Extension of the Mass Operator in the Symmetric Quark Model for Negative-Parity Baryon Resonances

MARVIN RESNIKOFF*
Department of Physics, State University of New York at Buffalo, Buffalo, New York 14214
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The two-body mass operators $M_{189}{}^{8S\,\mathbf{L}\cdot\mathbf{S}}$, $M_{189}{}^{8A\,\mathbf{L}\cdot\mathbf{S}}$, and three-body mass operators $M_{35_F}{}^{1\mathbf{L}\cdot\mathbf{S}}$, $M_{35_D}{}^{1\mathbf{L}\cdot\mathbf{S}}$, are considered in fitting the negative-parity baryon resonances in the $SU(6)\times O(3)$ multiplet $(\mathbf{70}, 1^-)$. The latter operators split the degeneracy of the Δ resonances in the decuplet. A least-squares fit to the latest experimental data is performed, and the results are in excellent agreement. It is suggested that a search be made for the two Ω^{-*}'s.

I. INTRODUCTION

It was first suggested by Dalitz[1] that the negative-parity baryon resonances could be fitted in the $(\mathbf{70},1^-)=(SU(6),L^P)$ multiplet. Employing the symmetric quark model with orbital excitation,[2] Greenberg and Resnikoff[3] carried out a detailed calculation and fit of the seven negative-parity baryon resonances which were known at the time. With the same two-body mass operators, Divgi and Greenberg[4] reanalyzed the results one year later with a best fit to 14 baryon resonances.

It is reasonable to ask whether yet another analysis of the negative-partly baryon resonances is required, so we start with a justification for this article. There now exist 20 established baryon resonances[5,6] in the energy range 1400–2000 MeV; the possibility arises, therefore, of predicting the remaining ten isomultiplets of the supermultiplet $(\mathbf{70},1^-)$ with a greater degree of precision. In addition to the number of new resonances which have been established since the publication of Ref. 4 and the revised values of the masses of the old resonances, there is now clear experimental evidence for the existence of two Δ resonances[7] with $J^P=\frac{1}{2}^-,\frac{3}{2}^-$ at 1650 and 1670 MeV, respectively. As was pointed out in Ref. 4, the two-body spin-orbit (or tensor) forces will not split the $j=\frac{1}{2}$, $\frac{3}{2}$ Δ (or Ω) resonances of the decuplet. The three-body spin-orbit operators $M_{35_F}{}^{1\mathbf{L}\cdot\mathbf{S}}$ and $M_{35_D}{}^{1\mathbf{L}\cdot\mathbf{S}}$, however, do split these resonances, and are calculated in this article. For completeness, we have also calculated the two-body spin-orbit mass operators[8] $M_{189}{}^{8A\,\mathbf{L}\cdot\mathbf{S}}$ and $M_{189}{}^{8S\,\mathbf{L}\cdot\mathbf{S}}$, which are not contained in Refs. 3 and 4.

A fit of the 14 established resonances contained in Table I is attempted with both a seven- and eight-parameter mass formula, the 4 $S=0$ antisymmetric mass operators $M_1{}^1$, $M_{35}{}^8$, $M_{189}{}^8$, the two spin-orbit mass operators $M_{35}{}^{1\mathbf{L}\cdot\mathbf{S}}$ and $M_{35}{}^{8\mathbf{L}\cdot\mathbf{S}}$, and the spin-orbit operators $M_{189}{}^{8A\,\mathbf{L}\cdot\mathbf{S}}$, $M_{189}{}^{8S\,\mathbf{L}\cdot\mathbf{S}}$, $M_{35_F}{}^{1\mathbf{L}\cdot\mathbf{S}}$, and $M_{35_D}{}^{1\mathbf{L}\cdot\mathbf{S}}$ acting separately or in pairs. Though the two-body spin-orbit operators $M_{189}{}^{8A\,\mathbf{L}\cdot\mathbf{S}}$ or $M_{189}{}^{8S\,\mathbf{L}\cdot\mathbf{S}}$ produce a good fit to the experimental resonances, either separately or together with the three-body operators, they also predict Σ resonances at 1588 MeV or below. (A possible $\Sigma\pi$ resonance[9] $P_{11}{}^-$ exists at 1560 MeV with positive parity.) The next highest Σ resonance, a $\Lambda\pi$ enhancement at 1618, is 30 MeV above the lowest predicted Σ resonance; hence we reject the **189** spin-orbit operators. We find that a small three-body spin-orbit contribution from the $\mathbf{35}_F$ or $\mathbf{35}_D$ operators provides an excellent fit

TABLE I. Experimental data for resonances in the $(\mathbf{70},1^-)$. The partial wave, J^P assignments, and mass values are from Refs. 5 and 6.

	Resonance	J^P		Resonance	J^P
	$D_{13}'(1520)$	$\frac{3}{2}^-$	Λ	$S_{01}(1405)$	$\frac{1}{2}^-$
	$S_{11}'(1535)$	$\frac{1}{2}^-$		$D_{03}'(1520)$	$\frac{3}{2}^-$
	$D_{15}(1670)$	$\frac{5}{2}^-$		$S_{01}'(1670)$	$\frac{1}{2}^-$
	$S_{11}''(1700)$	$\frac{1}{2}^-$		$D_{03}''(1690)$	$\frac{3}{2}^-$
	$D_{13}''(1700)$	$\frac{3}{2}^-$		$D_{05}(1830)$	$\frac{5}{2}^-$
Δ	$S_{31}(1650)$	$\frac{1}{2}^-$	Σ	1618	?
	$D_{33}(1670)$	$\frac{3}{2}^-$		$D_{13}(1670)$	$\frac{3}{2}^-$
Ξ	1635	?		1690	?
	1930	$\frac{5}{2}^-$?		$S_{11}(1750)$?	$\frac{1}{2}^-$
	1815	$\frac{3}{2}^-$?		$D_{15}(1765)$	$\frac{5}{2}^-$

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[1] R. H. Dalitz, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, California, 1966* (University of California Press, Berkeley, 1967), pp. 215–236; in *Proceedings of the Oxford International Conference on Elementary Particles* (Rutherford High-Energy Laboratory, Chilton, Berkshire, England, 1966). The simple fact that negative-parity baryons could be placed in the (**70**, 1⁻) multiplet was first made by Greenberg (Ref. 2). However, Dalitz, using two-body mass operators, first exhibited a detailed qualitative agreement with the negative-parity baryon mass spectrum.

[2] O. W. Greenberg, Phys. Rev. Letters **13**, 598 (1964).

[3] O. W. Greenberg and M. Resnikoff, Phys. Rev. **163**, 1844 (1967).

[4] D. R. Divgi and O. W. Greenberg, Phys. Rev. **175**, 2024 (1968).

[5] Particle Data Group, Rev. Mod. Phys. **42**, 87 (1970).

[6] R. Levi-Setti, rapporteur's talk in *Proceedings of the Lund International Conference on Elementary Particles, 1969*, edited by G. von Dardel (Berlingska, Lund, Sweden, 1969).

[7] A. Donnachie, rapporteur's talk in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).

