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Lower Bound on $W^{\pm}$-Meson Production in High-Energy Proton-Proton Collisions*

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A lower bound is obtained for differential cross sections of $W^{\pm}$-meson production in high-energy proton-proton collisions. Use is made of a current-algebra sum rule derived from time components of weak currents, and of a Regge argument based on duality. Additional physical assumptions enable us to calculate a lower bound on the sum of the total production cross sections for W^+ and W^- mesons. The bound turns out large enough to be of practical significance in future experiments, although it is not applicable to data at presently available energies.

I. INTRODUCTION

THE existence of bosons which mediate the weak interactions has been discussed for many years.[1] From a theoretical viewpoint, it has a rather close analogy with the electromagnetic interaction, and, after suitable modification, it might be possible to make the weak interactions less singular than four-fermion interactions. But there is so far no evidence for the intermediate weak bosons (W mesons), and only a lower bound has been set on the mass of the W mesons ($m_W \gtrsim 2$ GeV), if they exist, through high-energy neutrino experiments.[2–4] Another attempt has been made to detect the W mesons, by measuring energetic muons in high-energy proton-proton collisions.[5,6] The energetic muons would originate (see Fig. 1) from the $\mu\nu$ decay of W mesons produced in the collision

$$p+p \to W^{\pm}+\text{anything} \\ \searrow \mu^{\pm}+\nu_\mu(\bar{\nu}_\mu). \tag{1.1}$$

Because of background muons due to electromagnetic pair production,

$$p+p \to \gamma+\text{anything} \\ \searrow \mu^{+}+\mu^{-}, \tag{1.2}$$

and because of limitations on incident proton energies, they succeeded only in setting an upper bound on the production cross sections of the W mesons.[7]

Now, however, the muon pairs arising from virtual photons have been studied for the $(\mu\bar{\mu})$ invariant mass

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[7] In spite of the muon pair of electromagnetic origin, the limits set in Ref. 6 are still valid since no muons from short-lived parents were observed. It is important to note that the virtual photon competition depends sensitively on the effective resolution of the W-meson search.

up to about 6.7 GeV in proton-proton collisions,[8] and available proton energies will soon be as high as 500 GeV after the National Accelerator Laboratory (NAL) machine is completed, and even higher at the Intersecting Storage Ring (ISR) machine (~60 GeV in the c.m. frame).[9] The same experiments as (1.1) will then be repeated with more success at NAL and ISR. It is therefore very desirable to give a theoretical estimate of the W-meson production cross sections for this process. There exists practically no quantitative discussion except for its comparison with electromagnetic muon pairs.[10]

In the present paper we try to extract quantitative information on the W-meson-production cross sections from knowledge of the weak currents alone. The weak currents are fairly well established experimentally and equal-time commutators between their time components are the most well-known part of the current-algebra hypothesis. With this ingredient together with Regge asymptotic behavior at high energies, we derive a lower bound on the sum of the W^+- and W^--production cross sections in high-energy proton-proton collisions.

We use an equal-time commutator of two time components of the weak currents, which is the part of the current algebras best established.[11] Therefore, our sum rule derived from it is a version of the Fubini (Adler–Dashen–Gell-Mann–Fubini) sum rule[12] with a two-particle scattering state as a target. When we rewrite it in terms of the $W^{\pm}$-meson-production cross sections, we resort to Regge high-energy behavior based on a duality argument. The resulting sum rule leads to a lower bound on a sum of integrals involving differential cross sections at 90° for the W-meson production. By introducing a physical assumption on an angular distribution of the W-meson production by analogy with hadron productions, we obtain a modest lower bound on a sum of the total $W^{\pm}$-meson-production cross sections. The lower bound turns out to increase linearly in s (the square of the c.m. energy). However, the weak interactions are taken into account to the lowest order, and so our lower bound should not be taken seriously at energies higher than the unitarity cutoff (≃300-GeV c.m. energy).

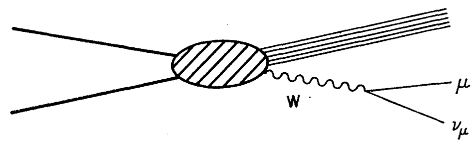

FIG. 1. W-meson production in proton-proton collision.

II. KINEMATICS

The cross section for the lepton-pair production through the $W^{\pm}$ meson in proton-proton collisions is given by

$$d\sigma^{\pm}=\frac{2}{v_{\rm rel}}\left(\frac{m_l m_{\bar{l}}}{l_0\bar{l}_0}\right)\frac{g^4}{(q^2+m_W^2)^2+m_W^2\Gamma^2}\times j_\mu^{\pm} j_\nu^{\mp} W_{\mu\nu}^{\pm}\frac{d^3l}{(2\pi)^3}\frac{d^3\bar{l}}{(2\pi)^3}, \qquad (2.1)$$

where the matrix element $W_{\mu\nu}^{\pm}$ is defined as

$$W_{\mu\nu}^{\pm}=\pi(2\pi)^3\sum_n \langle \mathbf{p}_1\mathbf{p}_2^{\rm in}|J_\mu^{\pm}(0)|n\rangle \times\langle n|J_\nu^{\mp}(0)|\mathbf{p}_1\mathbf{p}_2^{\rm in}\rangle\delta(p_1+p_2+q-P_n). \qquad (2.2)$$

The notation is as usual; the weak Hamiltonian is given in the current×current form

$$H_w=g(J_\mu^+ + j_\mu^+)W_\mu+\text{H.c.}, \qquad (2.3)$$

$j_\mu^{\pm}$ is the lepton current,

$$j_\mu^-=i\bar{u}_l(1)\gamma_\mu(1+\gamma_5)v_\nu(\bar{1}), \qquad (2.4)$$

$$j_\nu^+=i\bar{u}_\nu(1)\gamma_\mu(1+\gamma_5)v_l(\bar{1}), \qquad (2.5)$$

and $J_\mu^{\pm}$ is the weak hadronic current,

$$J_\mu^{\pm}=(\mathfrak{F}_{1\mu}\pm i\mathfrak{F}_{2\mu}+\mathfrak{F}_{1\mu}^5\pm i\mathfrak{F}_{2\mu}^5)\cos\theta_C +(\mathfrak{F}_{4\mu}\pm i\mathfrak{F}_{5\mu}+\mathfrak{F}_{4\mu}^5\pm i\mathfrak{F}_{5\mu}^5)\sin\theta_C, \qquad (2.6)$$

where $\mathfrak{F}_{i\mu}$ and $\mathfrak{F}_{j\nu}^5$ are the vector and axial-vector unitary-spin currents which obey the ordinary algebra of currents proposed by Gell-Mann,[11] and θ_C is the Cabibbo angle ($\theta_C=0.21\sim0.27$).[13] Considering actual experimental situations, the protons are understood as averaged over spin and the lepton-pair spins are summed over.

