S-Matrix Calculations of the Pion-Nucleon P-Wave Scattering Length a_{13} by the N/D Method*

V. F. Šachl
Department of Mathematics, University of Durham, Durham, England
and
Institute of Physics, Czechoslovak Academy of Sciences, Prague, Czechoslovakia

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The S-matrix calculations for the P-wave scattering process of pions on nucleons are carried out with the help of the N/D method. According to the Balázs method the numerator function N is considered in the three-pole effective-range approximation. We choose one of the poles in the nucleon pole, where the residue is known. The remaining two residues (b_i, $i=1$, 2) are to be calculated. We use the following input values: the ρ-meson and pion-nucleon (3,3) isobar N^* discontinuities and the high-energy contributions in the crossed channels, and the nucleon pole in the direct channel. The matching procedure is the usual one: The right-hand side of one of the determining equations for the effective-range parameters b_i is analytically established by the projections of the invariant amplitudes with the aid of the fixed-s dispersion relations in a regular point s_R in the unphysical region. Then the complete determining equations are obtained by comparing these projections and their derivatives, respectively, in the matching point s_R. Having thus determined the parameters b_i and the effective-range N/D amplitude of the state $J=\frac{3}{2}$, $I=\frac{1}{2}$, we compute the scattering length a_{13} on the basis of the noncorrelated physical parameters (g^2,m,W_R^{33},μ). Its magnitude $a_{13}=-0.019$ is in excellent agreement with the experimental value of Barnes *et al.*

I. INTRODUCTION

In some earlier papers on strongly interacting particles,[1-3] the S-matrix calculations of the low-energy partial-wave amplitudes, which are consistent with the requirements of analyticity, elastic unitarity, and crossing symmetry, have been carried out. For special processes, which generally depend on physical quantities at the input and output, the natural requirement of self-consistency in these calculations must be added. However, the N/D method used here usually leads to integral or integro-differential equations which may, as a rule, be solved only numerically.

An approximation scheme which enables us to obtain expressions for the partial-wave amplitudes which are more suitable for analytical evaluation and consistent with the aforementioned requirements, in a certain region, has been worked out by Balázs.[4] In contrast to the former approach, the nearby singularities of the left-hand cut in this scheme—called the effective-range approximation—are replaced by several (in our case two) poles.

The N/D equations are treated therefore in such a way that the numerator function N is given in the form of a sum of fractions $\sum_i b_i/(s-s_i)$ in which s_i denote the chosen poles in the s plane and the effective-range parameters b_i coincide with the corresponding residues. The essence of this method, a modification of which is used here,[5] consists in the determination of the residues. The detailed procedure for that purpose is described below. To check all these assumptions the resulting N/D partial-wave amplitude is then applied in determining the P-wave scattering length a_{13}.

The equations and their effective-range parameters (residues) b_i in our approach are determined by comparison with the theoretically given input values in the point s_R in the unphysical region. For more details, we have to say that the basic idea of the present paper lies in the special choice of these values. We use the analytical continuation of expressions which represent the contributions of the discontinuities of the ρ meson and pion-nucleon (3,3) isobar N^* in the crossed channels and the nucleon pole in the direct channel. These expressions are well known for the low-energy region and are discussed by Frautschi and Walecka.[6] The high-energy contributions in the crossed channels are also taken into account and expressed by the relationship between the states in various channels, i.e., between the high-energy states of the high angular momenta in the crossed channels and the low-energy resonance in the direct channel. The corresponding contributions to the direct channel are then added in the sense of their analytical continuations into the regular point s_R.

The choice of the matching point s_R is important in two respects: (1) The contributions of the other singularities not considered on the input must be relatively unimportant, so that the residues may be determined in a sufficiently high approximation. (2) It is necessary that the expressions for the partial-wave amplitude are convergent. The point must lie in the unphysical region where the unsubtracted dispersion relations are valid.

After carrying out the corresponding projections into the point s_R, which represent the input values, the residues b_i for the N/D partial-wave amplitude $J=\frac{3}{2}$, $I=\frac{1}{2}$, are then found by the following method.

The system of the n determining equations for n un-

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known parameters b_i is represented in the point s_R by equations $(N/D)^{(j)}=f^{(j)}(s_R)$, $j=0, 1, \ldots, n-1$, where j denotes the degree of the derivative. We use three poles at one of which (the nucleon pole) the residuum (b_3) is known, and therefore we have to determine only two unknown parameters b_i, $i=1, 2$.