[8] A linear combination of the two $M_{189}{}^{8\mathbf{L}\cdot\mathbf{S}}$ mass operators has also been calculated by D. R. Divgi, thesis, Part I, University of Maryland, 1969 (unpublished).

[9] L. Armenteros *et al.* (CERN-Heidelberg-Saclay collaboration), in *Proceedings of the Lund International Conference on Elementary Particles, 1969*, edited by G. von Dardel (Berlingska, Lund, Sweden, 1969); also referred to in Ref. 6.

with the data. Thus, one new spin-orbit operator and parameter is added to the previous results.

We also note that the method of calculation and fit with the experimental data differ from Refs. 3 and 4. Extensive use is made of $SU(2)$ 6-j coefficients, and $SU(3)$ and $SU(6)$ isoscalars, rather than Bose annihilation and creation operators. The derivation is more transparent and the results can be presented in compact form. The required coefficients and isoscalars are presented in tables, but the explicit calculations are not carried out in this paper. The mass formulas are fitted on the CDC 6400 with a program which minimizes the sum of squares of the differences between the experimental and theoretical numbers. This represents an improvement over previous calculations in that the computer rapidly finds *the* best fit, to the accuracy desired. It is established that the solution is not a local minima of the least-squares function.

Since the results agree so well with experiment, it is important that the missing masses of the $(\mathbf{70},1^-)$ be found. To this end, we discuss in some detail the properties of the Ω^{-*} resonances.

In Sec. II we derive the general two-body and three-body spin-orbit mass formulas, and apply them to the special cases under consideration. In Sec. III, we discuss the experimental situation and predictions of the symmetric quark model, with special attention to the Ω^{-*} resonances.

II. DERIVATION OF MASS FORMULAS

In this section we derive the required mass formulas for the symmetric quark model in the $(\mathbf{70},1^-)$ multiplet. We briefly summarize first the assumptions of the symmetric quark model: (a) symmetric statistics for the spin-$\frac{1}{2}$ quarks, (b) the predominance of two-body forces, and (c) octet dominance for the mass operators. Arguments for assumption (a) have previously been presented.[2,3] The most reasonable assumption regarding the ground-state space wave function of a three-particle system is that it has orbital configuration s^3, $L=0$, and that it is a symmetric state.[2] This is supported by the experimental fact that the neutron and proton form factors are smooth and always positive,[10,11] whereas an antisymmetric wave function implies a zero in the form factor.[12] Since the spin-unitary spin part of the wave function, the $\mathbf{56}$ representation of $SU(6)$, is also completely symmetric, the total $(\mathbf{56},0^+)$ wave function is symmetric under permutations of the quarks. We conclude that the quarks obey Bose statistics[13] and that

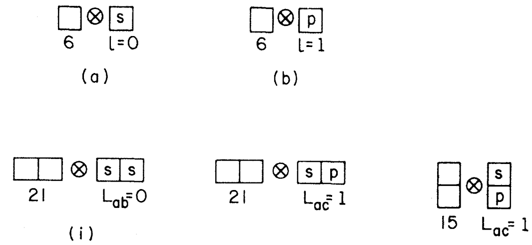

FIG. 1. One-body and symmetric two-body states (ab). The boxes represent Young tableaux.

the wave function for the $(\mathbf{70},1^-)$ multiplet must also be completely symmetric. Assumption (b) is also physically reasonable, but it must be relaxed to include a small three-body spin-orbit contribution in order to break the mass degeneracy of the $j=\frac{1}{2}, \frac{3}{2}$, Δ (and Ω) resonances of the decuplet. Assumption (c) is a restriction on the $SU(3)$ tensor properties of the mass operators, that they transform only as $SU(3)$ singlet and octet representations, and that contributions due to the 27-dimensional representation are ignored. This requirement leads to the Gell-Mann–Okubo formula for unmixed states.

Under assumption (a) then, we proceed to construct a completely symmetric wave function for the $(\mathbf{70},1^-)$ multiplet. By analogy with nuclear calculations, we must determine the fractional parentage coefficients for symmetric states of the group $SU(6)\times O(3)$. With no loss of generality, let us label the three single-particle states in the orbital configuration s^2p, $a=b=(6,\ l=0)$, and $c=(6,\ l=1)$. The possible symmetric two-body states (ab) [and (ac)] are listed in Fig. 1. Assuming relative coordinates with c.m. motion removed, only two kinds (not three, as in Fig. 1) of space states may occur: $l=0$, symmetric in space, and $l=1$, antisymmetric in space. We take this fact into account in the evaluation of the mass operator $M(\mathbf{70},1;\alpha,\alpha')$, Eq. (2.5a). An unsymmetrized wave function for the $(\mathbf{70},1^-)$ multiplet may be written

$$|\mathbf{70},1;\alpha\rangle=\sum_{\alpha_{ab},\alpha_c}\begin{pmatrix}(ab) & c & (\mathbf{70},1)\\ \alpha_{ab} & \alpha_c & \alpha\end{pmatrix}\times|(ab);\alpha_{ab}\rangle_{12}|c,\alpha_c\rangle_3, \quad (2.1)$$

where

$$\begin{pmatrix}(ab) & c & (\mathbf{70},1)\\ \alpha_{ab} & \alpha_c & \alpha\end{pmatrix}$$

is the $SU(3)\times O(3)$ coupling coefficient and α represents

TABLE II. $SU(6)$ recoupling coefficients from the state $(ab)\times c$ to $a\times(bc)$.

	6×15	6×21
21×6	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$
15×6	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$

[10] S. Ishida, K. Konno, and H. Shimodaira, Nuovo Cimento **46A**, 189 (1966); A. N. Mitra and R. Majumdar, Phys. Rev. **150**, 1194 (1966).

[11] W. Albrecht *et al.*, Phys. Rev. Letters **17**, 1192 (1966). These latest data show a positive form factor up to 245 F^{-2}.

[12] R. E. Kreps and J. J. de Swart, Phys. Rev. **162**, 1729 (1967). By introducing four parameters and a rather complicated antisymmetric wave function, it is possible to push the zero beyond the experimental region, though the form factor is no longer "smooth."

[13] Instead of parastatistics (Ref. 2), it is possible to use the three-triplet model of M. Y. Han and Y. Nambu, Phys. Rev. **139**, B1006 (1965).

the $SU(3)\times SU(2)$, $O(3)$ row labels, $\alpha=(\rho,ii_zy;ss_z;m)$, and ρ is the $SU(3)$ representation. The symbol (ab) here represents the coupled state $(ab)=(\mathbf{21},\ L_{ab}=0)$, and $|(ab);\alpha_{ab}\rangle_{12}$ represents the coupled state of particles 1 and 2, $|c,\alpha_c\rangle_3$ the state of particle 3. Contained in the coupling coefficient of Eq. (2.1) is the fact that we have first coupled three quarks to obtain a state of the **70**, and three single-particle[14] angular momenta l_i, $i=a,\ b,\ c$, to obtain total $L=1$, and then coupled total L to S of the **70** to give the spin J. This will be made clearer in the derivation of the mass formulas below.