The relevant momenta are as follows:

p_1,p_2,	initial protons
$-q$,	W meson
$l,\bar{l}$,	lepton and antilepton.

The coupling constant g is related to the Fermi coupling constant ($G=1.023\times10^{-5}m_p^{-2}$) through

$$g^2/m_W^2=G/\sqrt{2}, \qquad (2.7)$$

where m_W is the mass of the W meson.

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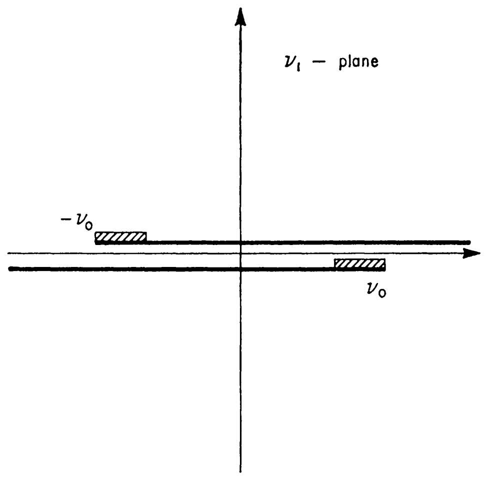

FIG. 2. Analytic structure on the complex ν_1 plane of a $p_1p_2 W \to p_1p_2W$ scattering amplitude. The locations of the branch points are given by $\nu_0=\frac{1}{2}(s-q^2-4m_N^2)$ and $-\nu_0$. The hatched regions near the branch points are the "low-energy" regions where some structure is expected in the s channels. Possible anomalous thresholds are not considered here.

In order to simplify the following calculation, we make the approximation that the weak hadronic currents are strictly conserved. The strangeness-changing currents are largely suppressed through the Cabibbo angle ($\tan^2\theta_C\simeq 0.05$), and the pion mass is small enough. Our approximation is therefore expected to be very good. In this approximation, namely,

$$q_\mu W_{\mu\nu}{}^{\pm}=q_\nu W_{\mu\nu}{}^{\pm}=0\,, \tag{2.8}$$

we can write down the matrix element $W_{\mu\nu}$ as

$$\begin{aligned}W_{\mu\nu}{}^{\pm}=(m_N{}^2/p_{10}p_{20})\{&(-\delta_{\mu\nu}q^2+q_\mu q_\nu)W_1{}^{\pm}+X_\mu X_\nu W_2{}^{\pm}\\&+Y_\mu Y_\nu W_3{}^{\pm}+\tfrac{1}{2}(X_\mu Y_\nu+X_\nu Y_\mu)W_4{}^{\pm}\\&+\tfrac{1}{2}[\epsilon_{\mu\alpha\beta\gamma}q_\alpha X_\beta Y_\gamma X_\nu+(\mu\leftrightarrow\nu)]W_5{}^{\pm}\\&+\tfrac{1}{2}[\epsilon_{\mu\alpha\beta\gamma}q_\alpha X_\beta Y_\gamma Y_\nu+(\mu\leftrightarrow\nu)]W_6{}^{\pm}\\&+\tfrac{1}{2}(X_\mu Y_\nu-X_\nu Y_\mu)W_7{}^{\pm}+\tfrac{1}{2}[\epsilon_{\mu\alpha\beta\gamma}q_\alpha X_\beta Y_\gamma X_\nu\\&-(\mu\leftrightarrow\nu)]W_8{}^{\pm}+\tfrac{1}{2}[\epsilon_{\mu\alpha\beta\gamma}q_\alpha X_\beta Y_\gamma Y_\nu\\&-(\mu\leftrightarrow\nu)]W_9{}^{\pm}\}\,,\end{aligned} \tag{2.9}$$

where $P=p_1+p_2$, $\Delta=p_1-p_2$, $X_\mu=P_\mu-q_\mu(P\cdot q)/q^2$, and $Y_\mu=\Delta_\mu-q_\mu(\Delta\cdot q)/q^2$. The invariant amplitude W_i's are functions of invariant variables $\nu_1=-(P\cdot q)$ $(=P_0q_0-\mathbf{Pq})$, $\nu_2=-(\Delta\cdot q)$, $s=-P^2$, and $-q^2$. Since the protons are averaged over spin, $W_{\mu\nu}{}^{\pm}$ is symmetric under interchange of p_1 and p_2, or under $P_\mu\to P_\mu$ and $\Delta_\mu\to-\Delta_\mu$.

Integrating the cross sections over the angle of the emitted lepton pair, one finds in the c.m. frame of the proton-proton collision ($\mathbf{P}=0$)

$$\frac{d\sigma^{\pm}}{dq_0d(-q^2)d\cos\theta}=\frac{m_N{}^2|\mathbf{q}|}{6\pi^4[s(s-4m_N{}^2)]^{1/2}}\frac{g^4}{(q^2+m_W{}^2)^2+m_W{}^2\Gamma^2}W^{\pm}, \tag{2.10}$$

$$W^{\pm}=3(q^2)^2W_1{}^{\pm}-q^2X^2W_2{}^{\pm}-q^2Y^2W_3{}^{\pm}-q^2(X\cdot Y)W_4{}^{\pm}, \tag{2.11}$$

or, writing it more explicitly,

$$=3(q^2)^2W_1{}^{\pm}+s|\mathbf{q}|^2W_2{}^{\pm}+(s-4m_N{}^2)(|\mathbf{q}^2|\cos\theta-q^2)W_3{}^{\pm}-[s(s-4m_N{}^2)]^{1/2}q_0|\mathbf{q}|\cos\theta\,W_4{}^{\pm},$$

where all quantities are those in the $\mathbf{P}=0$ frame, and θ is the production angle of the $W^{\pm}$ meson with respect to the collision axis $\mathbf{p}_1=-\mathbf{p}_2$. We have neglected the lepton masses in the above formula. In terms of the invariant variables ν_1, ν_2, s, and q^2, (2.11) may also be written as

$$W^{\pm}=3(q^2)^2W_1{}^{\pm}+(\nu_1{}^2+sq^2)W_2{}^{\pm}+[\nu_2{}^2-(s-4m_N{}^2)q^2]W_3{}^{\pm}+\nu_1\nu_2W_4{}^{\pm}. \tag{2.12}$$

