Covariant Derivatives in Broken Nonlinear Chiral Symmetry

MARJAN BAĆE
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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If the symmetry breaking in nonlinear Lagrangian theories depends on derivatives of boson fields, then the covariant derivatives $D_\mu\varphi_a$ lose their nice transformation properties.

In the construction of nonlinear chiral Lagrangians[1] the following procedure is universally followed: Using covariant derivatives of boson and fermion fields, and the fermion fields themselves, but *not* the boson fields, one constructs a Lagrangian density $\mathcal{L}_0(x)$ invariant under SU_n ($n=2$, 3). As shown, for example, by Coleman, Wess, and Zumino,[2] such a Lagrangian density will then be covariant under $SU_n\times SU_n$, also. One now adds a symmetry-breaking term $\epsilon\mathcal{L}'(x)$ in order to obtain partial conservation of axial-vector current (PCAC). The constant ϵ is the symmetry-breaking parameter. If the initial symmetry is realized in terms of Goldstone bosons, as is the case for a Lagrangian density $\mathcal{L}_0$ independent of the boson fields φ_a, and if PCAC is taken to mean the dominance of all matrix elements of $\partial_\mu\mathcal{A}_a{}^\mu(x)$ by the corresponding boson pole, then $\mathcal{L}'(x)$ need not be a function of the fields φ_a only, but can depend on boson and fermion fields as well as their derivatives.[3] Lam and Lee[4] have used this freedom in order to obtain an $SU_3\times SU_3$ Hamiltonian density breaking term $\mathcal{H}'(x)$ with the transformation properties suggested by Gell-Mann, Oakes, and Renner,[5]

$$\mathcal{H}'(x)=u_0(x)+cu_8(x)\,, \tag{1}$$

and depending on the boson fields φ_a, the fermion fields ψ, and the covariant derivatives of φ_a, $D_\mu\varphi_a$, in the following way:

$$\mathcal{H}'(x)=\mathcal{F}(\varphi_a,\psi)+g_{ab}D_\mu\varphi_aD^\mu\varphi_b\,. \tag{2}$$

As Okubo[6] has pointed out Lam and Lee[4] as well as the authors whose work was alluded to above assume that the objects

$$D_\mu\varphi_a\equiv d_{ab}(\varphi)\partial_\mu\varphi_b\,, \tag{3a}$$

$$D_\mu\psi\equiv\partial_\mu\psi+iM_a(\varphi)(\partial_\mu\varphi_a)\psi \tag{3b}$$

defined to have formal covariant transformation properties in the initial, symmetric theory, continue to have these properties in the case of broken symmetry. This, however, is not obviously true since, in order to find the functions $d_{ab}(\varphi)$ and $M_a(\varphi)$, one[1] uses the interchangeability of the vector and axial-vector charges V_a and A_a with ∂_μ, which one cannot do when the symmetry is broken and the A_a are time dependent.

In fact, we show below that $D_\mu\varphi_a$ and $D_\mu\psi$ retain the correct transformation properties

$$[A_a,D_\mu\varphi_b]=-iv_{ac}(\varphi)c_{cbd}D_\mu\varphi_d\,, \tag{4a}$$

$$[A_a,D_\mu\psi]=v_{ab}(\varphi)t_bD_\mu\psi \tag{4b}$$

in the case of time-dependent A_a, only if the theory satisfies[7]

$$\partial(\partial_\mu\mathcal{A}_a{}^\mu)/\partial\pi_b=0 \tag{5a}$$

and

$$\partial(\partial_\mu\mathcal{A}_a{}^\mu)/\partial\bar\psi=0\,, \tag{5b}$$

where π_b is the field canonically conjugate to φ_b. The notation is close to that of Ref. 1; $v_{ab}(\varphi)$ is the fermion field transformation function

