,q)\\ c^J(q,p) & c^J(q,p) & f^J(q,p) & e^J(p,q)\end{pmatrix};\tag{16}$$

thus, there are five independent amplitudes.

The fundamental relationship between the helicity and the JlS amplitude is given by

$$\begin{aligned}\langle JMlS|JM\lambda_1\lambda_2\rangle&=[(2l+1)/(2J+1)]^{1/2}\\&\quad\times C(\tfrac{1}{2}\,\tfrac{1}{2}S;\lambda_1\,-\lambda_2)C(lSJ;0\lambda),\end{aligned}\tag{17}$$

where $\lambda=\lambda_1-\lambda_2$ and the C's are the standard Clebsch-Gordan coefficients.[12] Then the interaction kernel in the JlS representation is given by

$$\begin{aligned}&W^{J,S}(p,l_f;q,l_i)\\&\quad=\sum_{\lambda_1\lambda_2\lambda_3\lambda_4}[(2l_f+1)^{1/2}(2l_i+1)^{1/2}/(2J+1)]\\&\qquad\times C(\tfrac{1}{2}\,\tfrac{1}{2}S;\lambda_1-\lambda_2)C(l_fSJ;0\lambda_f)\\&\qquad\times C(\tfrac{1}{2}\,\tfrac{1}{2}S;\lambda_3-\lambda_4)C(l_iSJ;0\lambda_i)\\&\qquad\times W^J(p\lambda_4\lambda_3;q\lambda_2\lambda_1),\end{aligned}\tag{18}$$

where $\lambda_f=\lambda_4-\lambda_3$, $\lambda_i=\lambda_2-\lambda_1$. Evaluating the sum appearing in Eq. (18) using the explicit values for the Clebsch-Gordan coefficients,[12] the following results are obtained:

(singlet) $$W^{J,0}(p,J;q,J)=a^J(p,q)-b^J(p,q),\tag{19a}$$

(triplet $l=J$) $$W^{J,1}(p,J;q,J)=e^J(p,q)-f^J(p,q),\tag{19b}$$

(coupled triplet)

$$\begin{aligned}&W^{J,1}(p,J+1;q,J+1)\\&\quad=\{(J+1)[a^J(p,q)+b^J(p,q)]+J[e^J(p,q)+f^J(p,q)]\\&\qquad-2[J(J+1)]^{1/2}[c(p,q)+c(q,p)]\}/(2J+1),\end{aligned}\tag{19c}$$

$$\begin{aligned}&W^{J,1}(p,J-1;q,J-1)\\&\quad=\{J[a^J(p,q)+b^J(p,q)]+(J+1)[e^J(p,q)+f^J(p,q)]\\&\qquad+2[J(J+1)]^{1/2}[c(p,q)+c(q,p)]\}/(2J+1),\end{aligned}\tag{19d}$$

$$\begin{aligned}&W^{J,1}(p,J+1;q,J-1)\\&\quad=\{2Jc(q,p)-2(J+1)c(p,q)+[J(J+1)]^{1/2}[e^J(p,q)\\&\qquad+f^J(p,q)-a^J(p,q)-b^J(p,q)]\}/(2J+1),\end{aligned}\tag{19e}$$

$$W^{J,1}(p,J-1;q,J+1)=\{2Jc(p,q)-2(J+1)c(q,p)+[J(J+1)]^{1/2}[e^J(p,q)+f^J(p,q)-a^J(p,q)-b^J(p,q)]\}/(2J+1). \quad (19f)$$

Integration formulas which are useful in performing the integration over $\cos\theta$ in Eq. (9) are given by

$$\frac{1}{2}\int_{-1}^{1}d\mu\frac{\frac{1}{2}(1+\mu)d_{11}{}^J-\frac{1}{2}(1-\mu)d_{-11}{}^J}{z-\mu}=Q_J(z), \quad (20a)$$

$$\frac{1}{2}\int_{-1}^{1}d\mu\frac{\frac{1}{2}(1+\mu)d_{11}{}^J+\frac{1}{2}(1-\mu)d_{-11}{}^J}{z-\mu}=\tfrac{1}{2}[(J+1)Q_{J-1}(z)+JQ_{J+1}(z)], \quad (20b)$$

$$\frac{1}{2}\int_{-1}^{1}d\mu\frac{(1-\mu^2)^{1/2}d_{10}{}^J}{z-\mu}=\frac{[J(J+1)]^{1/2}[Q_{J+1}(z)-Q_{J-1}(z)]}{2J+1}, \quad (20c)$$

$$\frac{1}{2}\int_{-1}^{1}\mu d\mu\frac{d_{00}{}^J}{z-\mu}=\frac{[(J+1)Q_{J+1}(z)+JQ_{J-1}(z)]}{2J+1}, \quad (20d)$$

where $Q_J(z)$ is a Legendre function of the second kind of order J, and z is a parameter to be defined later.