We see later that the current-algebra sum rule is relevant to the W_2 amplitude. Therefore, we have to restrict ourselves to a specific configuration in which the $W^{\pm}$ meson is produced along the direction perpendicular to the collision axis in the c.m. frame. In this configuration ($\nu_2=0$), one can prove a positive-definiteness condition based on the positivity of W_{ii},

$$-q^2W_1+\Delta^2W_3|_{\nu_2=0}\geqq 0\,, \tag{2.13}$$

which follows if one chooses in (2.9) μ $(=\nu)$ as a space component parallel to $\mathbf{\Delta}$ in the $\mathbf{P}=0$ frame. One can also show similarly

$$W_1\geqq 0\,, \tag{2.14}$$

which is valid also for $\nu_2\neq 0$. As a consequence, one has

$$\left.\frac{d\sigma^{\pm}}{dq_0d(-q^2)d\cos\theta}\right|_{\nu_2=0}\geqq\frac{m_N{}^2|\mathbf{q}|}{6\pi^4[s(s-4m_N{}^2)]^{1/2}}\frac{q^4}{(q^2+m_W{}^2)^2+m_W{}^2\Gamma^2}\times[2(q^2)^2W_1{}^{(\pm)}+s|\mathbf{q}|^2W_2{}^{\pm}]. \tag{2.15}$$

This estimate is utilized later when we derive an inequality for the production cross sections.

The current-algebra sum rule is derived for a $ppW^{\pm}\to ppW^{\pm}$ "scattering" amplitude

$$T_{\mu\nu}{}^{\pm}=\int d^4x\,e^{-iq\cdot x}\langle\mathbf{p}_1\mathbf{p}_2{}^{\text{out}}|[J_\mu{}^{\pm}(x),J_\nu{}^{\mp}(0)]\theta(x_0)\times|\mathbf{p}_1\mathbf{p}_2{}^{\text{in}}\rangle. \tag{2.16}$$

Note that the state vector on the left is an outgoing wave. Corresponding to (2.9), we decompose its absorptive part $A_{\mu\nu}$ as

$$\begin{aligned}A_{\mu\nu}{}^{\pm}=(m_N{}^2/p_{10}p_{20})\{&(-\delta_{\mu\nu}q^2+q_\mu q_\nu)A_1{}^{\pm}+X_\mu X_\nu A_2{}^{\pm}\\&+Y_\mu Y_\nu A_3{}^{\pm}+\tfrac{1}{2}(X_\mu Y_\nu+X_\nu Y_\mu)A_4{}^{\pm}\\&+\tfrac{1}{2}[\epsilon_{\mu\alpha\beta\gamma}q_\alpha X_\beta Y_\gamma X_\nu+(\mu\leftrightarrow\nu)]A_5{}^{\pm}\\&+\tfrac{1}{2}[\epsilon_{\mu\alpha\beta\gamma}q_\alpha X_\beta Y_\gamma Y_\nu+(\mu\leftrightarrow\nu)]A_6{}^{\pm}\\&+\tfrac{1}{2}(X_\mu Y_\nu-X_\nu Y_\mu)A_7{}^{\pm}+\tfrac{1}{2}[\epsilon_{\mu\alpha\beta\gamma}q_\alpha X_\beta Y_\gamma X_\nu\\&-(\mu\leftrightarrow\nu)]A_8{}^{\pm}+\tfrac{1}{2}[\epsilon_{\mu\alpha\beta\gamma}q_\alpha X_\beta Y_\gamma Y_\nu\\&-(\mu\leftrightarrow\nu)]A_9{}^{\pm}\}\,.\end{aligned} \tag{2.17}$$

The analytic structure of the scattering amplitudes on the complex plane of ν_1 is drawn in Fig. 2. Note that, because of the large "target mass" $\sqrt{s}$, the absorptive cuts extend deep into the opposite half-planes. The masses of intermediate states contributing to the discontinuities across the cuts are easily calculated to be

$$M_n^2=s\pm 2\nu_1-q^2. \quad (2.18)$$

This relation indicates, for instance, that the region of a finite ν_1 corresponds to intermediate states of very high mass (as high as $\sim\sqrt{s}$) with the baryon number 2. We would probably not find any resonance or structure in such a region. It means that the scattering amplitudes already reach a smooth asymptotic region there. It may be worthwhile to remark in this connection that the regions near the two branch points, although $|\nu_1|$ may become as large as $\frac{1}{2}s$ there, are not the Regge asymptotic region of three body scattering. In fact these, are the regions where we should expect resonances, bumps, and other structures in ν_1 if any, and Regge expansion should be made in the variable $q_0-(q_0)_{\min}$ instead of q_0 itself (see also a discussion in Sec. III).

III. CURRENT-ALGEBRA SUM RULES

We derive a sum rule from an equal-time commutation relation between two time components of the weak hadronic currents. The technique of deriving sum rules is essentially the same as that of the Fubini (Adler–Dashen–Gell-Mann–Fubini) sum rule.[12] However, the fact that state vectors are two-particle scattering states brings in some amount of complications.

We sandwich the equal-time commutator,

$$\left[\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}J_0^+(\mathbf{x},t),\ \int d^3y\, e^{+i\mathbf{q}'\cdot\mathbf{y}}J_0^-(\mathbf{y},t)\right]=\int d^3x\, e^{-i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}K_0(\mathbf{x},t), \quad (3.1)$$

$$K_\mu(x)\equiv 4J_{3\mu}(x)\cos^2\theta_C-4J_{6\mu}(x)\cos\theta_C\sin\theta_C+2[J_{3\mu}(x)+\sqrt{3}J_{8\mu}(x)]\sin^2\theta_C, \quad (3.2)$$

between $\langle\mathbf{p}_1',\mathbf{p}_2'^{\text{out}}|$ and $|\mathbf{p}_1,\mathbf{p}_2^{\text{in}}\rangle$ $(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}=\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}')$. The current $J_{i\mu}(x)$ is defined as $J_{i\mu}(x)=\mathfrak{F}_{i\mu}(x)+\mathfrak{F}_{i\mu}{}^5(x)$.