$$[A_a,\psi]=v_{ab}(\varphi)t_b\psi\,,$$

where t_b is the isospin matrix appropriate to ψ, the c_{abc} are the structure constants of $SU_n\times SU_n$ ($n=2$, 3), and $\mathcal{A}_a{}^\mu(x)$ is the ath component of the axial-vector current density. It is clear[8] from conditions (5) that if one wants the transformation properties of $D_\mu\varphi_a$ and $D_\mu\psi$ to be those of Eqs. (4), then neither the breaking suggested by Lam and Lee, Eq. (2), nor that by Dashen,[3] in which

$$\partial_\mu\mathcal{A}_a{}^\mu(x)\propto\epsilon\bar\psi\gamma_5\tau_a\psi\,,$$

is acceptable. In fact, if "covariant derivatives" are to remain covariant in the broken theory, we must have conventional partial conservation of axial-vector current (PCAC),[9]

$$\partial_\mu\mathcal{A}_a{}^\mu(x)=\epsilon\varphi_a(x)\,. \tag{6}$$

[1] S. Weinberg, Phys. Rev. **166**, 1568 (1968).

[2] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. Callan, S. Coleman, J. Wess, and B. Zumino, *ibid.* **177**, 2248 (1969).

[3] This has been especially stressed by R. Dashen, Phys. Rev. **183**, 1245 (1969).

[4] Y. M. P. Lam and Y. Y. Lee, Phys. Rev. D **2**, 2976 (1970); Phys. Rev. Letters **23**, 734 (1969).

[5] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

[6] S. Okubo, Phys. Rev. D **2**, 3005 (1970).

[7] All commutators in this paper are assumed to stand for equal-time commutators.

[8] We can clarify this a little more by noting that for a general type of breaking in which $\mathcal{L}'=\mathcal{L}'(\varphi,\psi,\partial_\mu\varphi,\partial_\mu\psi)$, one has

$$\partial_\mu\mathcal{A}_a{}^\mu=-\frac{\partial\mathcal{L}}{\partial\varphi_b}f_{ab}-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_b)}\partial_\mu f_{ab}-\frac{\partial\mathcal{L}}{\partial\psi}v_{ab}t_b\psi-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\mu(v_{ab}t_b\psi),$$

where $f_{ab}(\varphi)$ is defined by Eq. (8). Here $\mathcal{L}=\mathcal{L}_0+\epsilon\mathcal{L}'$, and one should be referred to D. K. Campbell, Nuovo Cimento **58A**, 547 (1968).

[9] Two minor points should be made here. First, the right-hand side of Eq. (6) is not unique in that any function of the field φ_a with appropriate transformation and normalization properties can replace it. For us this merely amounts to a redefinition of the boson field. Second, it is implied that we are working with the fields φ_a and ψ only. If canonical commutation relations are assumed, one can add anomalous terms depending on fields other than φ_a and ψ to the right-hand side of (6) without affecting the covariant derivatives $D_\mu\varphi_a$ and $D_\mu\psi$.

We now proceed to show how Eqs. (5) follow from the demand that Eqs. (4) should hold when the A_a are time dependent. Using (3a) in (4a), we find

$$[A_a,d_{bc}(\varphi)]\partial_\mu\varphi_c+d_{bc}(\varphi)\partial_\mu[A_a,\varphi_c]-d_{bc}(\varphi)[\partial_\mu A_a,\varphi_c]=-iv_{ac}(\varphi)c_{cbd}d_{de}\partial_\mu\varphi_e. \quad (7)$$

For $\mu=1, 2, 3$ this leads to

$$\frac{\partial d_{bc}}{\partial\varphi_e}f_{ae}\partial_\mu\varphi_c+d_{bc}\frac{\partial f_{ac}}{\partial\varphi_e}\partial_\mu\varphi_e=v_{ac}c_{cbd}d_{de}\partial_\mu\varphi_e, \quad (8)$$

since in that case $\partial_\mu A_a\equiv 0$. In Eq. (8), $f_{ab}(\varphi)$ is the boson transformation function satisfying

$$[A_a,\varphi_b]=-if_{ab}(\varphi). \quad (9)$$

For $\mu=0$, Eq. (8) becomes

$$\frac{\partial d_{bc}}{\partial\varphi_e}f_{ae}\dot\varphi_c+d_{be}\frac{\partial f_{ac}}{\partial\varphi_e}\dot\varphi_e-d_{bc}[\dot A_a,\varphi_c]=v_{ac}c_{cbd}d_{de}\dot\varphi_e. \quad (10)$$