Let us now turn to the specific cases relevant to this work, i.e., the partial-wave projections from the interaction kernel plane-wave matrix elements arising from the exchange of one scalar, pseudoscalar, or vector meson.

Scalar meson. The interaction Hamiltonian coupling the nucleon and scalar-meson fields is given by

$$H_I=g_S\bar{\psi}_N\psi_N\phi_S, \quad (21)$$

where ψ_N is the nucleon field operator, $\bar{\psi}_N=\psi_N{}^\dagger\gamma_0$, g_S is the coupling strength, and ϕ_S is the scalar-meson field.[14] Using the rules described in I, the interaction kernel to order $g_S{}^2$ is determined to be

$$W(\mathbf{q}\lambda_4\lambda_3;\mathbf{k}\lambda_2\lambda_1)=\frac{g_S{}^2[\bar{\omega}_{\lambda_4}(\mathbf{q})\omega_{\lambda_2}(\mathbf{k})][\bar{\omega}_{\lambda_3}(-\mathbf{q})\omega_{\lambda_1}(-\mathbf{k})]}{(2\pi)^6[(\mathbf{q}-\mathbf{k})^2+m_S{}^2]}, \quad (22)$$

where

$$\omega_\lambda(\mathbf{q})=\left(\frac{E(\mathbf{q})+m}{2m}\right)^{1/2}\begin{bmatrix}1\\ 2\lambda q/[E(\mathbf{q})+m]\end{bmatrix}\chi_q{}^\lambda \quad (23)$$

is the Dirac four-spinor, $\bar{\omega}_\lambda(\mathbf{q})=\omega_\lambda{}^\dagger(\mathbf{q})\gamma_0$, and

$$\chi_q{}^\lambda=\begin{bmatrix}(\lambda+\frac{1}{2})\cos\frac{1}{2}\theta+(\lambda-\frac{1}{2})\sin\frac{1}{2}\theta\, e^{-i\phi}\\ (\lambda+\frac{1}{2})\sin\frac{1}{2}\theta\, e^{i\phi}-(\lambda-\frac{1}{2})\cos\frac{1}{2}\theta\end{bmatrix} \quad (24)$$

is the Pauli two-spinor. Here θ and ϕ are the polar and azimuthal angles defining the direction of $\mathbf{q}$ with respect to the fixed z axis.

Writing out the specific components of Eq. (22) using Eqs. (23) and (24) and employing $\pm$ for $\lambda=\pm\frac{1}{2}$, the following results are obtained:

$$W(\mathbf{q}++;\mathbf{k}++)=\tfrac{1}{2}\alpha^2(++)(1+\mu), \quad (25a)$$

$$W(\mathbf{q}+-;\mathbf{k}+-)=\tfrac{1}{2}\alpha(++)\alpha(--)(1+\mu), \quad (25b)$$

$$W(\mathbf{q}++;\mathbf{k}--)=-\tfrac{1}{2}\alpha^2(+-)(1-\mu), \quad (25c)$$

$$W(\mathbf{q}+-;\mathbf{k}-+)=\tfrac{1}{2}\alpha(+-)\alpha(-+)e^{-2i\phi}\times(1-\mu), \quad (25d)$$

$$W(\mathbf{q}++;\mathbf{k}+-)=-\tfrac{1}{2}\alpha(++)\alpha(+-)e^{i\phi}\times(1-\mu^2)^{1/2}, \quad (25e)$$

where

$$\alpha(\lambda_2\lambda_1)=q_S{}^2\frac{[E(\mathbf{q})+m]^{1/2}[E(\mathbf{k})+m]^{1/2}-4\lambda_2\lambda_1kq/[E(\mathbf{q})+m]^{1/2}[E(\mathbf{k})+m]^{1/2}}{2(2\pi)^6m(z-\mu)(2qk)}, \quad (25f)$$

$$z=(k^2+q^2+m_S{}^2)/(2qk), \quad \text{and} \quad \mu=\cos\theta. \quad (25g)$$