The state vectors are two-proton states of the momenta specified, and it is understood that the protons are averaged over spin. To be explicit, let us again go to the c.m. frame of the proton-proton collision ($\mathbf{P}=0$). Just as in the derivation of the ordinary Fubini sum rule, we boost the states along the direction keeping the condition $(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_1'+\mathbf{p}_2'\cdot\mathbf{q})=0$. Then, going to the forward direction, we obtain a current-algebra sum rule.

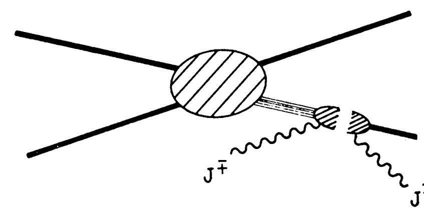

FIG. 3. Partially connected diagrams.

The resulting sum rule is written as[14]

$$\frac{1}{\pi}\left[\int_{-\nu_0}^{\infty}d\nu_1 A_2^+(\nu_1,\nu_2,q^2,s)-\int_{-\nu_0}^{\infty}d\nu_1 A_2^-(\nu_1,\nu_2,q^2,s)\right]=F(s), \quad (3.3)$$

$$\nu_0=\tfrac{1}{2}(s-q^2-4m_N^2),\qquad F(s)\equiv\lim_{\mathbf{p}_i=\mathbf{p}_i'}\lim_{\mathbf{q}=\mathbf{q}'}F_1, \quad (3.4)$$

$$\langle\mathbf{p}_1'\mathbf{p}_2'^{\text{out}}|\int d^3x\, e^{-i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}}K_\mu(\mathbf{x},t)|\mathbf{p}_1\mathbf{p}_2^{\text{in}}\rangle=(m_N^4/p_{10}p_{20}p_{10}'p_{20}')^{1/2}\{\tfrac{1}{2}(p_1+p_2+p_1'+p_2')_\mu F_1+\tfrac{1}{2}(p_1-p_2+p_1'-p_2')_\mu F_2+\cdots\}, \quad (3.5)$$

where the rest of { } in (3.5) vanishes in the limit of forward scattering. The invariant function F_i's are functions of six variables constructed out of the five vectors, p_1, p_2, p_1', p_2', and $q-q'\equiv k$. In the sum rule (3.3), ν_2 as well as q^2 and s is kept fixed. We can evaluate $F(s)$ using the low-energy theorem due to Low.[15] It results in

$$F(s)=\lim_{\mathbf{p}_i\to\mathbf{p}_i'}\lim_{k\to 0}(1+\sin^2\theta_C)\times\left[\left(\frac{1}{p_1\cdot k}-\frac{1}{p_1'\cdot k}+\frac{1}{p_2\cdot k}-\frac{1}{p_2'\cdot k}\right)T+\left(\frac{p_1\cdot k}{p_2\cdot k}+\frac{p_1'\cdot k}{p_2'\cdot k}+\frac{p_2\cdot k}{p_1\cdot k}+\frac{p_2'\cdot k}{p_1'\cdot k}-4\right)\frac{\partial T}{\partial s}\right], \quad (3.6)$$

where T is the p-p forward-scattering amplitude averaged over spins. This is the Fubini sum rule in the case of a two-particle scattering state as a target. The expression (3.6) for $F(s)$ contains terms which are divergent or not unambiguous in the limit of $k\to 0$. This divergence originates in the diagrams drawn in Figs. 3 and 4, which we call from now on "partially connected" diagrams. We argue below with the help of duality that

[14] By analogy with two-body scattering, $A_2^\pm$ would behave like $q_0^{\alpha-2}$ as $q_0\to\infty$ with s fixed for a Regge exchange of intercept α. This is not necessarily true when one goes back along the W-emission regions, since they have nothing to do with Regge asymptotic regions. The same Regge argument indicates that $A_1^\pm\sim q_0^\alpha$ in this limit. We expect the same asymptotic behavior for corresponding W's.

[15] F. E. Low, Phys. Rev. 110, 974 (1958).

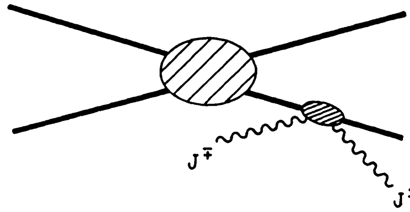

FIG. 4. A singular diagram among partially connected graphs.

the amplitude of "ppW" scattering should exhibit ordinary Regge behavior with respect to the W-meson energy after we subtract from it the partially connected diagrams. The contributions of the partially connected diagrams to our Fubini sum rule amount to "K_μ-current" emissions from external proton lines on the right-hand side of the sum rule (3.3). Subtraction of such a diagram has been performed in the Appendix. The contribution that should be subtracted is given as

$$\begin{aligned} F(s)_{\mathrm{p.c.}} &= \lim_{p_i\to p_i'} \lim_{k\to 0} (1+\sin^2\theta_C) \\ &\quad \times\left[\left(\frac{1}{p_1\cdot k}-\frac{1}{p_1'\cdot k}+\frac{1}{p_2\cdot k}-\frac{1}{p_2'\cdot k}\right)T \right. \\ &\quad \left. +\left(\frac{p_1\cdot k}{p_2\cdot k}+\frac{p_1'\cdot k}{p_2'\cdot k}+\frac{p_2\cdot k}{p_1\cdot k}+\frac{p_2'\cdot k}{p_1'\cdot k}\right)\frac{\partial T}{\partial s}\right], \\ &= F(s)+4(1+\sin^2\theta_C)\frac{\partial T}{\partial s}. \end{aligned} \qquad (3.7)$$

We thus rewrite our current-algebra sum rule as[16]

$$\frac{1}{\pi}\left[\int_{-\nu_0}^{\infty} d\nu_1 \tilde{A}_2^{+}(\nu_1,\nu_2,q^2,s) - \int_{-\nu_0}^{\infty} d\nu_1 \tilde{A}_2^{-}(\nu_1,\nu_2,q^2,s)\right] = \tilde{F}(s)\,, \qquad (3.8)$$

$$\begin{aligned} \tilde{F}(s) &= -4(1+\sin^2\theta_C)\frac{\partial T}{\partial s} \\ &= -i(1+\sin^2\theta_C)\sigma_T(s)/m_N^2\,, \end{aligned} \qquad (3.9)$$

where, by the optical theorem, $\sigma_T(s)$ is the total cross section for proton-proton scattering. The amplitudes $\tilde{A}_2^+$ and $\tilde{A}_2^-$ are understood as A_2^+ and A_2^- of which "partially connected" diagrams are subtracted out. The derivation of (3.7) is given in the Appendix.

