Self-Consistent Dispersion-Theoretic Electron-Positron Scattering Amplitude

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A method is outlined for the construction of dispersion-theoretic electromagnetic scattering amplitudes for processes in which bound states may appear; in particular, the electron-positron system is examined. A Lorentz-covariant spinor amplitude is exhibited which is analytic, cutoff independent, and crossing symmetric, which has the correct double-spectral functions through second order in the fine-structure constant α, and which reduces to the usual Born approximation in lowest order. Moreover, the amplitude displays Regge asymptotic behavior and the positronium Regge poles. Self-consistency requires that the positronium poles appear at the correct position and with the proper residue, and that the amplitude possesses a well-defined Jacob-Wick expansion. It is found that this second-order amplitude provides a suitable basis for iterative calculation of the higher-order terms using the same procedure.

I. INTRODUCTION

ALTHOUGH the usefulness of the Feynman-Dyson perturbation theory in quantum electrodynamics is beyond question, there is at least one area in which it is expected *a priori* that the usual formulation will fail —in the description of scattering processes for which there is the possibility of bound states.[1] In this case, it is known that the perturbation theory will not yield the correct scattering amplitudes. Of course, it is possible to modify the formalism in such a way that bound states may be accommodated,[2] but the calculational difficulties which are thereby introduced more than offset the gain in applicability. Thus, in spite of the advanced status of the perturbation calculation of bound-state energy shifts, the corresponding scattering problems remain essentially unresolved. This deficiency becomes acute if one attempts to construct a dispersion theory of electromagnetic interactions which is independent of perturbation theory and which can compare with perturbation theory in the accuracy to which, for example, the Lamb shift or the electron anomalous magnetic moment is determined. In that case it is necessary that the problems concerning scattering processes for which there are bound states be faced, since these scattering amplitudes will appear explicitly in the dispersion integrals used to evaluate the form factors[3] or the bound-state perturbations.[4] There is little doubt that an alternative to perturbation theory would be desirable, particularly if it offers the possibility of increased simplicity. Since a necessary preliminary step in the evolution of an independent dispersion theory of electromagnetic interactions is the construction of accurate scattering amplitudes when there are bound-state poles, the procedure which is outlined on the following pages, aside from any intrinsic appeal, should prove to be of considerable utility.

We will consider here only the elastic electron-positron scattering amplitude. This amplitude is necessary for the dispersion calculation of the Lamb shift[5] and is convenient for the dispersion calculation of the electron anomalous magnetic moment.[3] Moreover, it represents a highly symmetric and therefore considerably simpler system than that which results when the masses are not equal. It should be made clear, however, that our method can be extended to the unequal-mass case with some additional labor. We would like to construct a Lorentz-covariant spinor amplitude which is analytic, crossing symmetric, and cutoff independent, which exhibits Regge asymptotic behavior and the bound-state Regge poles, displays the correct double-spectral functions through second order in the fine-structure constant α, and reduces to the usual Born approximation in lowest order. Finally, we require that the amplitude which results be self-consistent. We will show that for electron-positron scattering these conditions can be satisfied by a relatively simple analytic form, although the inclusion of spin greatly increases the apparent complexity. While it was not our intention originally, we have, in a sense, effected a bootstrap, since the only pole whose position and residue is inserted *a priori* is that of the electron, while our result can be shown to be self-consistent only if it has positronium poles at the correct position and with the proper residue.

Basically, our program is very simple, although there are significant details which can only be appreciated by means of an example. We consider the three t-channel unitarity diagrams of Fig. 1. These three diagrams plus the three exchange diagrams obtained from u-channel unitarity represent a complete set through second order

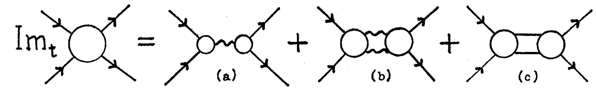

FIG. 1. Imaginary part of the elastic electron-positron scattering amplitude, through second order in α, given by these three t-channel unitarity diagrams.