The described procedure is carried out in Sec. II. Then to verify the correctness of the calculations of the P-wave N/D partial-wave amplitude with the isospin $I=\frac{1}{2}$ in this approximation, we calculate the pion-nucleon a_{13} scattering length, which we compare with the experimental value.

We use the following noncorrelated physical constants as the input values: coupling constant g, the nucleon mass m, the mass W_R, and the width γ_{33} of the (3,3) resonance N^*. Further, we chose the magnitudes of the π-meson mass equal to unity, and the natural system of units, i.e., $\hbar=c=\mu=1$. In these units the notation is the same as in Ref. 6.

II. S-MATRIX CALCULATIONS AND N/D METHOD

In the S-matrix theory of strong interactions we want to calculate here the effective-range formula for the scattering of π mesons on nucleons for the state $J=\frac{3}{2}$, $I=\frac{1}{2}$. Of special interest to us will be the determinations of the parameters in the formula and then with the aid of it, the calculation of the scattering length a_{13}.

In order to avoid a cut along the imaginary axis, we work with the N/D method in the complex plane of the total energy squared $W^2(=s)$. The barycentric system is used for two independent variables of the pion-nucleon scattering as the energy s and angle ϑ. Because of the chosen units $\hbar=c=\mu=1$, where μ is the pion mass, we denote the nucleon mass as m, which is identical with what is denoted M/μ in Ref. 6.

The leading idea of our approach is the comparison of the integral N/D representation on the one hand with the partial-wave projections ${}^If_l(s)$, $(l=J-\frac{1}{2})$, derived theoretically on the basis of the S-matrix theory, on the other hand. Meanwhile, the former is a function of the effective-range parameters b_i, which are to be determined; the latter we shall construct in the spirit of the pole theory with the aid of invariant Mandelstam amplitudes ${}^IA_l{}^{\pm}$, ${}^IB_l{}^{\pm}$:

$$^{I}N_l(s,b_i)/{}^ID_l(s,b_i)={}^If_l({}^IA_l{}^{\pm}(s),{}^IB_l{}^{\pm}(s)). \qquad (1)$$

On the basis of the pole theory of the process under consideration, the invariant amplitudes ${}^IA_l{}^{\pm}$, ${}^IB_l{}^{\pm}$ may be derived here from the fixed-s dispersion relations in a known way. In particular, limiting ourselves to pion-nucleon scattering, some of the projections ${}^If_l(s)$ (given in terms of the invariant amplitudes ${}^IA_l{}^{\pm}$, ${}^IB_l{}^{\pm}$) are identical with those given in Ref. 6. However, contrary to these authors, our approach is more general, owing to the fact that the high-energy states of the high angular momenta in the crossing channels t and u are also included.

The approximations we shall make in the N/D integral representation of Eq. (1) are the following: On the left-hand side the numerator function N is considered in the framework of the effective-range theory and generally in the n-pole approximation, as a sum of n fractions

$$^{I}N_l(s)=\sum_{i=1}^{n}\frac{b_i}{s-s_i}, \quad i=1,2,\cdots,n. \qquad (2)$$

Here s_i are the poles in the complex s plane which need to be chosen, and the parameters b_i are the unknown residues which belong to them.

For processes whose amplitude fulfil the requirement of analyticity in the energy variable and which are expressed with the aid of dispersion relations with one subtraction, the denominator function D, in the elastic approximation, may be written as

$$^{I}D_l(s)=1-\frac{s-s_0}{\pi}\int_{s_t}^{\infty}ds'\frac{\rho(s')}{(s'-s_0)(s'-s)}\,{}^IN_l(s'), \quad (3)$$

where the numerator function ${}^IN_l(s)$ is defined by Eq. (2),[4] s_t is the square threshold energy, s_0 is the subtraction point, and $\rho(s')$ is the discontinuity on the right-hand cut, free of kinematical singularities, which begins at s_t.

Suppose now that we know the right-hand side of Eq. (1) expressed analytically in a part of the physical region. By an analytical continuation into the neighboring unphysical region, in which the influence of unconsidered singularities has become very small, it represents a satisfactorily high approximation for the physical partial-wave amplitude ${}^If_l(s_R)$. In this way we can obtain from Eq. (1) the determining equations for the unknown parameters b_i.