To construct the completely symmetrized wave function of the $(\mathbf{70},1^-)$ representation, we permute the particles 1, 2, and 3, then employ the recoupling coefficients for $SU(6)\times O(3)$ to write the symmetrized wave function in the form

$$|\mathbf{70},1;\alpha\rangle_{\rm sym}=\frac{1}{\sqrt{3}}\Bigg\{\sum_{\alpha_{ab},\alpha_c}\begin{pmatrix}\mathbf{21},l=0 & c & \mathbf{70},1\\ \alpha_{ab} & \alpha_c & \alpha\end{pmatrix}|\mathbf{21},\alpha_{ab}\rangle_{12}|c,\alpha_c\rangle_3 + \sum_{(bc),\alpha_{bc},\alpha_a}(6\text{-}j)_{(bc)}\begin{pmatrix}(bc) & a & \mathbf{70},1\\ \alpha_{bc} & \alpha_a & \alpha\end{pmatrix}|(bc),\alpha_{bc}\rangle_{12}|a,\alpha_a\rangle_3 + \sum_{(ca),\alpha_{ca},\alpha_b}(6\text{-}j)_{(ca)}\begin{pmatrix}(ca) & b & \mathbf{70},1\\ \alpha_{ca} & \alpha_b & \alpha\end{pmatrix}|(ca),\alpha_{ca}\rangle_{12}|b,\alpha_b\rangle_3\Bigg\}, \tag{2.2}$$

where $(6\text{-}j)_{(bc)}$ and $(6\text{-}j)_{(ca)}$ are the $SU(6)\times O(3)$ recoupling coefficients listed in Table II. The coefficient $(6\text{-}j)_{(bc)}$ takes us from the coupling $(a\times b)\times c$ to $(b\times c)\times a$, or, in Young tableau form,

$$(i)\otimes(b)=(ii)\otimes(a)+(iii)\otimes(a), \tag{2.3}$$

where (i)–(iii), (a), and (b) are given in Fig. 1 and the coupled state (bc) represents the two possible $L_{bc}=1$ states on the right-hand side of Eq. (2.3).

We first consider the scalar $S=0$ two-body forces. The mass operator $M_{(12)}$ which acts on the two-body states $|(ab),\alpha_{ab}\rangle_{12}$ [or (bc)] is constructed from the direct products $\mathbf{15}\otimes\mathbf{15}^*$, $\mathbf{21}\otimes\mathbf{21}^*$ and is of the form

$$M_{(12)}=M_1+M_{35}{}^8+M_{405}{}^1+M_{405}{}^8+M_{189}{}^1+M_{189}{}^8, \tag{2.4}$$

assuming octet dominance, where $i=Y=0$ and $L=S=0$. The matrix elements of the mass operator (2.4) between states of the $(\mathbf{70},1^-)$ may be immediately set down using the results of Bég and Singh[15] to determine the two-body matrix elements

$$\langle\mathbf{70},1;\alpha|M_{(12)}|\mathbf{70},1;\alpha'\rangle=M(\mathbf{70},1;\alpha,\alpha')=\tfrac{1}{2}\Bigg\{\sum_{\alpha_{ab},\alpha_c}\begin{pmatrix}\mathbf{21},0 & c & \mathbf{70},1\\ \alpha_{ab} & \alpha_c & \alpha\end{pmatrix}\begin{pmatrix}\mathbf{21},0 & c & \mathbf{70},1\\ \alpha_{ab} & \alpha_c & \alpha'\end{pmatrix}\langle\mathbf{21},0|M_{(12)}|\mathbf{21},0\rangle + \sum_{\alpha_{bc},\alpha_a}\begin{pmatrix}\mathbf{15},1 & a & \mathbf{70},1\\ \alpha_{bc} & \alpha_a & \alpha\end{pmatrix}\begin{pmatrix}\mathbf{15},1 & a & \mathbf{70},1\\ \alpha_{bc} & \alpha_a & \alpha'\end{pmatrix}\langle\mathbf{15},1|M_{(12)}|\mathbf{15},1\rangle\Bigg\}, \tag{2.5a}$$

where

$$\begin{aligned}\langle\mathbf{21},0|M_{(12)}|\mathbf{21},0\rangle&=m_0+m_1Y+m_2[2S(S+1)+C_2{}^{(3)}]+m_3[i(i+1)-\tfrac{1}{4}Y^2],\\ \langle\mathbf{15},1|M_{(12)}|\mathbf{15},1\rangle&=m_0'+m_1'Y+m_2'[2S(S+1)-C_2{}^{(3)}]+m_3'[i(i+1)-\tfrac{1}{4}Y^2],\end{aligned} \tag{2.5b}$$

and the m_i, m_i' are linear combinations of the reduced matrix elements $\langle\mathbf{21},\ l=0||(\mu,\rho)||\mathbf{21},\ l=0\rangle$, $\langle\mathbf{15},\ l=1|\times|(\mu,\rho)||\mathbf{15},\ l=1\rangle$, respectively. Because of parity considerations, the matrix element $\langle\mathbf{15},\ l=1|M_{(12)}|\mathbf{21},\ l=1\rangle$ vanishes; the matrix element $\langle\mathbf{21},\ l=1|M_{(12)}|\mathbf{21},\ l=1\rangle$ is set equal to the matrix element $\langle\mathbf{21},\ l=0|M_{(12)}|\mathbf{21},\ l=0\rangle$, since there exists only a symmetric $l=0$ relative space state. The four reduced matrix elements m_i are determined from the multiplet $(\mathbf{56},0^+)$ by a least-squares fit to the experimental data. The values $m_0=739.23$, $m_1=-336.165$, $m_2=49.707$, and $m_3=104.4$ yield a sum of squares between experimental and theoretical numbers of 204.3 MeV^2 or a rms difference of approximately 5 MeV. The four reduced matrix elements m_i', and those due to spin-orbit effects, will be adjusted to give a best fit to the $(\mathbf{70},\ 1^-)$ data.

The mass operator $M_{(12)}$ [Eq. (2.4)] has $i=Y=0$ and $L=S=0$, and the two-body matrix elements (2.5b) do not depend on the row labels (i_z,s_z,m) of $SU(6)\times O(3)$. We may therefore perform a trivial sum over the $SU(2)$ coupling coefficients, and only the product of $SU(3)$ and $SU(6)$ isoscalars remain,

$$\begin{pmatrix}(ab) & \mathbf{6} & \mathbf{70}\\ \alpha_{ab} & \alpha_c & \alpha\end{pmatrix}=\begin{pmatrix}\rho_{ab} & \mathbf{3} & \rho\\ (IY)_{ab} & (IY)_c & (IY)\end{pmatrix}\begin{pmatrix}(ab) & \mathbf{6} & \mathbf{70}\\ (\rho,2S+1)_{ab} & (\mathbf{3},2) & (\rho,2S+1)\end{pmatrix}, \tag{2.6}$$

[14] The single-particle states are somewhat artificial, but the exact relative states may be found rather easily (see Ref. 3).
[15] M. A. B. Bég and V. Singh, Phys. Rev. Letters **13**, 418 (1964).