We now go on to see what is an important region of integration in (3.6). As was mentioned briefly in Sec. II, the region where ν_1 is far away from $-\nu_0$ is the Regge asymptotic region. In accord with duality of direct-channel structure and Reggeons, the Regge expansion should be made in the variable $(q_0-(q_0)_{\min})$. We must be somewhat careful about treatment of the ρ-Regge exchange contribution because of the reason to be stated below. To make our statement more precise, let us consider a $\pi^- pW^{\pm}$ "scattering" instead of a $ppW^{\pm}$ "scattering" for a while. From our experience[17] in other current-algebra sum rules such as the Adler-Weisberger sum rule or the Cabibbo-Radicati sum rule, we may expect that smooth asymptotic regions of the ρ Regge exchange contribute to isospin-antisymmetric sum rules only by a small amount, numerically negligible. In the case of ordinary current-algebra sum rules with a one-particle state as a target, resonance regions (below $M_n = M^* \simeq 3$ GeV) give a main contribution, while even a *lower-energy part* of the ρ Regge exchange is not very large. However, in our sum rule the resonance regions correspond to a very short interval in q_0 of the order of $\frac{1}{2}M^{*2}/\sqrt{s}$ while the low-energy portion of the ρ Regge exchange means $q_0-(q_0)_{\min}\lesssim m^*$ (a few GeV, finite as $s\to\infty$). Although the amplitudes are already smooth there, we are not allowed to neglect the ρ Regge exchange entirely as compared with the resonances. In consequence we should cut off the integral over q_0 at $(q_0)_{\min}+$(a few GeV). We might have some difficulty in applying the same argument to the $ppW^{\pm}$ channels, since there may be no resonance in the channel of baryon number two. If there is no resonance in the s channel, then duality indicates that the ρ Regge coupling is weak. Since this difficulty is inherent in duality argument, we leave the statement in the case of the $ppW^{\pm}$ channel as part of an assumption for the moment.

This argument, however, does not apply to partially connected diagrams because resonances dual to t-channel Regge exchanges are pW resonances, not three-body ppW states in these diagrams. Neglect of higher portions of integral regions is justified only after one subtracts partially connected diagrams. Cutting off the integrals in (2.8) at $\nu_1 = -\nu_0 + m^*\sqrt{s}$ [or at $q_0-(q_0)_{\min}=m^*$], we obtain

$$\frac{1}{\pi}\left[\int_{-\nu_0}^{-\nu_0+m^*\sqrt{s}} d\nu_1 A_2^{+}(\nu_1,\nu_2,q^2,s) - \int_{-\nu_0}^{-\nu_0+m^*\sqrt{s}} d\nu_1 A_2^{-}(\nu_1,\nu_2,q^2,s)\right] = \tilde{F}(s)\,. \qquad (3.10)$$

We have removed the tilde above $A_2^{\pm}$ since $A_2^{\pm}=\tilde{A}_2^{\pm}$ for a negative value of ν_1. The value of m^* is not fixed definitely, but its precise value is not needed in the high-energy limit, as far as it does not grow as s becomes large.

Finally we like to make a comment on the region of q^2. The current-algebra sum rule has been derived for a spacelike q^2 in the present method. But one can continue it without difficulty to the timelike region, since all relevant convergence properties in the variable ν_1 will not be affected by the continuation. In fact, the

[16] We have assumed that the proton-proton forward scattering amplitude becomes dominantly imaginary at sufficiently high energies. If it turns out false, we must modify the right-hand side accordingly. Since the p-p total cross section will be available prior to the W-meson experiment, we shall know how to modify the sum rule in this case.

[17] S. L. Adler and F. J. Gilman, Phys. Rev. **156**, 1598 (1967); see also F. J. Gilman and H. Schnitzer, *ibid.* **150**, 1362 (1966).

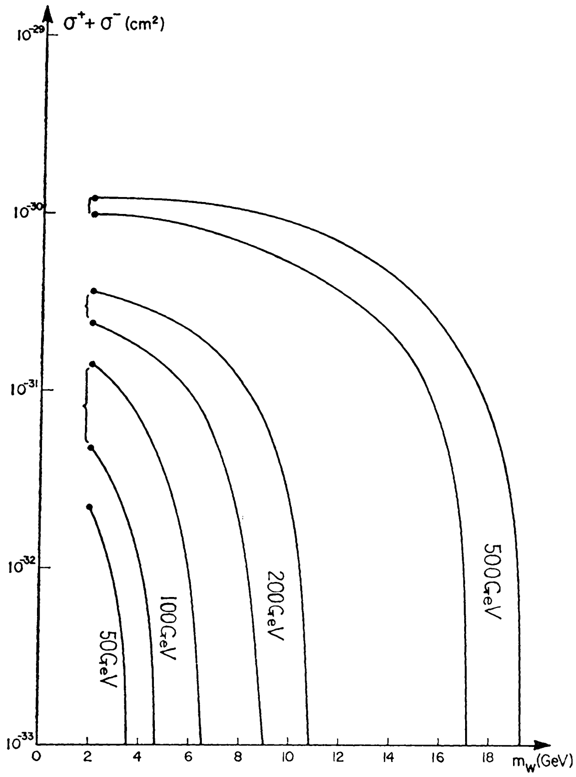

FIG. 5. Lower bound on the sum of the W^+- and W^--production cross sections as a function of the incident energy $\omega_{\rm lab}$ and m_W. The cutoff parameter has been chosen as 2 GeV (upper curves) and 3 GeV (lower curves).

covariant derivation by Fubini[12] leads directly to the sum rule for a timelike q^2.