In the n-pole approximation[7] for the numerator function ${}^IN_l(s)$, we generally get the equations for the parameters b_i by taking the derivatives up to the $(n-1)$st order on both sides of Eq. (1).

By means of the N/D amplitude analytically continued back into the energy range of the physical region, we may obtain the partial-wave amplitudes which fulfil all the above-mentioned requirements.

Further, we limit ourselves to the elastic scattering of π mesons on nucleons specified by the quantum numbers $J=\frac{3}{2}$ and $I=\frac{1}{2}$ $(l=1+)$.

We use the normalization of the amplitude ${}^{1/2}f_{1+}$ in the form

$${}^{1/2}f_{1+}=\frac{W^2}{q^3}e^{i^{1/2}\delta_{1+}}\sin{}^{1/2}\delta_{1+}, \qquad (4)$$

which is the same as that of Ref. 6. Here q denotes the three-momentum of the pion-nucleon scattering and is given below by Eq. (8).

In two-particle unitarity the specification of symbols in the N/D representation in Eqs. (1)–(3) is the following:

[7] For the details of the used method see Ref. 4.

We choose the poles s_i in the three-pole approximation for the ${}^{1/2}N_{1+}$ function in the points

$$s_1=-m^2,\quad s_2=-16m^2,\quad s_3=m^2, \tag{5}$$

where m denotes the nucleon mass. The parameters b_i are then the corresponding residues.

The residue b_3 of the nucleon pole is, however, well known in the literature,[6,8]

$$b_3=(8/3)\mathfrak{f}^2m^2\,{}^{1/2}D_{1+}(m^2),\quad \mathfrak{f}^2=(g/2m)^2=0.081, \tag{6}$$

so that we have two unknown residues b_1 and b_2 which completely determine the N/D amplitude in the low-energy region.

In the ${}^{1/2}D_{1+}$ function (3), the threshold energy of the pion-nucleon scattering is $s_t=\alpha_+$, $\alpha_\pm=(m\pm1)^2$, and the subtraction point may therefore be suitably chosen in $s_0=\alpha_-$. The discontinuity $-\rho(s)$, which is also related to the imaginary part of the amplitude ${}^{1/2}f_{1+}(s)={}^{1/2}A_{1+}(s)$, is given in pion-nucleon elastic unitarity as

$$\operatorname{Im}\,{}^{1/2}A_{1+}^{-1}(s)=-\rho(s)=-q^3(s)/s. \tag{7}$$

Here the square of the three-momentum $q^2(s)$ may be derived in an explicit form:

$$q^2(s)=(4s)^{-1}(s-\alpha_+)(s-\alpha_-),\quad \alpha_\pm=(m\pm1)^2. \tag{8}$$

Note that Eq. (8) is identical with (2.9) of Ref. 6.

On the right-hand side of Eq. (1), the invariant amplitudes ${}^IA^i$ $(\equiv{}^IA^\pm,\ {}^IB^\pm)$ fulfil the fixed-s dispersion relations represented by

$${}^IA^i(s,u,t)=\frac{R_s{}^i}{m^2-s}+\frac{R_u{}^i}{m^2-u}+\frac{1}{\pi}\int_{\alpha_+}^{\infty}\frac{A_u{}^i(u',s)}{u'-u}du' +\frac{1}{\pi}\int_4^{\infty}\frac{A_t{}^i(t',s)}{t'-t}dt'. \tag{9}$$

Of all states in channels u and t in the low-energy region we retain only the states described by the quantum numbers $J=\frac{3}{2}$, $I=\frac{3}{2}$ in the u channel and $J=1$, $I=1$ in the t channel. They are well known from the previous study.[6]

The contributions of these states are well represented by the first and second integrals in Eq. (9) with integration limits restricted to the low-energy intervals in the corresponding crossed channels u and t, i.e., by the (3,3) resonance and the ρ meson, respectively. The remaining parts of the integrals may be expressed in terms of the low-energy resonances in the direct channels as

$$\frac{1}{\pi}\int_{\text{high }u}\frac{A_u(u',s)}{u'-u}du'+\frac{1}{\pi}\int_{\text{high }t}\frac{A_t(t',s)}{t'-t}dt' \simeq\frac{1}{\pi}\int_{(3,3)}\frac{A_s(s',t)}{s'-s}ds'. \tag{10}$$