[1] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1962), pp. 642–643.
[2] Ref. 1, Sec. 17f.
[3] S. D. Drell and H. R. Pagels, Phys. Rev. **140**, B397 (1965).
[4] R. F. Dashen and S. C. Frautschi, Phys. Rev. **135**, B1190 (1964).
[5] J. McEnnan, Phys. Rev. **181**, 1967 (1969).

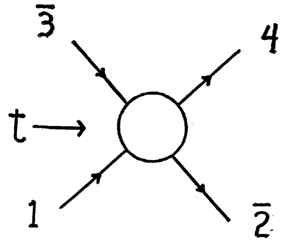

FIG. 2. Kinematics for our t-channel scattering amplitude.

in α. (Note that in perturbation theory there are 20 Feynman diagrams through second order.) We explicitly calculate the second-order contribution of each diagram to the double-spectral functions of the elastic electron-positron scattering amplitude. We then assume the invariant amplitudes can essentially be written in the form[6]

$$A_i(s,t,u)=\sum_k \gamma_k{}^i(s,t,u)\left(-\frac{u}{4q_t{}^2}\right)^{\alpha_k{}^i(t)} +(\text{single-spectral-function terms})+(\text{crossing})\,, \quad (1)$$

where the $\gamma_k{}^i(s,t,u)$ and the $\alpha_k{}^i(t)$ are to be determined by requiring that Eq. (1) give the correct second-order double-spectral functions and reduce to the Born term in lowest order; and that the amplitude be self-consistent. In practice, the intermediate stages in the evaluation of the double-spectral functions will strongly imply the specific form of the $\gamma_k{}^i(s,t,u)$ and of the $\alpha_k{}^i(t)$ which satisfy these requirements.

We must point out, however, that in one respect our evaluation of the double-spectral functions will differ from the usual procedure. Diagrams 1(a) and 1(b) can be evaluated without complication. Indeed, 1(a) is equivalent to the usual Born approximation with vertex corrections due to the electron anomalous magnetic moment. This term will not contribute to the double-spectral functions. Diagram 1(b) can be evaluated directly by employing the usual pole approximation to the two-photon annihilation amplitude. This term will exhibit cuts in u and s so that there will be nonvanishing contributions to each of the double-spectral functions. The point of departure arises in the consideration of the remaining diagram 1(c). This term cannot be evaluated using the Born approximation, since the integral over the angles of the photon pole is divergent. In order to evaluate the integral without introducing a cutoff, we replace the pole term by a simple function which exhibits Regge behavior and which reduces to the Born term in lowest order, and whose form is suggested by the form of the Coulomb scattering amplitude obtained from the Schrödinger equation. In a sense, this substitution amounts to a recognition of the fact that Fig. 1 represents a singular, inhomogeneous, nonlinear integral equation for the elastic electron-positron scattering amplitude, and that the Born term does not comprise a suitable solution. This substitution, which is related to the form of Eq. (1) and will be discussed more fully in subsequent pages, is crucial to the success of our program, since it is this feature of our calculation which will admit a process of iteration.

At this point, we should indicate the reasons for our emphasis of the double-spectral functions. It is certainly true that we cannot simply insert the discontinuity function, which we obtain from the evaluation of the diagrams in Fig. 1, into a dispersion integral. This procedure would (aside from the fact that our substitution for the photon pole obviates the necessity for an infrared cutoff) simply reproduce the perturbation theory results, which we know must be incorrect. In order to arrive at a resolution of this difficulty, we turn to the familiar example of the Coulomb scattering amplitude in nonrelativistic quantum mechanics. By separating the Schrödinger equation in parabolic coordinates, it is possible to obtain the exact Coulomb amplitude in closed form. As is well known, the first Born term is identical to the lowest-order term in an expansion of the exact amplitude in a power series in α. However, the second Born approximation is not equal to the second-order term of the exact amplitude. (As a matter of fact, the second Born term involves an integral similar to that which would result from a dispersion calculation in the relativistic case.) We do find, however, that with sufficient care the contribution of the second Born term to the double-spectral function can be evaluated, and this is identical to the second-order term in the expansion of the exact double-spectral function. It is this result which has prompted our elevation of the double-spectral functions to primacy. To be sure, there is no guarantee that in the relativistic case the double-spectral functions calculated by evaluating low-order unitarity diagrams will be identical to the exact double-spectral functions to that order, but we must have some basis for our calculation and our results are so satisfactory that it is entirely reasonable that this supposition is correct. We will thus postulate that, although a unitarity sum of the form given in Fig. 1 may not yield the correct single-spectral functions (when truncated at any finite order, at least), the double-spectral functions which can be derived from this expansion are correct to each order. Our results here strongly indicate that this is true through second order in α. It is, however, a matter for conjecture whether it remains true for higher orders, although it is at least plausible to assume that this is the case.