IV. DERIVATION OF BOUND

The amplitude $A_{\mu\nu}^{(\pm)}$ defined by (2.16) and (2.17) is not directly related to the $W^\pm$-meson-production cross sections. But, since the only difference between $A_{\mu\nu}^\pm$ and $W_{\mu\nu}^\pm$ in (2.2) consists in the boundary condition on the state vector on the left, one can write the Schwarz inequality between them:

$$\Big|\sum_n \langle \mathbf{p}_1\mathbf{p}_2^{\rm out}|J_\mu^\pm(0)|n\rangle\langle n|J_\mu^\mp(0)|\mathbf{p}_1\mathbf{p}_2^{\rm in}\rangle \times\delta(p_1+p_2+q-P_n)\Big| \quad \text{(not summed over } \mu) \leqq \sum_n |\langle \mathbf{p}_1\mathbf{p}_2^{\rm in}|J_\mu^\pm(0)|n\rangle|^2\delta(p_1+p_2+q-P_n). \quad (4.1)$$

For our present purpose, we take the configuration in which $\mathbf{q}$ is perpendicular to $\mathbf{\Delta}$, and boost to the rest frame of q_μ. Choosing in (4.1) the space component parallel to the direction of boost, we find

$$|(q^2)^2A_1+s|\mathbf{q}|^2A_2|\leqq(q^2)^2W_1+s|\mathbf{q}|^2W_2.$$

In addition, we have

$$|A_1|\leqq W_1,$$

which follows from the space component perpendicular to both $\mathbf{\Delta}$ and $\mathbf{q}$. By summing these two inequalities, we obtain for $\nu_2=0$

$$|s|\mathbf{q}|^2A_2|\leqq 2(q^2)^2W_1+s|\mathbf{q}|^2W_2. \quad (4.2)$$

The right-hand side is exactly what we had in the formula (2.14) for the cross section.

The current-algebra sum rule (3.10) thus leads us to

$$\frac{1}{\pi}\int_{-\nu_0}^{-\nu_0+m^*\sqrt{s}} d\nu_1\left[\frac{2(q^2)^2}{s|\mathbf{q}|^2}(W_1^++W_1^-)+(W_2^++W_2^-)\right]_{\nu_2=0}\geqq|\tilde{F}(s)|. \quad (4.3)$$

By substituting (2.15) into (4.3), we find that the sum rule is rewritten for large s as

$$\int_{-\nu_0/\sqrt{s}}^{-\nu_0/\sqrt{s}+m^*} dq_0|\mathbf{q}|^{-3}\left[\frac{d\sigma^+}{d|q_0|d\cos\theta}+\frac{d\sigma^-}{d|q_0|d\cos\theta}\right]_{\theta=90°} \geqq\frac{1}{12\pi^2}\frac{1}{(s-4m_N^2)^{1/2}}\frac{G^2m_W^3}{\Gamma}\sigma_T(\infty). \quad (4.4)$$

We obtain rather a conservative bound on the cross sections by replacing $|\mathbf{q}|$ by its minimum value in the integral region. We factor out $|\mathbf{q}|^{-3}$ and carry out the integration over q_0:

$$\left[\frac{d\sigma^+}{d\cos\theta}+\frac{d\sigma^-}{d\cos\theta}\right]_{\theta=90°}\gtrsim\frac{1}{12\pi^2}\frac{Gm_W^3}{\Gamma}\sigma_T(\infty)\frac{|\mathbf{q}|_{\rm min}^3}{\sqrt{s}},\qquad |q_0|\gtrsim\tfrac{1}{2}\sqrt{s}-m^*, \quad (4.5)$$

where

$$|\mathbf{q}|_{\rm min}=\left[\left(\frac{s+m_W^2-4m_N^2}{2\sqrt{s}}-m^*\right)^2-m_W^2\right]^{1/2}. \quad (4.6)$$

It may be somewhat surprising to see that the right-hand side increases linearly in s as $s\to\infty$. We recall that our weak amplitude does not take account of higher-order weak interactions. At energies where the higher-order weak interactions are not negligible ($\sqrt{s}\gtrsim 300$ GeV), we must impose unitarity including the higher orders. Our formula would not work in those regions. It is worthwhile pointing out that for purely hadronic processes this linearly increasing term is absent because there is no fixed singularity on the J plane, thus leading to a superconvergence sum rule. It is also interesting to note that Altarelli, Brandt, and Preparata[18] have obtained the same energy dependence of the cross sections for the lepton-pair production due to a virtual photon in the p-p collision which is essentially identical to our process. Their analysis is based upon some heuristic assumptions on singularities of

[18] G. Altarelli, R. A. Brandt, and G. Preparata, Rockefeller University report (unpublished); see also S. D. Drell and T. M. Yan, Phys. Rev. Letters 25, 316 (1970).

commutators near the light cone. We discuss the electromagnetic muon pairs in detail in a separate paper.

The inequality we have obtained involves the differential cross sections in which the W mesons are emitted perpendicularly to the collision axis in the $\mathbf{P}=0$ frame. It also involves the width of the W meson. In order to obtain a numerical result for the production cross sections, therefore, we have to supplement some physical assumptions. First we need to know the decay width of the $W^{\pm}$ meson. There have been many attempts to estimate it, but without much reliability. We choose here an estimate by Yamaguchi[19] for the baryonic decays

$$\Gamma \simeq (g^2/6\pi)\cos^2\theta_C\, m_W(1-4m_N^2/m_W^2)^{1/2} \times(1-m_N^2/m_W^2), \quad (4.7)$$

which is obtained for m_W well above $2m_N$ in a sort of closure approximation. We use it by multiplying by a factor of 3 due to leptonic decay modes,

$$\Gamma \simeq (2\sqrt{2}\pi)^{-1} G m_W^3. \quad (4.8)$$

This formula gives, for instance, $\simeq 0.14$ MeV for $m_W=5m_N$, and $\simeq 1.1$ MeV for $m_W=10m_N$. Another thing we have to know is the angular distribution of the W meson. We take here the analogy of inelastic hadron scattering; particles are produced fairly distinctively into either a forward or background cone with additional secondaries emitted more or less spherically symmetrically from the center, and so the differential cross sections at 90° in the c.m. frame are likely to be minimum [more precisely, we assume $\sigma_T \geqq 4\pi\, d\sigma/d\Omega$ $(\theta=90°)$]. This will be a reasonable guess for the W mesons produced at high energies. Therefore, we integrate (4.5) with respect to $\cos\theta$ over all emission angles. Then we obtain in the high-energy limit

$$(\sigma^+ + \sigma^-) \gtrsim (24\sqrt{2}\pi)^{-1} G s \sigma_T(\infty) I(s), \quad (4.9)$$

$$I(s) = \{[1-2m^*/\sqrt{s} + (m_W^2-4m_N^2)/s]^2 - 4m_W^2/s\}^{3/2}. \quad (4.10)$$

The factor $I(s)$ tends to unity as $s\to\infty$. Numerically it turns out to be

$$(\sigma^+ + \sigma^-) \gtrsim 3.8\times 10^{-33}(\omega_{\rm lab}/m_N) I(\omega_{\rm lab})\ \text{cm}^2, \quad (4.11)$$

where $\sigma_T(\infty)$ has been chosen ~ 20 mb, $\omega_{\rm lab}=s/2m_N - m_N$, and

$$I(\omega_{\rm lab}) = \left[\left(1 - \frac{\sqrt{2}m^*}{(m_N\omega_{\rm lab})^{1/2}} + \frac{m_W^2-4m_N^2}{2m_N\omega_{\rm lab}}\right)^3 - \frac{2m_W^2}{m_N\omega_{\rm lab}}\right]^{3/2}. \quad (4.12)$$

The quantity within the square brackets above must be positive for our formula (4.11) to be meaningful.

