Chiral-Invariant Perturbation Theory

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Charap has found a particular system of pion coordinates leading to chiral-invariant amplitudes, if the usual Feynman rules are applied. It was shown that this system is determined by the condition $\det(g_{ik})=1$, where g_{ik} is the metric tensor of the pion space. In the general case the Feynman rules are modified by a counterterm that can be derived from an invariant measure in function space. The counterterm vanishes for $g=1$.

In a recent article, Charap[1] has discussed the perturbation theory of the chiral-invariant Lagrangian for pions. Although we do not believe that it is likely to be useful from a physical point of view to pursue the perturbation theory of chiral-invariant Lagrangians beyond the tree approximation, we would like to make some technical remarks which answer the questions posed by Charap's article in a completely satisfactory way.

The pion Lagrangian invariant under the nonlinear realization of $SU(2)\times SU(2)$ may be written in the geometric form[2]

$$L(\boldsymbol{\pi})=\tfrac{1}{2}g_{ik}(\boldsymbol{\pi})\partial_\mu\pi^i\partial^\mu\pi^k\,, \tag{1}$$

where π^i ($i=1, 2, 3$) are the components of the pion field. They are components of an isovector. Hence the metric g_{ik} is an isotensor with the general decomposition

$$g_{ik}(\boldsymbol{\pi})=a(\boldsymbol{\pi}^2)\delta_{ik}+b(\boldsymbol{\pi}^2)\pi_i\pi_k. \tag{2}$$

The Riemann space associated with the metric is the homogeneous space $SU(2)\otimes SU(2)/SU(2)$, which can be identified with the three-dimensional sphere S_3.[2] This leads to the further condition

$$R_{ik}=(2/f_\pi^2)g_{ik}, \tag{3}$$

where R_{ik} is the Einstein-Ricci tensor, and f_π is the radius of the sphere. It may easily be checked that relation (3) is satisfied, if we express a and b in terms of Weinberg's[3] function $f(\boldsymbol{\pi}^2)$:

$$a(\boldsymbol{\pi}^2)=\frac{f_\pi^2}{f^2+\boldsymbol{\pi}^2},\quad b(\boldsymbol{\pi}^2)=\frac{4\boldsymbol{\pi}^2(f'^2-4ff'-1)}{f^2+\boldsymbol{\pi}^2}. \tag{4}$$

After all, we are free to choose an arbitrary function $f(\boldsymbol{\pi}^2)$, put it into the Lagrangian, and start perturbation theory. Charap[1] calls a change of f a gauge transformation. We prefer to speak of coordinate transformations, because any change of f may be achieved by a transformation of the pion coordinates

$$\pi'^i=\pi^i\phi(\boldsymbol{\pi}^2),\qquad i=1, 2, 3 \tag{5}$$

(ϕ arbitrary), as has already been shown by Weinberg.[3]

To apply perturbation theory one has to expand the functions a, b, and f into powers of $\boldsymbol{\pi}^2/f_\pi^2$. We use the same notation as Charap,[1]

$$\begin{aligned} a(\boldsymbol{\pi}^2)&=\tfrac{1}{2}\sum_{n=1}^{\infty}\alpha_n\left(\frac{\boldsymbol{\pi}^2}{f_\pi^2}\right)^n,\\ b(\boldsymbol{\pi}^2)&=\frac{1}{2}\frac{1}{f_\pi^2}\sum_{n=1}^{\infty}\beta_n\left(\frac{\boldsymbol{\pi}^2}{f_\pi^2}\right)^{n-1},\\ f(\boldsymbol{\pi}^2)&=f_\pi\sum_{n=0}^{\infty}a_n\left(\frac{\boldsymbol{\pi}^2}{f_\pi^2}\right)^n. \end{aligned} \tag{6}$$