where ρ is the $SU(3)$ representation, and (ab) represents here the **15** or **21** representation. To determine the masses in terms of the parameters m_i and m_i', it is necessary to know the appropriate $SU(6)$ and $SU(3)$ isoscalars which are listed in Tables III and IV, respectively. The absolute phase convention of the $SU(6)$ coupling coefficients is chosen such that

$$\begin{pmatrix} (ab) & 6 & \mathbf{70} \\ \alpha_{ab}|_{\max} & \alpha_c & \alpha|_{\max} \end{pmatrix}$$

is positive for each $[SU(3), SU(2)]$ multiplet, where $\alpha|_{\max}=(\rho,iiY_{\max};ss,l)$, with i being that isospin associated with the maximum y value. The relative phase convention for $SU(3)$ states is the same as de Swart's,[16] for $SU(2)$ states the same as Condon and Shortley's[17]; the relative phase between a state and its conjugate is $(-1)^{i_z+\frac{1}{2}Y}$ for $SU(3)$ states. The two-body spin-orbit mass operators split only the relative $l_{bc}=1$ space state, or the antisymmetric space state **15**, $l_{bc}=1$. The possible spin-orbit mass operators occur in the direct product $15\times 15^*$,

$$M_{(12)}{}^{\mathbf{L\cdot S}}=M_{35}{}^{1\mathbf{L\cdot S}}+M_{35}{}^{8\mathbf{L\cdot S}}+M_{189}{}^{8A\mathbf{L\cdot S}}+M_{189}{}^{8S\mathbf{L\cdot S}}. \tag{2.7}$$

The mass matrix element can be written as

$$\langle \mathbf{70},1;\alpha|M_{(12)}{}^{\mathbf{L\cdot S}}|\mathbf{70},1;\alpha'\rangle=\tfrac{2}{3}\sum_{\alpha_{bc},\alpha_a,\alpha_{bc}',\alpha_a'}\begin{pmatrix} 15 & 6 & 70 \\ \alpha_{bc} & \alpha_a & \alpha\end{pmatrix}\begin{pmatrix} L=1 & S & j \\ \sigma & s_z & m_j\end{pmatrix}\begin{pmatrix} 15 & 6 & 70 \\ \alpha_{bc}' & \alpha_a & \alpha'\end{pmatrix}\begin{pmatrix} L=1 & S' & j \\ \sigma' & s_z' & m_j\end{pmatrix}$$
$$\times(|\mathbf{15},\alpha_{bc}\rangle_{12},M_{(12)}{}^{\mathbf{L\cdot S}}|\mathbf{15},\alpha_{bc}'\rangle_{12}). \tag{2.8}$$

The spin-orbit mass operator $M_{(12)}{}^{\mathbf{L\cdot S}}$ may be factored into the product of $SU(6)$, $O(3)$ components

$$M_\mu{}^{\rho,J}=\sum_{s_z,l_z}M_{\mu s_z}{}^{(\rho,S)}M_{l_z}{}^{(L)}\begin{pmatrix} S & L & J \\ s_z & l_z & M_J\end{pmatrix},\quad M_\mu{}^{\rho,0}\equiv M_\mu{}^{\rho\mathbf{L\cdot S}}=\sum_{l_z}M_{\mu-l_z}{}^{(\rho,L)}M_{l_z}{}^L\frac{(-1)^{L-l_z}}{(2L+1)^{1/2}}. \tag{2.9}$$

Since the mass operator $M_{(12)}{}^{\mathbf{L\cdot S}}$ has $i=0$, $Y=0$, and is a scalar in L,S space, it is clear that the mass will not depend on the row labels i_z, s_z, m, and that they may be summed over. If we write the $SU(6)$ coupling coefficients as a product of $SU(6)$, $SU(3)$ isoscalars and $SU(2)$ coupling coefficients, we may sum over the $SU(2)$ coefficients; this yields a product of two $SU(2)$ recoupling coefficients.[18] The result is

$$\langle \mathbf{70},1;\alpha|M_\mu{}^{\rho\mathbf{L\cdot S}}|\mathbf{70},1;\alpha'\rangle=m_k'\sum\begin{pmatrix} 15 & 6 & 70 \\ (\rho,2S+1)_{ab} & (3,2) & (\rho,2S+1)\end{pmatrix}$$
$$\times\begin{pmatrix} \rho_{ab} & 3 & \rho \\ (IY)_{ab} & (IY)_c & (IY)\end{pmatrix}\begin{pmatrix} 15 & 6 & 70 \\ (\rho',2S'+1)_{ab} & (3,2) & (\rho',2S'+1)\end{pmatrix}\begin{pmatrix} \rho_{ab}' & 3 & \rho' \\ (IY)_{ab} & (IY_c) & (IY)'\end{pmatrix}$$
$$\times\begin{pmatrix} 15 & 15^* & \mu \\ (\rho',2S'+1)_{ab} & (\rho^*,2S+1)_{ab} & (\rho,3)\end{pmatrix}\begin{pmatrix} \rho_{ab}' & \rho_{ab}^* & \rho \\ (IY)_{ab} & (I,-Y)_{ab} & (00)\end{pmatrix}\frac{f(S_1,S_1')}{(2I_{ab}+1)^{1/2}}, \tag{2.10a}$$

where $f(S_1,S_1')$ is

$$f(S_1,S_1')=(-1)^{S_1'+1/2-j}[(2S+1)(2S'+1)]^{1/2}\times\begin{Bmatrix} S & S' & 1 \\ S_1' & S_1 & \frac{1}{2}\end{Bmatrix}\times\begin{Bmatrix} S & 1 & j \\ 1 & S' & 1\end{Bmatrix} \tag{2.10b}$$

and $\rho=1$ or 8, if the operator is singlet or octet. If $\rho=1$, then $\rho_{ab}=\rho_{ab}'$, and the isoscalar

$$\begin{pmatrix} \rho_{ab} & \rho_{ab}^* & 1 \\ (IY)_{ab} & (I-Y)_{ab} & (00)\end{pmatrix}$$

becomes $(2i_{ab}+1)^{1/2}$. The four coefficients m_k' represent the four reduced matrix elements for the four mass operators in Eq. (2.7). The braced coefficients are

TABLE III. $SU(6)$ isoscalar factors $\mathbf{15}\otimes\mathbf{6}$, $\mathbf{21}\otimes\mathbf{6}$. The phase convention is explained in the text.

15×6		20		70			
		(1,4)	(8,2)	(10,2)	(1,2)	(8,2)	(8,4)
(3*,3)	(3,2)	1	$-\sqrt{\frac{1}{2}}$	0	1	$\sqrt{\frac{1}{2}}$	1
(6,1)	(3,2)	0	$\sqrt{\frac{1}{2}}$	-1	0	$\sqrt{\frac{1}{2}}$	0
21×6		56		70			
		(10,4)	(8,2)	(10,2)	(1,2)	(8,2)	(8,4)
(3*,1)	(3,2)	0	$\sqrt{\frac{1}{2}}$	0	1	$-\sqrt{\frac{1}{2}}$	0
(6,3)	(3,2)	1	$\sqrt{\frac{1}{2}}$	-1	0	$\sqrt{\frac{1}{2}}$	1

[16] J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963). The highest-weight state is maximum isospin I, $I_z=I$ and associated Y.