[19] Y. Yamaguchi, Progr. Theoret. Phys. (Kyoto) **35**, 914 (1966); see also V. Namias and L. Wolfenstein, Nuovo Cimento **46**, 542 (1965).

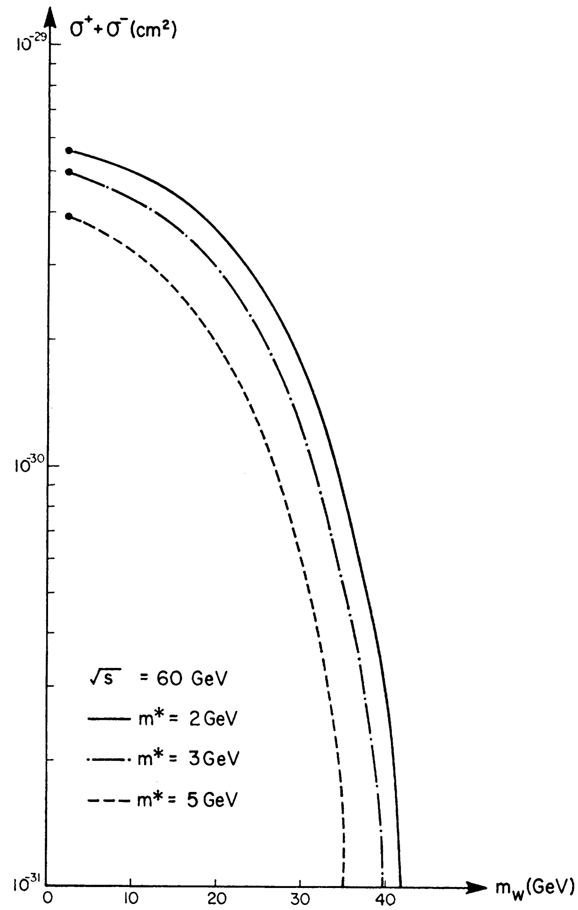

FIG. 6. Lower bounds on the sum of the W^+- and W^--production cross sections at $\sqrt{s}=60$ GeV as a function of m_W. This energy is available at the Intersecting Storage Ring machine. The parameter m^* is chosen to be 2, 3, and 5 GeV. It is highly unlikely that the cutoff parameter m^* is larger than 5 GeV.

At $\omega_{\rm lab}=100$ GeV, for instance, our formula is meaningful only if m_W is smaller than ~ 6 GeV with m^* chosen 2 GeV, while the kinematical limit is $m_W \simeq 12$ GeV. We have drawn in Fig. 5 the lower bounds given by (4.11) as a function of $\omega_{\rm lab}$ and m_W. Since the energy of $\sqrt{s}=60$ GeV will be available at ISR, we have drawn the lower bounds at this energy for various values of m^* (see Fig. 6). One can see that the bound is not sensitive to the value of m^* when energies are sufficiently high. The right-hand side of (4.11) gives a fairly large value in case the W meson is produced well above the production threshold. The current-algebra sum rule incorporating the same equal-time commutator has led us to the prediction that the inelastic electroproduction cross sections should be very large, larger than previously imagined. Subsequently, this turned out correct in the inelastic electroproduction experiment of SLAC. The large lower bound in our inequality is therefore more or less expected.

However, because the cross section grows linearly in s, we may not expect our lower bound to continue to be true. We have to be prepared for a breakdown of our inequality above $\sqrt{s}\approx 300$ GeV, simply because unitarity will be violated unless we take account of higher-order weak interactions.

V. DISCUSSION

The inequality we have derived from the current algebra and Regge argument indicates that the $W^{\pm}$-meson cross sections are very large if the W mesons exist. If one calculates the $W^{\pm}$-meson production through a specific diagram, its contribution would be at least one order of magnitude smaller in a matrix element than what we have obtained here, since we always have a strong damping factor due to the form factors associated with the weak currents (the momentum transfer is larger than at least 2 GeV). But, as the incident proton energy increases, more and more channels open and their contributions are summed up to compensate damping of each individual diagram. This phenomenon has been rather familiar in many aspects of atomic and low-energy nuclear physics. The similarity was, in fact, pointed out and emphasized by Dashen and Gell-Mann in the context of current commutators.[12] Therefore, the large lower-bound of the $W^{\pm}$-meson-production cross section is, in a sense, a necessary consequence of the fact that the W mesons interact with matter through vector and axial-vector currents. It should also be remarked that our estimate of the lower bounds is very conservative [see, for instance, the subscripts on the left-hand side of (4.5)]. It would not be surprising at all that actual cross sections, if the W mesons exist, may come out much larger than our lower bounds.

Although the $W^{\pm}$-meson production is large, the μ-pair production through a virtual γ will also be large. A similar calculation goes through for the electromagnetic μ-pair production, if one converts "charged photons" into the neutral photon using the charge independence. In consequence, the problem of background due to energetic muon pairs of electromagnetic origin may still exist.[20] One way to avoid muon background of electromagnetic origin is to detect the charge asymmetry of the produced $W^{\pm}$ mesons, or of the energetic muons, since the muons of electromagnetic origin must contain the same number of μ^{+} and μ^{-}.[21] Unfortunately, our inequality is not capable of telling about the charge asymmetry. However, it is quite natural to expect that, unless some miraculous cancellation takes place, the difference of the $W^{\pm}$-meson-production cross sections would not be smaller by orders of magnitude than the sum of them. If we insist on a duality picture of hadron dynamics, the difference of the two cross sections will be large at small M_n regions where energetic W mesons are produced, but smaller for low-energy W mesons. Observation of the charge asymmetry may be an easier way to detect the existence of the $W^{\pm}$ mesons.

[20] Y. Yamaguchi, Nuovo Cimento **53A**, 193 (1966).