Thus, we shall proceed as follows. In Sec. II we will review some aspects of spin-$\frac{1}{2}$–spin-$\frac{1}{2}$ equal-mass scattering and introduce our notation. In Sec. III we will discuss the Born approximation to the electron-positron elastic scattering amplitude. In Sec. IV we outline our evaluation of the double-spectral functions to second

[6] This particular form has been discussed by X. Artru, M. Fontannaz, and R. Omnès, Phys. Rev. 181, 2130 (1969).

order. Finally, an *ansatz* which satisfies our requirements will be presented in Sec. V.

II. NOTATION

We should now like to review briefly the general formalism of spin-$\frac{1}{2}$–spin-$\frac{1}{2}$ equal-mass scattering and establish our notation. There is a minor difficulty in that our calculation employed the two-component spinor formalism as discussed, for example, by Barut,[7] since this achieved a considerable reduction in the labor required to express the result of the unitarity integrations in standard form. However, since the four-component spinor formulation is generally more familiar, wehave translated our results into the usual notation. Thus, the scattering amplitude for two spin-$\frac{1}{2}$ identical particles with kinematics as in Fig. 2 can be written in the following form:

$$T(k_4,k_2;k_1,k_3)=\sum_i A_i(s,t,u)Y_i(k_4,k_2;k_1,k_3) +\tilde{A}_i(s,t,u)\tilde{Y}_i(k_4,k_2;k_1,k_3)\,, \quad (2)$$

where $t=(k_1+k_3)^2$, $s=(k_1-k_2)^2$, and $u=(k_1-k_4)^2$ are the usual Mandelstam variables, and the $A_i(s,t,u)$ are Lorentz-invariant scalar functions with

$$\tilde{A}_i(s,t,u)=-A_i(s,u,t)\,.$$

The spinor basis functions $Y_i(K)$ are given by

$$\begin{aligned}
Y_V &\equiv V=\bar{u}(k_4)\gamma^\mu v(k_2)\bar{v}(k_3)\gamma_\mu u(k_1)\,,\\
Y_S &\equiv S=\bar{u}(k_4)v(k_2)\bar{v}(k_3)u(k_1)\,,\\
Y_P &\equiv P=\bar{u}(k_4)\gamma_5 v(k_2)\bar{v}(k_3)\gamma_5 u(k_1)\,, \qquad (3)\\
Y_A &\equiv A=\bar{u}(k_4)\gamma_5\gamma^\mu v(k_2)\bar{v}(k_3)\gamma_5\gamma_\mu u(k_1)\,,\\
Y_T &\equiv T=\bar{u}(k_4)i\sigma^{\mu\nu}v(k_2)\bar{v}(k_3)i\sigma_{\mu\nu}u(k_1)\,,
\end{aligned}$$