Equations (25a)–(25e) are then inserted into Eq. (9) and using the integral formulas expressed in Eqs. (20a)–(20d) the quantities $W^J(q\lambda_4\lambda_3;k\lambda_2\lambda_1)$ are calculated. Then using Eqs. (14a)–(14f) and (19a)–(19f), the partial-wave projections of the interaction kernel are determined as

$$W^{J,0}(p,J;k,J)=\epsilon(\beta-z)Q_J(z), \quad (26a)$$

$$W^{J,1}(p,J;k,J)=\epsilon\{Q_J(z)-[(J+1)Q_{J-1}(z)+JQ_{J+1}(z)]/(2J+1)\}, \quad (26b)$$

$$W^{J,1}(p,J\pm1;k,J\pm1)=\epsilon\{Q_{J\pm1}(z)[2J(J+1)\beta+\beta+2\gamma J(J+1)]+2J(J+1)\times(\beta-\gamma)Q_{J\mp1}(z)-(2J+1)^2Q_J(z)\}/(2J+1)^2, \quad (26c)$$

$$W^{J,1}(p,J\pm1;k,J\mp1)=\epsilon[J(J+1)]^{1/2}(\beta-\gamma)[Q_{J-1}(z)-Q_{J+1}(z)]/(2J+1)^2, \quad (26d)$$

where

$$\beta=\frac{E(\mathbf{p})+m}{E(\mathbf{k})-m}\frac{k}{2p}+\frac{E(\mathbf{k})-m}{E(\mathbf{p})+m}\frac{p}{2k}, \quad (26e)$$

$$\gamma=\frac{E(\mathbf{p})+m}{E(\mathbf{k})-m}\frac{k}{2p}-\frac{E(\mathbf{k})-m}{E(\mathbf{p})+m}\frac{p}{2k}, \quad (26f)$$

and

$$\epsilon=g_S{}^2/[2(2\pi)^5m^2]. \quad (26g)$$

[14] In this work it is assumed that all interactions are invarient under rotations in isospin space. However, no explicit isospin dependence is shown in the interaction Hamiltonian for simplicity. The effect of exchanging an isovector meson is to multiply the interaction kernel by $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$, where $\boldsymbol{\tau}_i$ is the isospin operator for the ith nucleon (analogous to the Pauli spin operator $\boldsymbol{\sigma}$).

Terms involving the Kronecker δ have not been included in these formulas, since these terms cancel when a form factor of the type used in this work is introduced. Also it is easily verified that these results reduce to the usual on-shell results when $p=k$.[15]

Pseudoscalar meson. The interaction Hamiltonian is taken to be the standard direct coupling case,

$$H_I=g_P\bar{\psi}_N\gamma_5\psi_N\phi_P\,, \tag{27}$$

where $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$ and the γ_μ's obey the rule $\gamma_\mu{}^\dagger=\gamma_0\gamma_\mu\gamma_0$. The interaction kernel to order $g_P{}^2$ is given by

$$W(\mathbf{q}\lambda_4\lambda_3;\,\mathbf{k}_i\lambda_2\lambda_1)=\frac{g_P{}^2[\bar{\omega}_{\lambda_4}(\mathbf{q})\gamma_5\omega_{\lambda_2}(\mathbf{k})][\bar{\omega}_{\lambda_3}(-\mathbf{q})\gamma_5\omega_{\lambda_1}(-\mathbf{k})]}{(2\pi)^6[(\mathbf{q}-\mathbf{k})^2+m_P{}^2]}\,. \tag{28}$$

Since the partial-wave calculation is precisely analogous to the scalar-meson case, only the results are given here:

$$W^{J,0}(p,J;k,J)=\epsilon(z-\delta)Q_J(z)\,, \tag{29a}$$

$$W^{J,1}(p,J;k,J)=\epsilon[(2J+1)\delta Q_J(z)-(J+1)Q_{J-1}(z)-JQ_{J+1}(z)]/(2J+1)\,, \tag{29b}$$

$$W^{J,1}(p,J\pm1;k,J\pm1)=\epsilon[\pm Q_J(z)\mp Q_{J+1}(z)\delta]/(2J+1)\,, \tag{29c}$$

$$W^{J,1}(p,J\pm1;k,J\mp1)=\epsilon[J(J+1)]^{1/2}[(\delta\mp\rho)Q_{J-1}(z)-2Q_J(z)+(\delta\pm\rho)Q_{J+1}(z)]/(2J+1)\,, \tag{29d}$$

where

$$\delta=\frac{E(\mathbf{p})+m}{E(\mathbf{k})+m}\frac{k}{2p}+\frac{E(\mathbf{k})+m}{E(\mathbf{p})+m}\frac{p}{2k} \tag{29e}$$

and

$$\rho=\frac{E(\mathbf{p})+m}{E(\mathbf{k})+m}\frac{k}{2p}-\frac{E(\mathbf{k})+m}{E(\mathbf{p})+m}\frac{p}{2k}\,. \tag{29f}$$