Relations (4) can then be used to express α_n and β_n in terms of $a_0, a_1, \ldots, a_n$. Having looked at the graphs up to order $(1/f_\pi)^6$, Charap arrives at the following conclusion: If we just use the simple Feynman rules, the resulting amplitudes do not satisfy soft-pion theorems like Adler's condition,[4] nor are the amplitudes independent of the choice of pion coordinates on the mass shell. In fact, Adler-type conditions are violated by the most badly divergent terms. But there is a unique choice of coordinates, for which the soft-pion theorems hold for all amplitudes. The corresponding function f has been given by Charap[1] up to order $(1/f_\pi)^6$:

$$f(\boldsymbol{\pi}^2)=f_\pi\left[1-\frac{2}{5}\frac{\boldsymbol{\pi}^2}{f_\pi^2}-\frac{9}{175}\left(\frac{\boldsymbol{\pi}^2}{f_\pi^2}\right)^2-\frac{184}{15750}\left(\frac{\boldsymbol{\pi}^2}{f_\pi^2}\right)^3\pm\cdots\right], \tag{7}$$

which gives to this order

$$g=\det(g_{ik})=1. \tag{8}$$

One wants to know why the soft-pion theorems hold only in the coordinate system with $g=1$. This can best be understood by functional methods. Let us first consider the trivial case of the free-field pion Lagrangian

$$L(\boldsymbol{\pi})=\tfrac{1}{2}\delta_{ik}\partial_\mu\pi^i\partial^\mu\pi^k. \tag{9}$$

The measure in function space, which has to be used to calculate the generating functional of the time-ordered Green's functions, reads

$$[d\boldsymbol{\pi}]=\frac{1}{Z}\exp\left(i\int d^4x\,L(\boldsymbol{\pi})\right)\prod_x d\boldsymbol{\pi}(x)\,, \tag{10}$$

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[1] J. M. Charap, Phys. Rev. D 2, 1554 (1970).
[2] K. Meetz, J. Math. Phys. 10, 589 (1969).
[3] S. Weinberg, Phys. Rev. 166, 1568 (1968).
[4] S. L. Adler, Phys. Rev. 137, B1022 (1965).

where

$$Z=\int \exp\left(i\int d^4x\, L(\boldsymbol{\pi})\right)\prod_x d\boldsymbol{\pi}(x) \tag{11}$$

and the product is to be extended over all space-time points x. The measure (10) is correct only if we use Euclidean pion coordinates for which

$$g_{ik}=\delta_{ik}. \tag{12}$$

Let us now transform to curvilinear coordinates,

$$\pi'^i=\pi'^i(\boldsymbol{\pi}),$$
$$g_{ik}'(\boldsymbol{\pi}')=\delta_{mn}\frac{\partial\pi^m}{\partial\pi'^i}\frac{\partial\pi^n}{\partial\pi'^k}. \tag{13}$$

The functional differential is transformed as

$$\prod_x d\boldsymbol{\pi}'(x)=\det D\prod_x d\boldsymbol{\pi}(x). \tag{14}$$

Here D is the functional transformation matrix

$$D_{ik}(x,x')=\frac{\delta\pi^i(x)}{\delta\pi'^k(x')}=\delta(x-x')\frac{\partial\pi^i}{\partial\pi'^k}(x)$$
$$=\delta(x-x')\Delta_{ik}. \tag{15}$$

In order to calculate the determinant of this matrix, we use a method given by Boulware.[5] We start with the general formula

$$\det D=e^{\mathrm{Tr}\ln D} \tag{16}$$

and define the matrix $\ln D$ by

$$[\ln D(x,x')]_{ik}=\delta(x-x')(\ln\Delta)_{ik}. \tag{17}$$

Hence,

$$\det D=e^{\mathrm{Tr}\ln D}=\exp\left(\delta(0)\int d^4x\,\mathrm{Tr}\ln\Delta\right)$$
$$=\exp\left(\delta(0)\int d^4x\ln\sqrt{g'}\right), \tag{18}$$

where

$$\sqrt{g'}=\det\Delta=\det\left(\frac{\partial\pi^i}{\partial\pi'^k}\right). \tag{19}$$