[21] One of the authors (M.S.) is indebted to Professor N. Christ for raising this possibility.

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APPENDIX

We describe the derivation of (3.6) and (3.7). The $A_{\mu\nu}^{\pm}$ amplitude is decomposed explicitly as

$$A_{\mu\nu}^{\pm}=\pi(2\pi)^3\sum_n \langle \mathbf{p}_1'\mathbf{p}_2'^{\text{out}}|J_\mu^{\pm}(0)|n\rangle\langle n|J_\nu^{\mp}(0)|\mathbf{p}_1\mathbf{p}_2^{\text{in}}\rangle \times\delta(p_1+p_2+q-P_n) \quad \text{(A1)}$$

in the nonforward scattering. Each of the matrix elements of $J^{\pm}(0)$ may contain disconnected diagrams. A product of two disconnected diagrams, which is a totally disconnected diagram when regarded as a $Wpp\to Wpp$ scattering process, can be rather trivially removed from the both sides of the sum rule. What we are now interested in is a product of connected and disconnected diagrams, as drawn in Fig. 3. We call it a partially connected diagram. One such example is written as

$$(A_{00}^{+})_{\text{p.c.}}=-i\pi(2\pi)^3(m_N/p_{10})^{1/2}\sum_{n'}\int d^4r\, e^{ip_1y}$$
$$\times\langle \mathbf{p}_1'\mathbf{p}_2'^{\text{out}}|[J_0^{+}(0),\bar\psi(y)]\theta(-y_0)|n'\rangle\langle n'|J_0^{+}(0)|\mathbf{p}_2\rangle \times(i\gamma p_1+m_N)u(\mathbf{p}_1)\delta(p_2+q-P_{n'}), \quad \text{(A2)}$$

where $\sum_{n'}$ includes averaging over the proton spins in the initial and final states. The difference $\int dq_0(A_{00}^{+}-A_{00}^{-})_{\text{p.c.}}$ is given as

$$\int dq_0(A_{00}^{+}-A_{00}^{-})_{\text{p.c.}}$$
$$=-i\pi(m_N/p_{10})^{1/2}\int d^4y\, e^{ip_1y}\Big\{-\int d^3x\, e^{-i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}}$$
$$\times[\langle \mathbf{p}_1'\mathbf{p}_2'^{\text{out}}|\bar\psi(y)K_0(\mathbf{x},0)|\mathbf{p}_2\rangle\theta(-y_0)]$$
$$+\langle \mathbf{p}_1'\mathbf{p}_2'^{\text{out}}|J_0^{\pm}(0)\bar\psi(y)J_0^{\mp}(0)|\mathbf{p}_2\rangle\theta(-y_0)$$
$$-\langle \mathbf{p}_1'\mathbf{p}_2'^{\text{out}}|J_0^{\pm}(0)\bar\psi(y)J_0^{\mp}(0)|\mathbf{p}_2\rangle\theta(-y_0)\Big\}$$
$$\times(i\gamma p_1+m_N)u(\mathbf{p}_1), \quad \text{(A3)}$$

where $K_\mu(x)$ is defined in (3.2). The first term on the right-hand side is calculated by inserting a complete set of intermediate states between $\bar{\psi}$ and K_0. Although K_μ is not associated with anything like a photon, K_μ is conserved. Therefore, the computation is the same as in the low-energy theorem by Low[15] except that all the protons are on the mass shell in our calculation. In other words, it is implicitly assumed here that the "K_μ"-proton channel contains no contact subtraction term. If one follows a Feynmann-diagram calculation, the off-shell effects of the internal proton lines would give rise to additional terms such as derivatives in the proton mass. We have three more diagrams in which K_μ is hooked on the other external proton lines. Summing these four terms together, we obtain

$$F_1(s)_{\text{p.c.}}=(1+\sin^2\theta_C)\times\left[\left(\frac{1}{p_1\cdot k}-\frac{1}{p_1'\cdot k}+\frac{1}{p_2\cdot k}-\frac{1}{p_2'\cdot k}\right)T\times\left(\frac{p_1\cdot k}{p_2\cdot k}+\frac{p_1'\cdot k}{p_2'\cdot k}+\frac{p_2\cdot k}{p_1\cdot k}+\frac{p_2'\cdot k}{p_1'\cdot k}\right)\frac{\partial T}{\partial w}\right]+O(k), \quad \text{(A4)}$$

$$F_2(s)_{\text{p.c.}}=O(k), \quad \text{(A5)}$$

where $k=(\mathbf{q}'-\mathbf{q},0)$ and $w=(p_1\cdot p_2)+(p_1'\cdot p_2')$, and T is the p-p forward scattering amplitude averaged over spin. The axial-vector currents in K_μ do not contribute to (A4) or (A5) in the relevant order of k. Given (A4) and (A5), we are now able to determine $\lim_{\mathbf{q}=\mathbf{q}'}[F_1-(F_1)_{\text{p.c.}}]$ according to Low's theorem as

$$\tilde{F}\equiv\lim_{\mathbf{q}\to\mathbf{q}'}[F_1-(F_1)_{\text{p.c.}}]=-4(1+\sin^2\theta_C)\frac{\partial T}{\partial w}. \quad \text{(A6)}$$

The left-hand side tends to a constant proportional to the total cross section in the high-energy limit ($s\to\infty$),

$$\frac{\partial T}{\partial w}=\frac{i\sigma_T(\infty)}{4m_N^2}+O(s^\delta), \quad \text{(A7)}$$

where $\sigma_T(\infty)$ is the high-energy limit of the proton-proton total cross section, and $\delta\sim-0.5$ according to analysis of high-energy data. As for the second and third terms in (A3), we should discard them according to our argument of ppW intermediate states with high mass (see Secs. II and III), for those diagrams always contain genuine ppW-scattering states somewhere between two $W^\pm$ lines. From the duality viewpoint, these should be considered as part of s-channel contributions which construct Regge trajectories in cross channels, so that we may discard them as negligible. Our sum rule finally becomes

$$\int_{-\nu_0}^{-\nu_0+m^*\sqrt{s}} d\nu_1 A_2^- - \int_{-\nu_0}^{-\nu_0+m^*\sqrt{s}} d\nu_1 A_2^+ = \tilde{F}(s), \quad \text{(A8)}$$

where we have cut off the integrals at $\nu_1=-\nu_0+m^*\sqrt{}$ for the reason stated in the text.