As in the case of the scalar meson, these results also check with the usual on-shell results when $p=k$.[15]

Vector meson. The interaction Hamiltonian is taken to be

$$H_I=g_V\bar{\psi}_N\gamma_\mu\psi_N\phi_V{}^\mu+(f_V/2m)\bar{\psi}_N\sigma_{\mu\nu}\psi_N f^{\mu\nu}\,, \tag{30a}$$

where

$$\sigma_{\mu\nu}=(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)/2i \tag{30b}$$

and

$$f^{\mu\nu}=\partial\phi^\nu/\partial x_\mu-\partial\phi^\mu/\partial x_\nu\,. \tag{30c}$$

The interaction kernel to order $g_V{}^2$ is given by

$$W(\mathbf{p}\lambda_4\lambda_3;\,\mathbf{q}\lambda_2\lambda_1)=-\{g_V{}^2[\bar{\omega}_{\lambda_4}(\mathbf{p})\gamma_\mu\omega_{\lambda_2}(\mathbf{k})][\bar{\omega}_{\lambda_3}(-\mathbf{p})\gamma^\mu\omega_{\lambda_1}(-\mathbf{k})]+i(f_Vg_V/2m)\{[\bar{\omega}_{\lambda_4}(\mathbf{p})\sigma_{\mu\nu}\omega_{\lambda_2}(\mathbf{k})]\times[\bar{\omega}_{\lambda_3}(-\mathbf{p})\gamma^\mu\omega_{\lambda_1}(-\mathbf{k})]-[\bar{\omega}_{\lambda_4}(\mathbf{p})\gamma^\mu\omega_{\lambda_2}(-\mathbf{k})][\bar{\omega}_{\lambda_3}(-\mathbf{p})\sigma_{\mu\nu}\omega_{\lambda_1}(-\mathbf{k})]\}q^\nu+(f_V/2m)^2[\bar{\omega}_{\lambda_4}(\mathbf{p})\sigma_{\mu\nu}\omega_{\lambda_2}(\mathbf{k})][\bar{\omega}_{\lambda_3}(-\mathbf{p})\sigma^{\lambda\nu}\omega_{\lambda_1}(-\mathbf{k})]q^\mu q_\lambda\}\{(2\pi)^6[(\mathbf{p}-\mathbf{q})^2+m_V{}^2]\}^{-1}, \tag{31}$$

where $q_\mu=(\mathbf{p}-\mathbf{q},\,0)$.

Define the quantities

$$A(p\lambda_4\lambda_3;\,k\lambda_2\lambda_1)=-2\epsilon'\{g_V{}^2[T(\mathbf{p})T(\mathbf{k})+4pk\lambda_3\lambda_1][T(\mathbf{p})T(\mathbf{k})+4pk\lambda_4\lambda_2]+(2f_Vg_V/m)\times\{16[T(\mathbf{p})+T(\mathbf{k})]\lambda_1\lambda_2\lambda_3\lambda_4p^2k^2-T(\mathbf{p})T(\mathbf{k})[T(\mathbf{k})p^2+T(\mathbf{p})k^2]\}+(2f_V/m)^2\times[T(\mathbf{p})\lambda_1k-T(\mathbf{k})\lambda_3p][T(\mathbf{p})\lambda_2k-T(\mathbf{k})p\lambda_4](\lambda_3p-\lambda_1k)(\lambda_4p-\lambda_2k)-(f_V/m)^2[T(\mathbf{p})T(\mathbf{k})-4\lambda_1\lambda_3pk][T(\mathbf{p})T(\mathbf{k})-4\lambda_2\lambda_4pk][2(\lambda_1\lambda_3+\lambda_2\lambda_4)pk-\tfrac{1}{4}m_V{}^2]\}/[8pkT(\mathbf{p})T(\mathbf{k})]\,, \tag{32a}$$