[17] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge U. P., London, 1935).

[18] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U. P., Princeton, N. J., 1957), p. 90.

TABLE IV. $SU(3)$ isoscalar factors **3***×**3**, **6**×**3**. The phase convention is explained in text.

			8		**1**			
3*×**3**	$(\frac{1}{2},1)$	$(1,0)$	$(0,0)$	$(\frac{1}{2},-1)$	$(0,0)$			
$(0,\frac{2}{3})$ $(\frac{1}{2},\frac{1}{3})$	1	0	0	0	0			
$(\frac{1}{2},\frac{2}{3})$ $(0,-\frac{2}{3})$	0	0	$\sqrt{\frac{2}{3}}$	0	$\sqrt{\frac{1}{3}}$			
$(\frac{1}{2},-\frac{1}{3})$ $(\frac{1}{2},\frac{1}{3})$	0	1	$-\sqrt{\frac{1}{3}}$	0	$\sqrt{\frac{2}{3}}$			
$(\frac{1}{2},-\frac{1}{3})$ $(0,-\frac{2}{3})$	0	0	0	1	0			
			10				**8**	
6×**3**	$(\frac{3}{2},1)$	$(1,0)$	$(\frac{1}{2},-1)$	$(0,-2)$	$(\frac{1}{2},1)$	$(1,0)$	$(0,0)$	$(\frac{1}{2},-1)$
$(1,\frac{2}{3})$ $(\frac{1}{2},\frac{1}{3})$	1	0	0	0	1	0	0	0
$(1,\frac{2}{3})$ $(0,-\frac{2}{3})$	0	$\sqrt{\frac{1}{3}}$	0	0	0	$-\sqrt{\frac{2}{3}}$	0	0
$(\frac{1}{2},-\frac{1}{3})$ $(\frac{1}{2},\frac{1}{3})$	0	$\sqrt{\frac{2}{3}}$	0	0	0	$\sqrt{\frac{1}{3}}$	+1	0
$(\frac{1}{2},-\frac{1}{3})$ $(0,-\frac{2}{3})$	0	0	$\sqrt{\frac{2}{3}}$	0	0	0	0	$-\sqrt{\frac{1}{3}}$
$(0,-\frac{4}{3})$ $(\frac{1}{2},\frac{1}{3})$	0	0	$\sqrt{\frac{1}{3}}$	0	0	0	0	$\sqrt{\frac{2}{3}}$
$(0,-\frac{4}{3})$ $(0,-\frac{2}{3})$	0	0	0	1	0	0	0	0

$SU(2)$ 6-j coefficients. To evaluate the above matrix elements, the isoscalar factors in Tables III–VI are required.

As we have previously noted, the two-body spin-orbit mass operators do not split the unmixed $j=\frac{1}{2}, \frac{3}{2}$ states of the decuplet, the Δ and Ω resonances. The reason can be seen directly from Tables III and V. To construct a state of the decuplet in the **70** representation, we require a (**6**,1) multiplet of the **15** in the direct product **15**×**6**=**70**+**20**. However, no spin-orbit mass operator can be constructed from (**6**,1)×(**6***,1). It is an experimental fact that the $j=\frac{1}{2}, \frac{3}{2}$ Δ resonances are split; thus it will be necessary to consider three-body spin-orbit mass operators.

The three-body spin-orbit mass operators occur in the direct product **70**×**70***,

$$\mathbf{70}\times\mathbf{70}^*=\mathbf{1}+\mathbf{35}_F+\mathbf{35}_D+\mathbf{189}+\mathbf{280}+\mathbf{280}^*+\mathbf{405}+\mathbf{3675}. \quad (2.11)$$

We choose the operators $M_{35_F}{}^{1\mathrm{L}\cdot\mathrm{S}}$, $M_{35_D}{}^{1\mathrm{L}\cdot\mathrm{S}}$, which have diagonal matrix elements in the decuplet; higher-representation choices, of course, are possible. We take matrix elements of these operators between states of the (**70**,1$^-$). These operators act on the three-body state and it is not required to decompose the wave function into two-body symmetric states. The symmetry of the wave function is not a consideration here. The operators may be written as in Eq. (2.9), and a summation performed over the magnetic quantum numbers. The resultant matrix elements may be written

$$\langle \mathbf{70},1;\alpha | M_{35_F}{}^{1\mathrm{L}\cdot\mathrm{S}} | \mathbf{70},1;\alpha' \rangle = m_F \frac{(-1)^{S'-j}}{(2S'+1)^{1/2}} \times \begin{pmatrix} \mathbf{35} & \mathbf{70} & \mathbf{70} \\ (1,3) & (\rho,2S'+1) & (\rho,2S+1) \end{pmatrix} \times \begin{Bmatrix} S' & 1 & j \\ 1 & S & 1 \end{Bmatrix} \quad (2.12)$$

and the same equation with F replaced by D. The required $SU(6)$ isoscalar factors[19] appear in Table VII; only the lower four rows of the table are required for the calculation. We note that the 35_F operator has only diagonal matrix elements, whereas the 35_D operator has off-diagonal matrix elements between the octets.

Configuration mixing between supermultiplets (**56**,0$^+$) and (**70**,1$^-$) would *a priori* be expected.[20,21] The average of the experimental masses of the isomultiplets in the (**56**,0$^+$) is 1299 MeV, with a spread from $N(939)$ to $\Omega^-(1673)$, and of the calculated masses of the isomultiplets in the (**70**,1$^-$) is 1734 MeV, with a spread from

TABLE V. $SU(6)$ isoscalar factors **15**×**15***. The phase convention is explained in the text.

		35		**189**	
15×**15***	(8,3)	(1,3)	(8,1)	$(8,3)_A$	$(8,3)_S$
(3*,3) (3,3)	$\frac{1}{2}$	1	$\sqrt{\frac{3}{8}}$	0	$\sqrt{\frac{3}{4}}$
(3*,3) (6*,1)	$\sqrt{\frac{3}{8}}$	0	0	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{8}}$
(6,1) (3,3)	$\sqrt{\frac{3}{8}}$	0	0	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{8}}$
(6,1) (6*,1)	0	0	$-\sqrt{\frac{5}{8}}$	0	0

TABLE VI. $SU(3)$ isoscalar factors **3***×**6***. The phase convention is explained in the text.

			8	
3*×**6***	$(1,\frac{1}{2})$	$(0,0)$	$(0,1)$	$(-1,\frac{1}{2})$
$(-\frac{1}{3},\frac{1}{2})$ $(\frac{4}{3},0)$	$-\sqrt{\frac{2}{3}}$	0	0	0
$(\frac{2}{3},0)$ $(\frac{1}{3},\frac{1}{2})$	$\sqrt{\frac{1}{3}}$	0	0	0
$(-\frac{1}{3},\frac{1}{2})$ $(\frac{1}{3},\frac{1}{2})$	0	-1	$-\sqrt{\frac{1}{3}}$	0
$(\frac{2}{3},0)$ $(-\frac{2}{3},1)$	0	0	$\sqrt{\frac{2}{3}}$	0
$(-\frac{1}{3},\frac{1}{2})$ $(-\frac{2}{3},1)$	0	0	0	-1

[19] An independent calculation of the $SU(6)$ isoscalar factors in the direct product **70**×**70***=2(**35**)+··· was first performed by E. Golowich, Phys. Rev. **184**, 1815 (1969).