Thus we get the amplitude (9) distributed into all three channels s, u, and t as follows:

$${}^IA^i(s,u,t)\simeq\frac{R_s{}^i}{m^2-s}+\frac{R_u{}^i}{m^2-u}+\frac{1}{\pi}\int_{(3,3)}\frac{A_u{}^i(u',s)}{u'-u}du' +\frac{1}{\pi}\int_{\rho}\frac{A_t{}^i(t',s)}{t'-t}dt'+\frac{1}{\pi}\int_{(3,3)}\frac{A_s{}^i(s',t)}{s'-s}ds'. \tag{11}$$

Analytical continuation into the point $s=\alpha_-=(m-1)^2$ in the unphysical region and projections of the invariant amplitudes ${}^IA^i$ yield then a sufficiently high approximation for the amplitude ${}^{1/2}f_{1+}(s)$ [Eq. (11)]:

$${}^{1/2}f_{1+}(s)={}^{1/2}f_{1+}{}^{(N)}(s)+{}^{1/2}f_{1+}{}^{(N^*)}(s)+{}^{1/2}f_{1+}{}^{(\rho)}(s) +{}^{1/2}f_{1+}{}^{(h)}(s);\quad s=s_R=\alpha_-. \tag{12}$$

Here the particular contributions to the amplitude come from the nucleon pole N in the direct channel s, the (3,3) resonance N^*, the ρ meson, and the high-energy contributions in the crossed channels u and t.

Their explicit forms have been partly derived.[6] We express them by the analytical formulas $(s=x^2)$

$${}^{1/2}f_{1+}{}^{(N)}(x)=\tfrac{1}{8}g^2[(x-m)\beta_+Q_1(\alpha_1)/q^4 +(x+m)\beta_-Q_2(\alpha_1)/q^4], \tag{13}$$

$${}^{1/2}f_{1+}{}^{(N^*)}(x)=-\frac{W_R{}^2}{18}\mathfrak{f}^2q_r{}^2\left\{[3x^*(W_R+2m-x)\beta_{R+}{}^{-1}+(W_R-2m+x)\beta_{R-}{}^{-1}]\beta_+\frac{Q_1(\alpha_2)}{q^4} +[3x^*(W_R+2m+x)\beta_{R+}{}^{-1}+(W_R-2m-x)\beta_{R-}{}^{-1}]\beta_-\frac{Q_2(\alpha_2)}{q^4}\right\}, \tag{14}$$

$${}^{1/2}f_{1+}{}^{(\rho)}(x)=\frac{3}{16\pi}\left\{\left[\frac{\gamma_2}{m}(x^2+\tfrac{1}{2}t_R-m^2-1)-(x-m)(\gamma_1+2\gamma_2)\right]\beta_+\frac{Q_1(\alpha_3)}{q^4} +\left[\frac{\gamma_2}{m}(x^2+\tfrac{1}{2}t_R-m^2-1)+(x+m)(\gamma_1+2\gamma_2)\right]\beta_-\frac{Q_2(\alpha_3)}{q^4}\right\}, \tag{15}$$

$${}^{1/2}f_{1+}{}^{(h)}(x)=\frac{\mathfrak{f}^2}{3}\frac{W_R{}^2}{W_R-x}\frac{\beta_+}{\beta_{R+}}. \tag{16}$$

[8] G. F. Chew and F. E. Low, Phys. Rev. 101, 1570 (1956).

Here the three-momentum q is given by Eq. (8), as well as the quantity q_τ; in the latter case the variable s is replaced by the square W_R^2, i.e., the position of the (3,3) resonance in the complex s plane. The quantities $Q_i(\alpha_j)$ $(i=1, 2)$ are Legendre polynomials of the second kind, of order i. The symbols $\beta_{R\pm}$, $\beta_\pm$, α_i, ξ_i $(i=1, 2, 3)$, and x^* denote

$$\beta_{R\pm}=(W_R\pm m)^2-1\,,\quad \beta_\pm=(x\pm m)^2-1\,, \qquad (17)$$

$$\alpha_k=\xi_k/q^2-1 \quad (k=1,2)\,,$$

$$2\xi_1=x^2-m^2-2\,,\quad 2\xi_2=x^2-2m^2-2+W_R^2\,, \qquad (18)$$

$$\alpha_3=\xi_3/q^2+1\,,\quad 2\xi_3=t_R\,,$$

$$-x^*=\xi_2/q_\tau^2-1\,, \qquad (19)$$

where W_R indicates the position of the (3,3) resonance with the width $\gamma_{33}=\frac{4}{3}f^2$ and t_R is the position of the π-π resonance. For the factors γ_1 and γ_2 we take theoretical values from the study of the electomagnetic structure of the nucleon, $\gamma_1=-4.91$ and $\gamma_2=-11.7$.[9]