$$B(p\lambda_4\lambda_3;\,k\lambda_2\lambda_1)=2\epsilon'\{4g_V{}^2[T(\mathbf{k})p\lambda_3+T(\mathbf{p})k\lambda_1][T(\mathbf{k})p\lambda_4+T(\mathbf{p})k\lambda_2]+(2f_Vg_V/m)[T(\mathbf{p})T(\mathbf{k})-4\lambda_3\lambda_1pk]\times[T(\mathbf{k})\lambda_4p+T(\mathbf{p})\lambda_2k](\lambda_3p+\lambda_1k)+(2f_Vg_V/m)[T(\mathbf{k})\lambda_3p+T(\mathbf{p})\lambda_1k][T(\mathbf{p})T(\mathbf{k})-4\lambda_2\lambda_4pk](\lambda_4p+\lambda_2k)+(f_V/m)^2[T(\mathbf{p})T(\mathbf{k})-4\lambda_1\lambda_3pk](\lambda_3p+\lambda_1k)[T(\mathbf{p})T(\mathbf{k})-4\lambda_2\lambda_4pk](\lambda_4p+\lambda_2k)\}/[8pkT(\mathbf{p})T(\mathbf{k})]\,, \tag{32b}$$

where

$$T(\mathbf{p})=E(\mathbf{p})+m \quad\text{and}\quad \epsilon'=1/[2(2\pi)^5m^2]. \tag{32c}$$

Then using Eq. (9) and Eqs. (14a)–(14f), the following relations are obtained:

$$a^J(p,k)=[(1+z)Q_J(z)-\delta_{J,0}]A(p++;\,k++)+(z-3)Q_J(z)B(p++;\,k++)\,, \tag{33a}$$

$$b^J(p,k)=[(z-1)Q_J(z)-\delta_{J,0}]A(p++;\,k--)+(z+3)Q_J(z)B(p++;\,k--)\,, \tag{33b}$$

$$e^J(p,k)=[A(p-+;\,k-+)+B(p-+;\,k-+)][JQ_{J+1}(z)/(2J+1)+Q_J(z)+(J+1)Q_{J-1}(z)/(2J+1)]\,, \tag{33c}$$

$$c^J(p,k)=-[J(J+1)]^{1/2}[A(p++;\,k-+)+B(p++;\,k-+)][Q_{J+1}(z)-Q_{J-1}(z)]\,, \tag{33d}$$

$$f^J(p,k)=[A(p-+;\,k+-)+B(p-+;\,k+-)]\{[JQ_{J+1}(z)/(2J+1)]-Q_J(z)+[(J+1)Q_{J-1}(z)/(2J+1)]\}\,. \tag{33e}$$

[15] S. Ogawa, S. Sawada, T. Ueda, W. Watari, and M. Yonezawa, Progr. Theoret. Phys. (Kyoto) Suppl. **39**, 140 (1967).

In order to obtain the partial-wave amplitudes these expressions are then inserted into Eqs. (19a)–(19f). Because these resulting expressions are unwieldy, no attempt is made to display them here. However, as a check, the on-shell limit of these expressions was taken, and a numerical comparison was made with the result of Arndt *et al.*[16]; it was found that our matrix elements agree with theirs.

IV. FORM FACTOR

In this work, the form factor given in the following equation is used[1]:

$$F(\mathbf{t})=\Lambda^2/(\Lambda^2+\mathbf{t}^2)\,, \tag{34}$$

where $\mathbf{t}=\mathbf{p}-\mathbf{q}$, and Λ is the cutoff mass. Hence the procedure used here is to insert a factor of $F(\mathbf{t})$ at each vertex of the irreducible diagram. This results in multiplying the $g_B{}^2$ interaction kernel by $[F(\mathbf{t})]^2$. This does not complicate the partial-wave projections because the quantity $[F(\mathbf{t})]^2(m_B+\mathbf{t}^2)^{-1}$ can be written in terms of partial fractions and then each of the three resulting terms treated separately.[17]

V. NUMERICAL METHODS

Equation (6) is solved numerically to obtain the K matrix and hence the phase parameters. The numerical methods and techniques are given in detail in II. Basically, Eq. (6) is reduced to a matrix problem using quadrature formulas of the Gauss type and the principal-value singularity is eliminated using the Kowalski-Noyes method.[18,19] Much time and effort was spent in ensuring that our numerical methods are accurate. In the limit of small momenta our equations reduce to the Lippman-Schwinger equation with OBEP as the potential. Hence as a check on the numerical methods this limit was taken and the phase parameters were calculated using the parameters of Bryan and Scott[2] and Ueda and Green.[1] Agreement with their results was obtained to approximately 0.1% with 20 integration points. Since in this work we wish to carry out a machine search on the parameters of our model, it is desirable to use as few integration points as possible. It was found that it is possible to obtain accuracy to better than ~1% using only eight integration points. Hence our procedure was to do the search with eight points and then check the results with 20 points.