where the spinor normalization is that of Bjorken and Drell.[8] The $\tilde{Y}_i(K)$ can be obtained from (3) by the exchange $k_4\leftrightarrow -k_3$, and will be denoted by $\tilde{V}$, $\tilde{S}$, $\tilde{P}$, $\tilde{A}$, $\tilde{T}$. These are essentially the five spinor basis functions used by Goldberger, Grisaru, MacDowell, and Wong[9] in their analysis of the nucleon-nucleon problem. Note that there are only five independent amplitudes, since the exchange spinor basis functions $\tilde{Y}_i(K)$ can be written as a linear combination of the $Y_i(K)$. We see that the amplitude (2) is antisymmetric with respect to the exchange $k_4\leftrightarrow -k_3$, as required by the Pauli principle. In the t channel, $W=\sqrt{t}$ is the total energy in the c.m. frame; $s=-2q_t^2(1+z)$ and $u=-2q_t^2(1-z)$ are momentum transfers, with z the cosine of the scattering angle; $q_t=\frac{1}{2}(t-4)^{1/2}$ is the c.m. relative momentum. As usual, we choose units in which $\hbar$, c, and the mass of the electron are unity.

[7] A. O. Barut, *The Theory of the Scattering Matrix* (MacMillan, New York, 1967).
[8] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[9] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

III. BORN APPROXIMATION

The evaluation of diagram 1(a) yields the one-photon-exchange contribution to the electron-positron elastic scattering amplitude, with vertex corrections due to the electron anomalous magnetic moment. It is essentially the Born term. Although this term does not contribute to the double-spectral functions, it must be the case that the photon pole be reproduced in our amplitude in lowest order. The contribution of diagram 1(a) to each of the five scalar amplitudes, assuming a general form for the electron current,

$$j^\mu(t)=(e/\sqrt{2})[\Gamma_1(t)\bar{u}\gamma^\mu u-\tfrac{1}{2}\Gamma_2(t)q^\mu\bar{u}u]\,, \qquad (4)$$

is given by

$$\begin{aligned}
A_V(s,t,u)_{\text{Born}} &=(4\pi\alpha/t)\Gamma_1(0)F_1(0)\,,\\
A_S(s,t,u)_{\text{Born}} &=-\tfrac{1}{4}\kappa(s-u)(4\pi\alpha/t)F_1(0)F_2(0)\,,\\
A_P(s,t,u)_{\text{Born}} &=-\tfrac{1}{4}\kappa(s-u)(4\pi\alpha/t)\Gamma_1(0)F_2(0)\,, \qquad (5)\\
A_A(s,t,u)_{\text{Born}} &=0\,,\\
A_T(s,t,u)_{\text{Born}} &=-\tfrac{1}{8}\kappa t(4\pi\alpha/t)\Gamma_1(0)F_2(0)\,.
\end{aligned}$$

In (5), $\Gamma_1(t)=F_1(t)+\Gamma_2(t)=F_1(t)+\kappa F_2(t)$, where $F_1(t)$ and $F_2(t)$ are the electron charge and magnetic moment form factors, respectively, with $F_1(0)=F_2(0)=1$; $\kappa=\frac{1}{2}(g-2)$ is the electron anomalous magnetic moment. To first order, $\kappa=\alpha/2\pi$. Note that for reasons of symmetry we have retained terms of order α^3 in (5). To obtain the complete one-photon contribution to the amplitude, it is, of course, necessary to antisymmetrize with respect to the exchange $k_4\leftrightarrow -k_3$. Thus, there will also be terms of the form

$$\tilde{A}_i(s,t,u)_{\text{Born}}=-A_i(s,u,t)_{\text{Born}}\,, \qquad (6)$$

which will be associated with the five exchange spinor basis functions $\tilde{Y}_i(K)$. Although the form of Eq. (2) is rather cumbersome for the representation of the Born term, it is in this form that comparison can most easily be made with our final amplitude.

IV. DOUBLE-SPECTRAL FUNCTIONS

Diagrams 1(b) and 1(c) of Fig. 1 both produce nonvanishing contributions to the double-spectral functions of the elastic electron-positron scattering amplitude.