Factoring out $\partial_\mu\varphi_c$ in (8), one gets

$$\frac{\partial d_{bc}}{\partial\varphi_e}f_{ae}+d_{be}\frac{\partial f_{ae}}{\partial\varphi_c}=v_{ac}c_{ebd}d_{dc},$$

and using this in Eq. (10), one ends up with

$$d_{bc}[\dot A_a,\varphi_c]=0.$$

It can be seen in general[2] that $d_{bc}(\varphi)$ is a nonsingular tensor[10] so that $d^{-1}{}_{ab}(\varphi)$ can be defined, and we are led to

$$[\dot A_a,\varphi_b]=0$$

or

$$[[A_a,H],\varphi_b]=0, \quad (11)$$

where H is the Hamiltonian of the theory. If one writes

$$H=\int d^3x\,\mathcal{H}(x),$$

then the equation

$$\partial_\mu\mathcal{A}_a{}^\mu(x)=i[\mathcal{H}(x),A_a]$$

(which is due to Gell-Mann,[11,5] and shown by Campbell[8] to be valid independently of the type of breaking used) allows Eq. (11) to be rewritten as

$$\int d^3x[\partial_\mu\mathcal{A}_a{}^\mu(x),\varphi_b(y)]=0. \quad (12)$$

The integrand of (12) is an equal-time commutator and can be expanded as follows:

$$[\partial_\mu\mathcal{A}_a{}^\mu(x),\varphi_b(y)]=\frac{\partial(\partial_\mu\mathcal{A}_a{}^\mu)}{\partial\varphi_c}[\varphi_c(x),\varphi_b(y)]+\frac{\partial(\partial_\mu\mathcal{A}_a{}^\mu)}{\partial\pi_c}[\pi_c(x),\varphi_b(y)]+\frac{\partial(\partial_\mu\mathcal{A}_a{}^\mu)}{\partial\psi}[\psi(x),\varphi_b(y)]+\cdots, \quad (13)$$

where the dots stand for commutators of φ_b with all of the remaining independent fields participating in the construction of $\partial_\mu\mathcal{A}_a{}^\mu(x)$.[12] If canonical commutation relations are assumed, all but the second term vanish and we are led to condition (5a). Equation (5b) is derived in a completely analogous fashion.

Using the fact that the Lam-Lee theory does not satisfy Eq. (5a), it is now a straightforward matter to show that their Hamiltonian density actually does not have the transformation properties (1), as they assumed.

It is interesting that condition (5a) was obtained by Gottlieb[13] in an independent way. He demands that a broken chiral theory produce conventional time-component–space-component current commutators starting from the usual time-component–time-component commutators

$$[\mathcal{A}_a{}^0(x),\mathcal{A}_b{}^0(y)]=ic_{abc}\mathcal{A}_c{}^0(x)\delta^3(x-y),$$

and finds this possible only if $\partial_\mu\mathcal{A}_a{}^\mu$ is independent of field derivatives.

We emphasize that, even though the use of breaking terms which are functions of field derivatives is, in principle, possible, they lead to a change of the transformation properties of most of the objects in the theory; thus, as an example, $\mathcal{L}_0$ will no longer be chiral symmetric, but will have a variation proportional to the symmetry-breaking parameter ϵ.

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[10] A tensor $T_{ab}(\varphi)$ is nonsingular if $T_{ab}(0)=\text{const}\times\delta_{ab}$.
[11] M. Gell-Mann, Phys. Rev. 125, 1067 (1962).
[12] It is assumed that $\partial_\mu\pi_a$ is not one of these.
[13] H. P. W. Gottlieb, Nuovo Cimento Letters 3, 693 (1970).