VI. SEARCH PROCEDURE

In order to obtain a best fit to the experimentally measured observables, i.e., differential cross sections, polarizations, etc., it is desirable to minimize the following χ^2:

$$\chi^2{}_{\rm expt}=\sum_{\rm data}[(p^i{}_{\rm expt}-p^i{}_{\rm theor})/\Delta p^i{}_{\rm expt}]^2\,, \tag{35}$$

where $p^i{}_{\rm expt}$ is the measured observable, $p^i{}_{\rm theor}$ is the same calculated observable, and $\Delta p^i{}_{\rm expt}$ is the error assigned to $p^i{}_{\rm expt}$. In practice, if one were to attempt to minimize $\chi^2{}_{\rm expt}$ directly using Eq. (35), the necessary amount of computing time would be extremely large. Therefore, what has been done frequently in the past is to minimize the χ^2 defined by

$$\chi^2=\sum_{\rm phases}[(\delta^i{}_{\rm ph}-\delta^i{}_{\rm theor})/\Delta\delta^i{}_{\rm ph}]^2\,, \tag{36}$$

where $\delta^i{}_{\rm ph}$ is the phenomenological phase parameter determined from experiment, $\delta^i{}_{\rm theor}$ is the same calculated phase parameter, and $\Delta\delta^i{}_{\rm ph}$ is the error assigned to $\delta^i{}_{\rm ph}$. Phase-shift analyses based on a large number of experiments centered around laboratory energies of 25, 50, 95, 142, 210, 330, and 425 MeV have been performed by both the Yale[20] and Livermore[21] groups, and the two analyses now appear to be in good general agreement. Thus, the adjustment of OBEP models with the aid of Eq. (36) usually leads to similar coupling constants or parameters, regardless of whether the Yale or the Livermore phenomenological phase shifts are used.

A difficulty, however, with the use of Eq. (36) is that it does not take into account correlations which might, for example, account for the differences in a number of the Yale and Livermore phase assignments. Such correlation problems can cause considerable difficulties in testing theoretical models which implicity have phase-shift correlations. In an attempt to circumvent this, we made use of a method based upon the Livermore second-derivative matrix[21] which was available to one of us (A.G.). These matrices are based upon 1292 pieces of experimental data. With these matrices one calculates an approximate experimental χ^2 for a model using

$$\chi^2{}_{\rm expt}\cong\chi^2{}_{\rm expt}\big|_{\rm min}+\tfrac{1}{2}\sum_{\rm phases}\left.\frac{\partial^2\chi^2{}_{\rm expt}}{\partial\delta^i{}_{\rm ph}\partial\delta^j{}_{\rm ph}}\right|_{\rm min}(\delta^i{}_{\rm ph}-\delta^i{}_{\rm theor})\times(\delta^j{}_{\rm ph}-\delta^j{}_{\rm theor})\,. \tag{37}$$

This expression gives an approximate description of the experimental χ^2 as a function of the calculated phase parameters near the minimum.

VII. RESULTS OF SEARCH

The mesons used in our relativistic-model calculation are given in Table I. Apart from the mass of the ϵ

[16] R. A. Arndt, R. A. Bryan, and M. H. MacGregor, Phys. Rev. **152**, 1490 (1966). Several mistakes in this article concerning the vector-meson partial-wave projections were pointed out to use by R. A. Bryan (private communication).

[17] See II for details on the inclusion of the form factor.

[18] K. L. Kowalski, Phys. Rev. Letters **15**, 798 (1965).

[19] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965).

[20] R. E. Seamon, K. A. Friedman, and G. Breit, Phys. Rev. **145**, 779 (1966).

[21] M. H. MacGregor, R. A. Arndt, and R. M. Wright, Phys. Rev. **169**, 1128 (1968); **182**, 1714 (1969).

TABLE I. Mesons and coupling constants. Rel denotes relativistic model. NR denotes nonrelativistic OBEP based upon masses used in relativistic model. Also shown are f_ρ/g_ρ, the regularization parameter Λ, and χ^2_{expt} per datum. The last column gives external values of coupling parameters.