Dropping the primes, we obtain the functional measure for curvilinear pion coordinates,

$$[d\boldsymbol{\pi}]=\frac{1}{Z}\exp\left(i\int d^4x L(\boldsymbol{\pi})+\delta(0)\int d^4x\ln\sqrt{g}\right)$$
$$\times\prod_x d\boldsymbol{\pi}(x), \tag{20}$$

where

$$L(\boldsymbol{\pi})=\tfrac{1}{2}g_{ik}(\boldsymbol{\pi})\partial_\mu\pi^i\partial^\mu\pi^k,$$
$$Z=\int\exp\left(i\int d^4x\,L(\boldsymbol{\pi})+\delta(0)\int d^4x\ln\sqrt{g}\right)$$
$$\times\prod_x d\boldsymbol{\pi}(x). \tag{21}$$

This measure is invariant under general coordinate transformations in the pion space.

The chiral-invariant Lagrangian (1) is related to a pion space of constant curvature.[2] In this case we may also use (20) as an invariant measure in function space. As is seen from (20), we should obtain an invariant perturbation series, if we apply the simple Feynman rules to the modified Lagrangian

$$L'(\boldsymbol{\pi})=L(\boldsymbol{\pi})-i\delta(0)\ln\sqrt{g(\boldsymbol{\pi})}. \tag{22}$$

We write

$$L'=\tfrac{1}{2}(\partial_\mu\boldsymbol{\pi})^2+L_I, \tag{23}$$

and use the expansions (6) to calculate the total interaction Lagrangian L_I:

$$\begin{aligned}L_I=&(1/f_\pi^2)\tfrac{1}{2}[\alpha_1\boldsymbol{\pi}^2(\partial_\mu\boldsymbol{\pi})^2\\&+\beta_1(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2-i\delta(0)(3\alpha_1+\beta_1)\boldsymbol{\pi}^2]+(1/f_\pi^4)\\&\times\tfrac{1}{2}\{\alpha_2(\boldsymbol{\pi}^2)^2(\partial_\mu\boldsymbol{\pi})^2+\beta_2(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\cdot\boldsymbol{\pi}^2-i\delta(0)\\&\times[3\alpha_2+\beta_2-\tfrac{1}{2}(3\alpha_1^2+2\alpha_1\beta_1+\beta_1^2)](\boldsymbol{\pi}^2)^2\}+\cdots.\end{aligned} \tag{24}$$

It is not difficult to check that the most divergent terms violating the soft-pion theorems are in fact canceled by corresponding terms of $-i\delta(0)\ln\sqrt{g}$. In second order of $1/f_\pi$ we have

$$\tfrac{1}{2}i\frac{1}{f_\pi^2}(3\alpha_1+\beta_1)\boldsymbol{\pi}^2 i\int\frac{d^4k}{(2\pi)^4},$$

as the most divergent contribution to the pion self-mass. It is canceled by

$$-\tfrac{1}{2}i(1/f_\pi^2)(3\alpha_1+\beta_1)\boldsymbol{\pi}^2 i\delta(0).$$

In fourth order, the constant term violating Adler's condition for the pion-pion amplitude is given by

$$\tfrac{1}{2}i\frac{1}{f_\pi^4}(3\alpha_2+\beta_2)(\boldsymbol{\pi}^2)^2 i\int\frac{d^4k}{(2\pi)^4}$$
$$+\tfrac{1}{8}i^2\frac{1}{f_\pi^4}2(3\alpha_1^2+2\alpha_1\beta_1+\beta_1^2)(\boldsymbol{\pi}^2)^2 i^2\int\frac{d^4k}{(2\pi)^4}.$$

Again it is canceled by

$$-\tfrac{1}{2}i(1/f_\pi^4)[3\alpha_2+\beta_2-\tfrac{1}{2}(3\alpha_1^2+2\alpha_1\beta_1+\beta_1^2)](\boldsymbol{\pi}^2)^2 i\delta(0),$$

etc.

We have thus seen that the most divergent terms in

[5] D. G. Boulwave, Ann. Phys. (N. Y.) 56, 140 (1970).

perturbation theory can be removed by a counterterm originating from the invariant functional measure. The counterterm vanishes in the coordinate system uniquely defined by the condition $g=1$. The question of renormalizability is not affected by this result. The arguments against renormalizability[5] remain untouched.