Two-Photon Intermediate State

We will consider first the two-photon intermediate state in the evaluation of the t-channel unitarity [diagram 1(b)]. This contribution may be written

$$(T-T^\dagger)_{2\gamma}=i\sum_{\text{spins}}\rho_{2\gamma}\int d\Omega' T_a{}^\dagger(k_f;q'')T_a(q';k_i)\,, \qquad (7)$$

where $\rho_{2\gamma}$ is the appropriate phase-space factor for two identical photons and $T_a(q_5,q_6;k_1,k_3)$ is simply the two-photon annihilation amplitude in pole approximation. The integration is over the angles of the four-momenta of the two intermediate state photons, q_j. If we expand the right-hand side of (7) in terms of the five spinor basis functions given in (3) and perform the integration over the angles, we find that the two-photon intermediate-state contribution to the imaginary part of the invariant amplitudes of the elastic electron-positron scattering matrix can be written in the form

$$\mathrm{Im}_t A_i(s,t,u)_{2\gamma}=D_t{}^i(s,t,u)_{2\gamma}=\{\theta(-t)\sum_{j=0}^{3} q_j{}^i(s,t,u)I_j(s,t,u)\}+(-1)^i\{s\leftrightarrow u\}, \quad (8)$$

where $i=V, S, P, A, T$, and

$$(-1)^i=\begin{cases}-1, & i=V,T\\ +1, & i=S,P,A.\end{cases} \quad (9)$$

The $q_j{}^i(s,t,u)$, which are essentially ratios of polynomials, and the integrals $I_j(s,t,u)$ will be discussed below. Note that in Eq. (8), contrary to the usual convention, we have placed the unitarity cut due to two-photon exchange along the negative real t axis, rather than along the positive real axis. The reason for this particular arrangement will be discussed after we have investigated the singularities in s and u of the $D_t{}^i(s,t,u)$. In detail, omitting the Heaviside functions $\theta(-t)$, we find that the five discontinuity functions of Eq. (8) can be written as follows:

$$\begin{aligned}
D_t{}^V(s,t,u)=\frac{1}{16}\Bigg\{&\left[\frac{4t}{s}-\frac{4(s-u)(u-4)}{s}-\frac{t^2(u-2)}{s^2}-\frac{t(t-4)(u-2)(u-4)}{s^2u}\right]I_0(s,t,u)\\
&+\left[\frac{8(s-u+1)}{s}-\frac{4t(u-2)}{s^2}+\frac{4t(t-4)(u-2)}{s^2u}\right]I_1(s,t,u)+\left[\frac{4(t-2)}{s^2}-\frac{8(t-4)}{su^2}\right]I_2(s,t,u)\\
&+\left[\frac{4(2-u)}{s^2}-\frac{8}{u^2}\right]I_3(s,t,u)\Bigg\}-\frac{1}{16}\{s\leftrightarrow u\},\\
D_t{}^S(s,t,u)=\frac{1}{16}\Bigg\{&\left[\frac{t(s-u-8)}{s}+\frac{t^2(2+s-u)}{s^2}-\frac{t(s-u)(u-2)(u-4)}{s^2u}\right]I_0(s,t,u)\\
&+\left[\frac{2(s-u-8)}{s}+\frac{4t(2+s-u)}{s^2}+\frac{4t(s-u)(u-2)}{s^2u}\right]I_1(s,t,u)+\left[\frac{2(4+s-2u)}{s^2}+\frac{2(u-4)}{u^2}\right]I_2(s,t,u)\\
&+\left[\frac{2(4-u)}{u^2}+\frac{4(u-2)}{su}+\frac{2(4+s-2u)}{s^2}\right]I_3(s,t,u)\Bigg\}+\frac{1}{16}\{s\leftrightarrow u\},\\
D_t{}^P(s,t,u)=\frac{1}{16}\Bigg\{&\left[\frac{t(8+s-u)}{s}+\frac{t^2(2+s-u)}{s^2}-\frac{t(s-u)(u-2)(u-4)}{s^2u}\right]I_0(s,t,u)\\
&+\left[\frac{2(s-u)}{s}+\frac{4t(2+s-u)}{s^2}+\frac{4t(s-u)(u-2)}{s^2u}\right]I_1(s,t,u)+\left[\frac{2(4+s-2u)}{s^2}+\frac{2(u-4)}{u^2}+\frac{16(t-4)}{stu}\right]\\
&\times I_2(s,t,u)+\left[\frac{2(4+s-2u)}{s^2}-\frac{2(u-4)}{u^2}+\frac{4(u-2)}{su}+\frac{16(s-u)}{stu}\right]I_3(s,t,u)\Bigg\}+\frac{1}{16}\{s\leftrightarrow u\},\\
D_t{}^A(s,t,u)=\frac{1}{16}\Bigg\{&\left[\frac{4t(4-u)}{s}-\frac{8t}{su}-\frac{2t^2(u-2)}{s^2}\right]I_0(s,t,u)+\left[\frac{8t}{s}-\frac{8t}{su}+\frac{4t(t-u)}{s^2}\right]I_1(s,t,u)\\
&+\left[\frac{4(t-2)}{s^2}-\frac{8}{u^2}\right]I_2(s,t,u)+\left[\frac{4(2-u)}{s^2}+\frac{8(s-u)}{su^2}\right]I_3(s,t,u)\Bigg\}+\frac{1}{16}\{s\leftrightarrow u\},\\
D_t{}^T(s,t,u)=\frac{1}{16}\Bigg\{&\left[\frac{t^2(2-u)}{s^2}-\frac{2t(u-2)}{su}\right]I_0(s,t,u)+\left[\frac{t}{s}+\frac{2t(t-2)}{s^2}+\frac{2t(t-4)(u-2)}{s^2u}\right]I_1(s,t,u)\\
&+\left[\frac{t-u}{s^2}-\frac{u-4}{u^2}-\frac{2(u-2)}{su}\right]I_2(s,t,u)+\left[\frac{t-u}{s^2}+\frac{u-4}{u^2}\right]I_3(s,t,u)\Bigg\}-\frac{1}{16}\{s\leftrightarrow u\}.
\end{aligned} \quad (10)$$