Sym	J^p	T	m (MeV)	Rel	N.R.	Ext.
π	0^-	1	138.7	14.19	13.03	14.6
η	0^-	0	548.5	3.09	5.07	2
ϵ	0^+	0	570*	6.97	7.32	
δ	0^+	1	960	0.33	0.48	
ω	1^-	0	782.8	9.92	11.05	9.1
ρ	1^-	1	763	0.43	0.48	0.65
f_ρ/g_ρ				6.38	5.53	4
Λ (MeV)				1414	1366	
χ^2/datum				4.6	5.3	

and the regularization parameter Λ which were varied in the search, all meson masses were taken at values given in recent tables of Rosenfeld *et al.*[9] Mesons with masses heavier than 1 BeV were omitted since our regularization procedure already parametrizes such short-range effects.

Using Eq. (37) and the second derivatives of MacGregor *et al.*[21] we found a minimum χ^2 per datum of 4.59 at the coupling parameters given in the column headed Rel. The theoretical phase parameters are given by the solid curve in Fig. 1 in relationship to the single-energy results of Livermore (points with error bars) and the energy-dependent phase shifts of Breit *et al.*[20] (open circles). Also shown in Fig. 1 by dashed curves are the phase parameters based upon a nonrelativistic OBEP with the same coupling constants and masses used in our relativistic model. The comparison is meant only to show that the relativistic model gives significantly different results than OBEP with the same input parameters in the energy region of interest.

If, using the same set of meson masses, we readjust the nonrelativistic OBEP by the matrix method, we find the coupling constants in the column headed OBEP in Table I. This gives a χ^2 per datum of 5.3, which is not as good as our relativistic model. We should note, however, that certain less-restricted variations of the nonrelativistic OBEP models[22] do achieve better fits than the restricted OBEP model specified in Table I. However, it would appear in general that the relativistic model helps improve fits to certain phase parameters ($^1D_2$, $^3P_2$, and $^3D_2$) which have characteristically been hard to fit with nonrelativistic models.

In this work Coulomb effects have been allowed for on the basis of earlier OBEP work in which accurate Coulomb corrections were made by including the Coulomb potential for the $T=1$ states and matching to Coulomb wave functions at some large distance.[23] The Coulomb correction was found to be significant only for the $^1S_0$ state, as was shown by running the OBEP code with and without the Coulomb potential. In the present work we simply added the Coulomb correction as determined from the OBEP code (see Table II) with the U.G.I parameters[1] to our $^1S_0$ state. This procedure is approximate but should be accurate enough beyond 25 MeV.

In this work the lowest energy considered in our data-fitting procedure is $E_{\text{lab}}=25$ MeV. Because of this it would not be surprising if the deuteron parameters calculated from our model did not agree precisely with experiment. Using effective-range theory,[24] our deuteron binding energy is estimated to be ~ 3 MeV as opposed to the experimental value of ~ 2.2 MeV. In

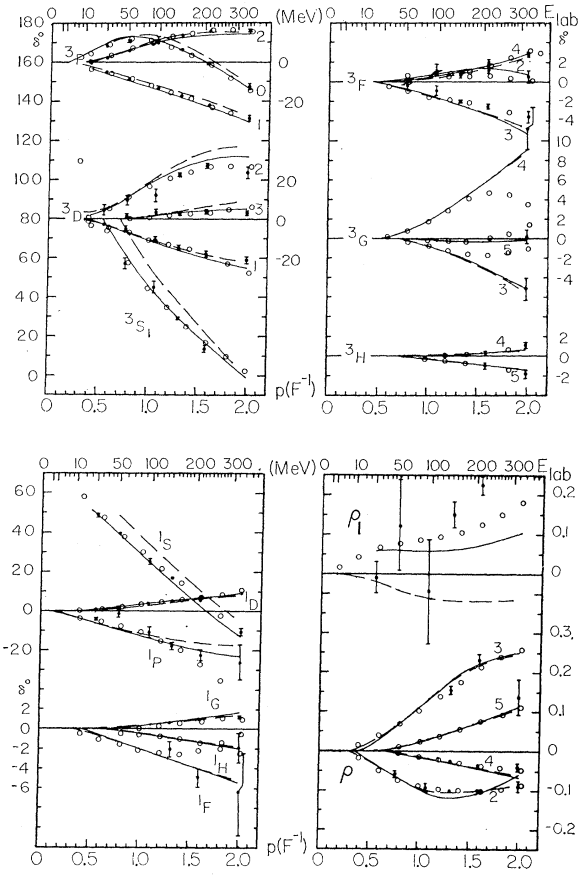

FIG. 1. *N-N* phase parameters are shown vs laboratory kinetic energy. The solid curves are calculated from the relativistic theory, the dashed curves are calculated from OBEP with the masses and coupling constants taken from the relativistic calculation. The experimental points indicated with error bars are from the energy-independent phase-parameter analysis of MacGregor *et al.* (Ref. 21). The open circles are the energy-dependent phase shifts of Breit *et al.* (Ref. 20).