Substituting relations (13)–(16) into Eq. (12), we obtain one of the determining equations for the unknown quantity b_1. Then we get the second one for the parameter b_2 by differentiation of both sides of Eq. (1).

By the analytical continuation back to the low-energy range of the physical region, we find the N/D P-wave amplitude of the state $I=\frac{1}{2}$. It is determined by the effective-range parameters b_1 and b_2 and by Eqs. (3) and (2) supplemented by Eqs. (6)–(8).

The numerical calculations have been carried out using the noncorrelated physical quantities

$$g^2=14.97,\quad m=6.7974,\quad W_R=8.9674 \quad (\mu=1) \qquad (20)$$

given by experiment.

To make sure about the validity of the derived formulas in the low-energy range of the physical region, we may test them by means of the pion-nucleon scattering length of the state $J=\frac{3}{2}$, $I=\frac{1}{2}$.

Generally, the scattering length for the pion-nucleon scattering process is defined with the help of the N/D amplitude [see Eqs. (1)–(3)]

$$a_{2I,2J}={}^IN_l(\alpha_+)/\alpha_+{}^ID_l(\alpha_+)\,,\quad \alpha_+=(m+1)^2. \qquad (21)$$

Formula (21) applied for the P-wave amplitude, $I=\frac{1}{2}$, yields, with the aid of Eqs. (2)–(7) and computed values of the parameters b_i $(i=1, 2, 3)$,

$$a_{13}={}^{1/2}N_{1+}(\alpha_+)/\alpha_+\,{}^{1/2}D_{1+}(\alpha_+)=-0.019. \qquad (22)$$

This value is in excellent agreement[10] with the experimental value of Barnes *et al.*[11]

$$a_{13}=-0.016 \quad (\pm 0.008). \qquad (23)$$

It is necessary to note that the S-matrix calculations of scattering length (22) have been based only on the axioms of the S-matrix theory and, of course, on several postulates of quantum field theory on which this theory is established, on the assumption about elastic unitarity of the amplitude, and on the approximation (10) for the high-energy states of high angular momenta in the crossed channels. Besides that, the only physical input quantities taken from experiment are the nucleon mass, coupling constant g^2, and the position of the (3,3) resonance in pion units. Otherwise there are no free parameters in the theory and no cutoff is introduced.

By means of the same procedure the other quantities of the state $J=\frac{3}{2}$, $I=\frac{1}{2}$ may also be determined.[12] From the N/D partial-wave amplitude it is possible to derive the general formula for the position of the (1,3) resonance. This will be dealt with elsewhere.

According to the simple theoretical relations of the partial-wave amplitudes $I=\frac{1}{2}$ and $I=\frac{3}{2}$, the analogous quantities of the state $J=\frac{3}{2}$, $I=\frac{3}{2}$, may be calculated.[13] In this respect two remarks are important.

The same input values for both states $I=\frac{1}{2}$ and $I=\frac{3}{2}$ may fail to yield the same degree of approximation unless the inelastic processes are sufficiently taken into account.[14]

In the calculation of the (3,3) resonance, another difficulty arises—the problem of self-consistency. The input and output values W_{33} and γ_{33} in this case must be equal and thus correlated with the input value throughout the calculations.

We assume here that theoretically and especially by including higher inelastic processes it would be in principle possible to obtain the limiting values of these quantities W_{33} and γ_{33}. For this reason we have started in this paper from sufficiently verified experimental values.

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[9] V. Singh and B. M. Udgaonkar, Phys. Rev. **128**, 1820 (1962).

[10] Although there are also other less accurate measurements and evaluations of the scattering length a_{13} by different, even less direct methods (see Refs. 10 and 13), it is necessary to say that our value a_{13} always lies within the limits of observational errors.

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