The integrals $I_j(s,t,u)$, which appear above, are given by

$$
\begin{aligned}
I_0(s,t,u) &= \alpha^2 \int d\Omega' \frac{1}{(s'-1)(s''-1)} \\
&= \frac{8\pi\alpha^2}{t[u(u-4)]^{1/2}} Q_0\left(\frac{2-u}{[u(u-4)]^{1/2}}\right), \\
I_1(s,t,u) &= \alpha^2 \int d\Omega' \frac{1}{s'-1} = \frac{-8\pi\alpha^2}{tv} Q_0\left(\frac{1}{v}\right), \\
I_2(s,t,u) &= \alpha^2 \int d\Omega' \frac{s'-1}{s''-1} \\
&= \frac{-8\pi\alpha^2}{4q_t^2 v}\left[uQ_0\left(\frac{1}{v}\right) - \tfrac{1}{2}v(u-s)\right], \\
I_3(s,t,u) &= \alpha^2 \int d\Omega' = 4\pi\alpha^2,
\end{aligned}
\tag{11}
$$

where $v=q/E=[(t-4)/t]^{1/2}$, and $Q_0(z)$ is the Legendre function of the second kind of degree zero. The phase-space factors have been absorbed into the $I_j(s,t,u)$. From (10) and (11) we see that only those terms proportional to $I_0(s,t,u)$ contribute to the double-spectral functions. We find, using $\mathrm{Im}Q_l(z) = -\frac{1}{2}\pi P_l(z)$, for z real and $-1 \leq z \leq 1$, that

$$
\begin{aligned}
\rho_{tu}{}^i(s,t,u)_{2\gamma} &= q_0{}^i(s,t,u)\left\{\frac{-4\pi^2\alpha^2}{t[u(u-4)]^{1/2}}\right\} \\
&= (-1)^i \rho_{ts}{}^i(u,t,s)_{2\gamma}.
\end{aligned}
\tag{12}
$$

An interesting and, at first sight, unexpected result is that the cuts in u (and s) of $D_t{}^i(s,t,u)_{2\gamma}$ run from $-\infty$ to 0 and from 4 to $+\infty$ along the real axis, so that, for example, $\rho_{tu}{}^i(s,t,u)_{2\gamma}$ is nonvanishing for real u which satisfy $u(u-4)\geq 0$. This particular cut structure may be unambiguously verified by noting that $I_0(s,t,u)$, while finite at $u=0$ and $u=4$, has an infinite first derivative at each point. However, this result could actually have been anticipated, since both the Klein-Gordon and Dirac Coulomb scattering amplitudes exhibit a cut structure of this sort. It was for this reason that we put the cut in t of the $A_i(s,t,u)$ along the negative real t axis; otherwise, the double-spectral functions would be nonvanishing in the physical region.