[22] R. W. Stagat, F. Riewe, and A. E. S. Green, Phys. Rev. (to be published).

[23] The original OBEP code due to T. Sawada was modified by him and one of the authors (A. Gersten) to include Coulomb effects.

[24] Alexander Gersten and A. E. S. Green, Phys. Rev. **176**, 1199 (1968).

TABLE II. Coulomb corrections used for the $^1S_0$ state.

E_{lab} (MeV)	25	50	95	142	210	330	425
$\delta^1S_0(np)-\delta^1S_0(pp)$ (deg)	2.0	1.5	1.9	2.4	3.2	4.2	5.0

our previous experience with OBEP models,[25] small changes in the parameters were found to be sufficient to correct a moderate discrepancy in the deuteron binding energy.

VIII. DISCUSSION

The last column of Table I gives values of the coupling constants obtained from experimental or theoretical investigations not involving N-N scattering. For example, g_ρ^2 is predicted to be 0.65 by Sakurai[26] and f_ρ/g_ρ is known to be approximately 4 from form-factor data.[27] These are not far removed from the values 0.43 and 6.4 found in our work. According to the calculation of Oakes and Sakurai,[25] the ratio of g_ω^2 to g_ρ^2 should be 14, which would give $g_\omega^2 \simeq 9$ if $f_\rho/g_\rho=4$ and $g_\rho^2=0.65$. This is quite comparable to our value of 9.9. The constant $g_\eta^2=2$ is based upon $SU(3)$ predictions. It is not too far removed from our $g_\eta^2=3.09$, although characteristically N-N predictions are rather insensitive to this constant. The π-meson coupling constant 14.6 is based upon π-nucleon scattering studies. It is in reasonable accord with the value 14.2 which we find. The δ, a newly discovered scalar meson,[9] does not play a large role in the N-N interaction except perhaps for the S waves. Thus there is considerable uncertainty as to its coupling strength.

The ϵ meson has perhaps been the most controversial of the recent resonances but now appears in the Rosenfeld tables.[9] It appears that this resonance has a very broad mass distribution. This broad distribution has recently been found to be helpful in explaining certain features of the N-N interaction in nonrelativistic treatments of OBEP.[22] In the present work the mass of the ϵ is taken as an adjustable parameter. The effective value (570 MeV) obtained is quite reasonable in view of the actual resonance value ($\sim$740 MeV) and the broad width. The coupling constant g_ϵ^2 is somewhat smaller than the g_ω^2. In view of the lighter mass of ϵ we still preserve the essential feature of OBEP models in the approximate cancellation of the static effects of the scalar and vector mesons.

[25] For a review of the coupling constants and masses used in various models of the N-N interaction, see G. Breit, in International Conference on Properties of Nuclear States, Montreal, 1969 (unpublished).

[26] R. J. Oakes and J. J. Sakurai, Phys. Rev. Letters 19, 1266 (1967).

[27] T. A. Griffy and L. I. Schiff, in *High Energy Physics*, edited by E. H. S. Burhop (Academic, New York, 1967), Vol. I.

IX. SUMMARY

In this work we have attempted to amend some of the deficiencies in the OBEP model of the N-N interaction. Specifically, the interaction is not expanded in powers of the momentum divided by m, but rather the momentum dependence is treated by solving the integral equation for the scattering amplitude directly in the momentum space. Also by working with a covariant approximation to the Bethe-Salpeter equation, relativistic effects are included in the calculation.

The fact that the χ^2_{expt} per datum of our relativistic treatment is somewhat lower than that given by the nonrelativistic treatment is probably not of great importance. More remarkable is the over-all resemblance of the parameters obtained in the two treatments. Furthermore, the fact that the parameters, i.e., masses and coupling strengths, obtained in this work as a result of searching on the data, are compatible with independent estimates and measurements of these parameters lends credence to it. This property has been lacking in some models of the N-N interaction.[25]

In the future, the relativistic calculation performed here should be particularly useful for extending the OBE hypothesis into the higher-energy region where relativistic effects become more profound. Of course, the application to higher energies requires the inclusion of inelastic effects. This could be approached through the coupled-channel formalism or in a more phenomenological way, say by introducing complex potentials. The Los Alamos meson factory should provide data in the high-energy region which might enable one to pursue such an approach.

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