[20] Configuration mixing was ignored in Ref. 3 (see Ref. 28).

[21] I thank Professor Roland Good for a useful discussion on this point.

TABLE VII. $SU(6)$ isoscalar factor $\mathbf{35}\times\mathbf{70}$.[a]

$\mathbf{35}\times\mathbf{70}$	70_D (10,2)	(8,4)	(8,2)	(1,2)	70_F (10,2)	(8,4)	(8,2)	(1,2)
(8,3) (10,2)	40	$-65\sqrt{2}$	$7\sqrt{5}$	0	2	$-\sqrt{5}$	$\sqrt{10}$	0
(8,3) (8,4)	$26\sqrt{2}$	$55\sqrt{5}, -35$	$4\sqrt{5}, 40$	6	$\sqrt{8}$	$\sqrt{10}$	$-\sqrt{10}, \sqrt{2}$	-4
(8,3) (8,2)	$-7\sqrt{2}$	$10\sqrt{2}, 20\sqrt{10}$	$22\sqrt{5}, -26$	5	$\sqrt{8}$	$-1, \sqrt{5}$	$\sqrt{2}$	4
(8,3) (1,2)	0	$9\sqrt{10}$	15	0	0	-1	$-\sqrt{2}$	0
(8,1) (10,2)	$-4\sqrt{3}$	0	$-11\sqrt{15}$	0	$-2\sqrt{3}$	0	0	0
(8,1) (8,4)	0	$-55\sqrt{3}, 7\sqrt{15}$	0	0	0	$-\sqrt{6}$	0	0
(8,1) (8,2)	$11\sqrt{6}$	0	$4\sqrt{3}$	$-11/\sqrt{3}$	0	0	$-\sqrt{6}$	0
(8,1) (1,2)	0	0	$-11\sqrt{3}$	0	0	0	0	0
(1,3) (10,2)	-20	0	0	0	1	0	0	0
(1,3) (8,4)	0	$10\sqrt{2}$	$22\sqrt{2}$	0	0	$+\sqrt{5}$	0	0
(1,3) (8,2)	0	$22\sqrt{5}$	$-2\sqrt{2}$	0	0	0	1	0
(1,3) (1,2)	0	0	0	4	0	0	0	1

[a] For normalization purposes, the isoscalars for 70_F must be multiplied by the factor $1/\sqrt{33}$; the isoscalars for 70_D must be multiplied by the following factors: $1/8\sqrt{66}$, $1/16\sqrt{165}$, $1/16\sqrt{33}$, $\frac{1}{4}(3/22)^{1/2}$, in the columns (10,2), (8,4, (8,2), and (1,2), respectively. Where two numbers appear in the same column, they represent the isoscalars for the symmetric, antisymmetric octet in the $SU(3)$ direct product $\mathbf{8}\times\mathbf{8}$, respectively. The phase convention is explained in the text.

$\Delta(1405)$ to $\Omega^{-*}(2012)$. Thus, there is a large overlap between the two supermultiplets, the splitting between supermultiplets comparable to the splitting within supermultiplets. Nevertheless, under the same assumptions employed to derive the two-body mass formulas, there is no configuration mixing by the two-body mass operators, as we now show.

We write a state of the $(\mathbf{56},0^+)$ in the following transparent form:

$$\Psi(\mathbf{56},0^+)=\chi(i)\chi(a)\,, \tag{2.13a}$$

where (a) and (i) are the one- and two-body states, respectively, in Fig. 1. A state of the $(\mathbf{70},1^-)$ may be written

$$\Psi(\mathbf{70},1)^-=\chi(i)\chi(b)+\chi(ii)\chi(a)+\chi(iii)\chi(a)\,. \tag{2.13b}$$

The explicit form of Eq. (2.13b) is given by Eq. (2.2). The states (ii) and (iii) have $L_{ac}=1$. The matrix element of the two-body mass operator, $M|_{\text{two body}}$, either Eq. (2.4) or (2.7), between $(\mathbf{56},0^+)$ and $(\mathbf{70},1^-)$ becomes

$$(\Psi(\mathbf{56},0^+),M|_{\text{two body}}\Psi(\mathbf{70},1^-))=(\psi(i),M|_{\text{two body}}\chi(ii)) \\ +(\chi(i),M|_{\text{two body}}\chi(iii))\,, \tag{2.14}$$

using the fact that the mass operator acts only on the two-body state and the single-particle states with $l=0$ and $l=1$ are orthogonal. The matrix element of the non-spin-orbit mass operator [Eq. (2.4)], with $L=0$, vanishes, using the Wigner-Eckart theorem on the space wave function. The spin-orbit two-body mass operator [Eq. (2.7)] acts only on the antisymmetric space state (iii); this matrix element has already been eliminated in the derivation of the two-body mass formulas [Eq. (2.5b)] by parity considerations.

There remains configuration mixing due to the three-body spin-orbit mass operator since the $\mathbf{56}$ representation is contained in the direct product $\mathbf{35}\times\mathbf{70}$. As we shall see in Sec. III, the maximum splitting within the $(\mathbf{70},1^-)$ multiplet due to three-body spin-orbit mass operators is 28 MeV, and the configuration mixing is smaller and can therefore be ignored.

III. RESULTS AND DISCUSSION

The mass formulas, Eqs. (2.5), and (2.10) and (2.12), are fitted to the 14 established (with no question marks) resonances listed in Table I. Isomultiplets with the same I, Y, and J values are mixed. In the case of the Σ, Ξ, and Λ resonances, 3×3 matrices are explicitly diagonalized, and expressed as functions of the reduced matrix elements m_k'; the 2×2 matrices of the N resonances are treated similarly. The $J^P=\frac{5}{2}^-$ octet and the Δ, Ω^{-*} resonances are unmixed. The sum of the squares of the differences[22] between the calculated and experimental masses,

$$F(m_k')=\sum_{i=1}^{14}(\Delta_i)^2\,, \tag{3.1a}$$

where

$$\Delta_i=M_i|_{\text{calc}}-M_i|_{\text{expt}} \tag{3.1b}$$

($M_i|_{\text{calc}}$ and $M_i|_{\text{expt}}$ are the calculated and experimental masses), is then minimized by adjustment of the reduced matrix elements. The computer program FMCG, an IBM subroutine, finds the minimum function $F_{\min}$, to the accuracy desired, by adjusting all the parameters m_k' in the direction, in m_k' space, of the greatest change in the function F towards the minimum $F_{\min}$. For an eight-parameter fit, summed over 14 resonances, the average required CDC 6400 time is 15 sec. The resonances were first fitted with six parameters, the same mass formula as in Refs. 3 and 4; the fit with eight parameters was then attempted with the six parameters as input values, and also with arbitrary input values. The number $F_{\min}$ was identical, though the amount of machine time, of course, varied. We therefore believe

[22] If the errors in the mass values were more precisely known, it would be possible to weight the terms in the sum inversely with the errors.