Although the double-spectral functions (12) have poles at $s=0$ $(u=0)$ due to the $q_0{}^i(s,t,u)$, $D_t{}^i(s,t,u)_{2\gamma}$ is, in fact, finite at these points. This may be determined by expanding the $I_j(s,t,u)$ about $s=0$ $(u=0)$ and collecting terms. This result is, of course, necessary if we are to assume that the amplitude has no spurious singularities, but it is reassuring to verify explicitly that the cancellations occur. [As an additional check, we have evaluated (7) independently at $t=4$ and compared the result with the limit of (8) as $t\to 4$, with the same value in each case.]

Equation (12) represents the contribution of the t-channel two-photon intermediate state to the double-spectral functions of the electron-positron elastic scattering amplitude. In order to complete the two-photon part, it is necessary to consider the u-channel unitarity, the addition of which is equivalent to antisymmetrization with respect to the exchange $k_4 \leftrightarrow -k_3$. Thus, in addition to (12), we will also have double-spectral-function terms of the form

$$
\tilde{\rho}_{ut}{}^i(s,t,u)_{2\gamma} = -\rho_{tu}{}^i(s,u,t)_{2\gamma}
$$
and
$$
\tilde{\rho}_{us}{}^i(s,t,u)_{2\gamma} = -\rho_{ts}{}^i(s,u,t)_{2\gamma}, \tag{13}
$$

where, as usual, the tilde indicates that these factors are associated with the exchange spinor basis functions $\tilde{V}$, $\tilde{S}$, $\tilde{P}$, $\tilde{A}$, $\tilde{T}$. Although, as we have indicated, the exchange spinor basis functions can be written as a linear combination of the direct spinor basis functions (V,S,P,A,T), it is convenient to keep them separate, since in this form crossing symmetry is easily maintained.

Elastic Unitarity

We now turn to the evaluation of diagram 1(c), which is equivalent to elastic unitarity. We have

$$
(T-T^\dagger)_{e^+e^-} = i \sum_{\text{spins}} \rho_2(t) \int d\Omega' T^\dagger(k_f;k'')T(k';k_i), \tag{14}
$$

where $\rho_2(t)$ is the two-body phase-space factor and the integration is over the angles of the four-momenta of the electron-positron pair intermediate state. If we attempt to approximate the amplitudes which appear in the unitarity integral (14) by their pole terms (5), we find that the integral will diverge due to the vanishing of the photon mass. Indeed, it is this circumstance which necessitates the introduction of an infrared cutoff in the perturbation calculation. There, while the amplitude has an explicit cutoff dependence, the cross section is finite due to cancellation with the bremsstrahlung contribution. In our case, however, we wish to construct an amplitude which is cutoff independent. Thus, our procedure must, perforce, become somewhat less straightforward than before. In a previous calculation,[5] we have shown that the substitution of a simple function with Regge behavior for the photon pole term will serve to remove the divergence difficulty. In a sense, that substitution here amounts to a recognition of the fact that Fig. 1 represents a singular, inhomogeneous, nonlinear integral equation for the electron-positron scattering amplitude. We are attempting to construct a solution which is accurate to second order, and it is the case that the pole approximation is not a convenient point