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Closed-Loop Calculations Using a Chiral-Invariant Lagrangian: An Addendum

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The significance of the condition on the metric $\det g_{ij}=1$, found in an earlier paper to lead to the elimination of the most divergent parts of the amplitudes, is explained on the basis of a paper by Salam and Strathdee.

IN a recent paper[1] we reported the result of application of naive Feynman rules derived from the chiral-invariant Lagrangian density for zero-mass pions,

$$L=\tfrac{1}{2}\partial_\mu\phi_i g_{ij}(\phi)\partial_\mu\phi_j\,, \qquad (1)$$

to closed-loop contributions to invariant amplitudes. We remarked on the presence of contributions which appeared to violate the equivalence theorem, in that they depended explicitly on the choice of pion "gauge," or Weinberg's function $f(\phi^2)$. The contributions are the most divergent parts of the amplitudes. They also fail to vanish in the soft-pion limit, in violation of Adler's condition. It was shown by explicit calculation that for the choice of gauge[2]

$$f(\phi^2)=f_\pi\left[1-\frac{2}{5}\left(\frac{\phi^2}{f_\pi^2}\right)-\frac{9}{175}\left(\frac{\phi^2}{f_\pi^2}\right)^2-\frac{184}{15\,750}\left(\frac{\phi^2}{f_\pi^2}\right)^3+\cdots\right], \qquad (2)$$

these contributions vanished, and the Adler conditions were satisfied.

In a note added in proof we remarked that the same choice of gauge leads to the condition

$$g\equiv\det g_{ij}=1 \qquad (3)$$

on the metric, although we could offer no interpretation of this condition.

We have since seen a paper by Salam and Strathdee[3] in which may be found the missing explanation. Suppose that for a general interaction one starts with a generating functional given in a canonical formulation by

$$Z(I,J)=\int(d\phi)(d\pi)\times\exp\left\{i\int dx[\pi\dot{\phi}-H(\phi,\nabla\phi,\pi)+I\phi+J\pi]\right\}, \qquad (4)$$

where I and J are external source functions, and ϕ and π are a set of fields and canonically conjugate momenta. Then, in passing to the manifestly covariant Lagrangian formulation by setting $J=0$ and performing the functional integration over π, there results

$$Z(I)=\int(d\phi)M(\phi)\exp\left\{i\int dx[L(\phi,\partial\phi)+I\phi]\right\}, \qquad (5)$$

in which occurs a factor $M(\phi)$ which is not usually included. This factor is given explicitly by

$$M(\phi)=\int(du)\exp\left\{-i\int dx[H(\phi,\nabla\phi,\pi_0+u)-H(\phi,\nabla\phi,\pi_0)]-u\frac{\partial H(\phi,\nabla\phi,\pi_0)}{\partial\pi_0}\right\}, \qquad (6)$$

and in fact is unity for theories in which the interaction has no more than one field derivative.

As shown by Salam and Strathdee, in the case of the chiral-invariant theory with Lagrangian density as in (1), the functional integration in (6) is Gaussian and leads to

$$M(\phi)=\exp\left[\tfrac{1}{2}\delta^{(4)}(0)\int dx\ln g(\phi)\right]. \qquad (7)$$

In fact, it plays the role of a measure on the space of the fields, so that $(d\phi)M(\phi)$ is a chiral invariant.

Now it is clear that the generating functional becomes

$$Z(I)=\int(d\phi)\exp\left\{i\int dx[\bar{L}(\phi,\partial\phi)+I\phi]\right\}, \qquad (8)$$

[1] J. M. Charap, Phys. Rev. D 2, 1554 (1970).
[2] An algebraic error in Ref. 1 has been corrected to give the new coefficient of the last term. We are grateful to J. Honerkamp and K. Meetz for pointing out our error.
[3] A. Salam and J. Strathdee, Phys. Rev. D 2, 2869 (1970).