TABLE VIII. Least-squares fit to experimental masses.[a]

	M	M+III	M+IV	M+I	M+II	M+I+III	M+I+IV	M+II+III	M+II+IV
$F_{\min}(m_k')$ (MeV^2)	1949	1320	1448	1124	1505	949	894	1448	1138
$\lvert\Delta_i\rvert_{\mathrm{rms}}$ (MeV)	11.8	9.7	10.2	9.0	10.4	8.2	8.0	9.1	9.0
Lowest calc. Σ res. (MeV)	1606	1603	1612	1562	1581	1560	1556	1577	1588

[a] $F_{\min}(m_k')$ is the minimum of the function $F(m_k')=\Sigma_{i=1}{}^{14}(\Delta_i)^2$, where $\Delta_i=(m_i|_{\mathrm{calc}}-m_i|_{\mathrm{expt}})$ and $m_i|_{\mathrm{calc}}$ and $m_i|_{\mathrm{expt}}$ are the calculated and experimental masses. M is the mass operator, $M=M_{(12)}+M_{35^1}{}^{\mathrm{L\cdot S}}+M_{35^8}{}^{\mathrm{L\cdot S}}$ and I $=M_{189^8A}{}^{\mathrm{L\cdot S}}$, II $=M_{189^8S}{}^{\mathrm{L\cdot S}}$, III $=M_{35^1F}{}^{\mathrm{L\cdot S}}$, IV $=M_{35D}{}^{\mathrm{1L\cdot S}}$.

TABLE IX. Masses in the (**70**,1^-) multiplet. Three-body operator M_{35_F}.[a]

	4P				2P	
$J^P=$	$\frac{5}{2}^-$	$\frac{3}{2}^-$	$\frac{1}{2}^-$		$\frac{3}{2}^-$	$\frac{1}{2}^-$
				Ω	2012	1991
				Ξ	1936 (1930?)	1921
				Σ	1847	1824
				Δ	1673 (1670)	1651 (1650)
Ξ	1906 (1930?)	1828 (1815)	1787			
Λ	1809 (1830)	1811	1797	Ξ	1775	1662 (1655?)
Σ	1765 (1765)	1670 (1670)	1640	Λ	1679 (1690)	1682 (1670)
N	1689 (1672)	1700 (1700)	1705 (1700)	Σ	1620 (1618?)	1603 (1618?)
				N	1530	1525 (1535)
				Λ	1528 (1520)	1405 (1405)

[a] Fit of 14 resonances with seven-parameter mass formula including a contribution from the three-body mass operator $M_{35F}{}^{\mathrm{1L\cdot S}}$. The left-hand columns are the calculated masses; the right-hand columns (in parentheses) are the experimental masses. The masses with question marks (not included in the 14 masses fitted) are placed closest to the calculated mass (or masses).

TABLE X. Masses in the (**70**,1^-) multiplet. Three-body operator $M_{35_D}{}^{\mathrm{1L\cdot S}}$.[a]

	4P				2P	
$J^P=$	$\frac{5}{2}^-$	$\frac{3}{2}^-$	$\frac{1}{2}^-$		$\frac{3}{2}^-$	$\frac{1}{2}^-$
				Ω	2015	1997
				Ξ	1934 (1930?)	1925 (1930?)
				Σ	1840	1825
				Δ	1670 (1670)	1652 (1650)
Ξ	1904 (1930?)	1831 (1815?)	1794			
Λ	1810 (1830)	1811	1797	Ξ	1772	1677 (1635?)
Σ	1761 (1765)	1674 (1670)	1646	Λ	1677 (1690)	1683 (1670)
N	1691 (1672)	1699 (1700)	1702 (1700)	Σ	1619 (1618?)	1612 (1618?)
				N	1525 (1520)	1532 (1535)
				Λ	1531 (1520)	1405 (1405)

[a] Fit of 14 resonances with seven-parameter mass formula including a contribution from the three-body mass operator $M_{35D}{}^{\mathrm{1L\cdot S}}$. The left-hand columns are the calculated masses; the right-hand columns (in parentheses) are experimental masses. The masses with question marks (not included in the 14 masses fitted) are placed closest to the calculated mass (or masses).

that the solutions obtained represent *the* best fit, and not a local minima. The minimum function $F_{\min}$, for the various mass operators, and combinations of such, is listed in Table VIII.

We note that the best fits[23] for the mass formulas, Eqs. (2.5), (2.10), and (2.12), vary from $|\Delta_i|_{\mathrm{rms}}=11.8$ MeV to 8.0 MeV. Also listed in Table VIII are the lowest predicted Σ resonances. The lowest experimentally observed Σ resonance[24] in the $\Lambda\pi$ and $\Lambda\pi\pi$ channels has mass 1618 MeV with uncertain spin parity. (There does exist a P_{11} partial-wave $\Sigma\pi$ resonance at 1560 MeV with $J^P=\frac{1}{2}^+$.) Since the rms difference of Δ_i is at most 10.4 MeV for the last six columns of Table VIII, and the difference $1618-M_i|_{\mathrm{calc}}$ is, at the least, 30 MeV for the last six columns, we reject them as fits to the data, and retain for consideration columns 2 and 3. This criterion eliminates the two-body spin-orbit mass operators $M_{189}{}^{8A\,\mathrm{L\cdot S}}$ and $M_{189}{}^{8S\,\mathrm{L\cdot S}}$, and is consistent with the expectation that the higher-dimensional representations do not contribute to the mass operator.

The calculated and experimental masses for the remaining cases of the three-body spin-orbit operators, $M_{35_F}{}^{\mathrm{1L\cdot S}}$ and $M_{35_D}{}^{\mathrm{1L\cdot S}}$, are listed in Tables IX and X, respectively. It is not possible to choose between these two solutions with the present experimental situation, particularly with regard to Σ and Ξ resonances. The predicted large overlap may make it difficult to resolve the Σ and Ξ resonances experimentally.

Let us consider Table IX. We note that the greatest differences between calculated and experimental masses for the established resonances is 21 and 17 MeV for the $J^P=\frac{5}{2}^-$ Λ and N resonances, respectively. The rms difference Δ_i is 9.7 MeV for the 14 established reso-

[23] This represents a considerable improvement over Ref. 4, where the rms difference was 18 MeV.

[24] D. J. Crennell *et al.*, [Phys. Rev. Letters **21**, 648 (1968)] report a mass of 1619 MeV; also summarized in Ref. 6.

nances in Table IX. This compares favorably with the mass fit of 5 MeV difference for the eight isomultiplets of the $(\mathbf{56},0^+)$, where the particle masses are well established. We further note that the Ξ resonances at 1635, 1815, and 1930 MeV, respectively, have a place in the table, and similarly for the $\Sigma(1618)$. However, the $\Sigma(1690)$, $J^P=\frac{3}{2}^-$, which strongly overlaps the $D_{13}(1670)$ Σ resonance, and which may not exist,[25] has no obvious place in the table. The $S_{11}(1750)$ Σ resonance also has no convenient location. Considerable experimental effort is underway to resolve the Σ, and Ξ resonance ambiguities.

The resonances are fitted well with the three-body spin-orbit mass operator $M_{35_F}{}^{1\mathrm{L}\cdot\mathrm{S}}$. It had been previously noted[4] that a three-body operator would be required to split the Δ resonances. The relative contributions of the two-body spin-orbit mass operators $M_{35}{}^{1\mathrm{L}\cdot\mathrm{S}}$ and $M_{35}{}^{8\mathrm{L}\cdot\mathrm{S}}$ and three-body spin-orbit mass operator $M_{35_F}{}^{1\mathrm{L}\cdot\mathrm{S}}$ can be determined by multiplying the reduced matrix elements by the change in mass eigenvalues for the three mass operators. The splitting ΔM (in MeV) for the three mass operators in this seven-parameter fit is

$$\Delta M_{35}{}^{1\mathrm{L}\cdot\mathrm{S}}=-42.6,\quad \Delta M_{35}{}^{8\mathrm{L}\cdot\mathrm{S}}=98.6,$$
$$\Delta M_{35_F}{}^{1\mathrm{L}\cdot\mathrm{S}}=-28.8.$$

The three-body spin-orbit contribution to the mass splitting is the smallest of the seven mass operators in the $(\mathbf{70},1^-)$; the largest contributions come from the non-spin-orbit mass operators,[3,4] as expected. Configuration mixing between the $(\mathbf{56},0^+)$ and $(\mathbf{70},1^-)$ due to this three-body operator is a much smaller effect. We observe that the contributions to the splitting of the two-body spin-orbit mass operators in the six-parameter fit is

$$\Delta M_{35}{}^{1\mathrm{L}\cdot\mathrm{S}}=-74.7,\quad \Delta M_{35}{}^{8\mathrm{L}\cdot\mathrm{S}}=97.5.$$

It appears that the splitting due to the two-body spin-orbit mass operator $M_{35}{}^{1\mathrm{L}\cdot\mathrm{S}}$ is taken up by the three-body spin-orbit operator $M_{35_F}{}^{1\mathrm{L}\cdot\mathrm{S}}$ in going from the six- to seven-parameter fit.

While the experimental situation clarifies itself with regard to the Σ and Ξ resonances (and the two missing Λ resonances), an important test of the symmetric quark model in the $(\mathbf{70},1^-)$ multiplet is the existence and discovery of the two Ω^{-*} resonances required by the model. The Ω^{-*}'s of $J^P=\frac{1}{2}^-$, $\frac{3}{2}^-$ are predicted at 1991 and 2012 MeV, respectively (Table IX) and should be produced in the reaction $K^-p\to\Omega^{-*}K^0K^+$, as with the $\Omega^-(1673)$. The topology of the weak decay[26] of the $\Omega^-(1673)$ is diagrammed in Fig. 2(a). The $\Omega^{-*}(\frac{1}{2}^-)$ can decay strongly in the channels $\Xi\bar{K}$ and $\Xi\bar{K}\pi$, since the thresholds for these decays are $\sim$1815 and 1950 MeV, respectively. The topology of the $\Xi\bar{K}$ s wave, highly favored over the $\Xi\bar{K}\pi$ p wave because of the greatly reduced phase space for this latter channel, is given by Fig. 2(b), and features no Ω^{-*} kink or track. The partial width for the $\Xi\bar{K}$ decay may be calculated using the results of Divgi[27] and of Mitra and Ross,[28] with the prediction

$$\Gamma_{\Omega^{-*}(\frac{1}{2}^-)\to\Xi\bar{K}}=68\text{ MeV},\quad s\text{ wave}.$$

With the input value chosen, this is probably an overestimate, but the predictions of the decay model are not reliable for s-wave decay,[27] and may be off by a factor of 2. Possibly the width of the $\Omega^{-*}(\frac{1}{2}^-)$ is too wide to be readily visible as claimed by Mitra and Ross.[28] The threshold of the channel $\Xi^*(1530)K$, 2025 MeV, is above the predicted mass of the $\Omega^{-*}(\frac{1}{2}^-)$. The partial widths[29] for the decay of the $\Omega^{-*}(\frac{3}{2}^-)$ are

$$\Gamma_{\Omega^{-*}(\frac{3}{2}^-)\to\Xi\bar{K}}=10\text{ MeV},\quad d\text{ wave},$$
$$\Gamma_{\Omega^{-*}(\frac{3}{2}^-)\to\Xi^*\bar{K}}=0.38q\text{ MeV},\quad s\text{ wave},$$

where q is the momentum, in MeV/c, of the emitted meson. The p-wave $\Xi\bar{K}\pi$ channel is assumed to be small. The $\Omega^{-*}(\frac{3}{2}^-)$ is predicted at 13 MeV below Ξ^*K threshold, but the predictive error, from the experimental input error, is, of course, much larger. The width is sensitive to the Ξ^*K threshold,[28] but it is possible that the $\Omega^{-*}(\frac{3}{2}^-)$ is visible. The discovery of the Ω^{-*} resonances would provide an important test of the symmetric quark model and it is hoped that experimentalists could check their K^-p film for observation of these particles.

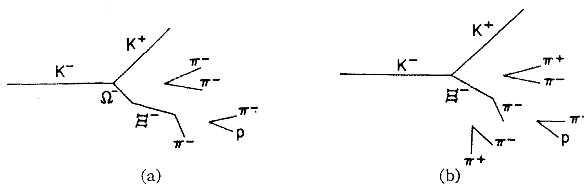

FIG. 2. (a) Topology of the Ω^- weak decay. Note the three-kink track and two V's. (b) Topology of the Ω^{-*} strong decay. The Ω^{-*} decay has three V's and no Ω^{-*} track.

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[25] P. Eberhard *et al.* [Phys. Rev. Letters **22**, 200 (1969)] report two $\Sigma(1660)$ states. A. C. Ammann *et al.* [*ibid.* **24**, 327 (1970)] report a $\Lambda\pi$ resonance at 1642 ± 12 MeV, different from either of the $\Sigma(1660)$ states on the basis of the branching ratios to $\Lambda\pi$, $\Sigma\pi$, and $\Lambda(1405)\pi$. However, the double $\Sigma(1660)$ states have been questioned by V. E. Barnes *et al.*, Brookhaven-Syracuse collaboration, BNL Report No. 13823 (unpublished).

[26] I thank Professor David Evans for a helpful discussion on the experimental questions.

[27] D. R. Divgi, Phys. Rev. **175**, 2027 (1968). The input value for s-wave decays was changed from $\Delta(1640)\to N\pi$, 54 MeV, to $\Lambda(1405)\to\Sigma\pi$, 40 MeV.

[28] A. N. Mitra and M. Ross, Phys. Rev. **158**, 1630 (1967).

[29] The input value from the d-wave decay $Y_1(1765)\to\Lambda\pi$ is changed from 14 MeV in Ref. 27 to 15.6 MeV. Inclusion of the baryon mass in the denominator changes the results of Ref. 28 from 6 to 10 MeV for the d-wave decay and from 0.5 to $0.38q$ for the s-wave decay.