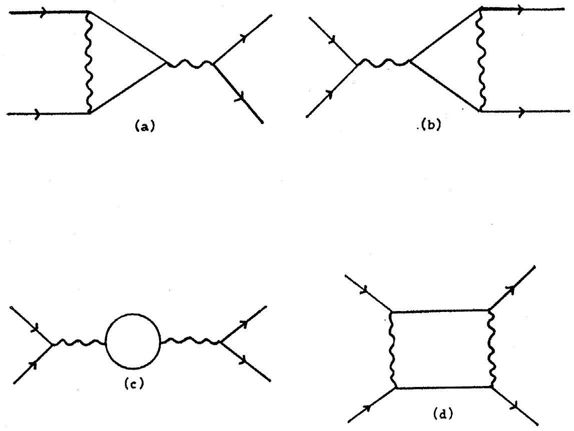

FIG. 3. These four Feynman diagrams are related to the elastic unitarity diagram 1(c).

of departure. Thus, we replace $A_i(s,t,u)_{\mathrm{Born}}$ by $A_i^0(s,t,u)$, where each $A_i^0(s,t,u)$ is identical to the corresponding Born term in Eq. (5), except for the substitution

$$\frac{4\pi\alpha}{t} \longrightarrow f_0(s,t,u) = \left(\frac{-4\pi\alpha}{4q_u^2}\right)\frac{\Gamma(1-i\eta(u))}{\Gamma(1+i\eta(u))}\left(\frac{-t}{4q_u^2}\right)^{-1+i\eta(u)} + \left|\frac{\alpha}{2q_uW_u\eta(u)}\right|\frac{\delta(-t/4q_u^2)}{2\pi i\rho_2(u)}, \quad (15)$$

where $\Gamma(z)$ is the Euler gamma function. The precise form of the trajectory function $\eta(u)$ is to be determined by requiring self-consistency. The particular form of Eq. (15) is motivated by the form of the exact Coulomb scattering amplitude obtained from the Schrödinger equation. It must be the case that our electron-positron amplitude reduces to the Schrödinger amplitude in a suitable limit, if we believe nonrelativistic quantum mechanics at all.

At this point, we should say a few words about the presence of a δ function in (15). This term is a necessary and unavoidable part of the Coulomb solution of the Schrödinger equation, although it is often overlooked in the derivation, and is a reflection of the little-remarked fact that for the Coulomb scattering process it is the *S matrix* which is analytic and *not* the transition amplitude $iT=S-I$. Thus, T contains an explicit δ function [$I=\delta(1-z)/2\pi\rho$ in the spinless case]. This peculiarity of Coulomb scattering will find a correspondence in our amplitude, which will also exhibit an explicit δ function. We will find (see Appendix C) that the rather odd-appearing coefficient of the δ function in Eq. (15) is such that, when combined with the spinor basis functions and crossing, the δ-function terms in our amplitude will reduce to exactly $-i$ times the identity matrix in the momentum representation in each channel. Thus, for electromagnetic processes in the relativistic case too, it is the S matrix which is explicitly analytic and not the transition amplitude.

If we insert the amplitude $T_0(k';k_i)$, which we obtain from the Born amplitude by means of substitution (15), into the right-hand side of the unitarity integral (14), and expand the product in terms of the spinor basis functions discussed in Sec. II, then we find that the result divides naturally into two parts which, for convenience, we will discuss separately. [Note that in each case, the integrations can be performed explicitly and analytically (see Appendix A).] The physical basis for this dichotomy can be understood as follows. The unitarity integral (14) is related to the four Feynman diagrams of Fig. 3. Of these diagrams, 3(a)–3(c) only represent vertex corrections, while box diagram 3(d) has a nonvanishing double-spectral function. In our evaluation of (14), we obtain homologous results. There is a further (mathematical) distinction, however, which may cause confusion. The vertex corrections obtained from (14) are multiplied by the direct spinor basis functions V, S, P, A, T, while the remainder is most simply expressed in terms of the crossed spinor basis functions $\tilde{V}$, $\tilde{S}$, $\tilde{P}$, $\tilde{A}$, $\tilde{T}$. This is independent of the fact that we will have to antisymmetrize these contributions to comply with the Pauli principle. With this caveat, we turn to the evaluation of (14).

The vertex corrections which arise from the elastic unitarity integral (14) can be expressed relatively simply. We find the following second-order contributions to the imaginary parts of the invariant amplitudes:

$$\begin{aligned}
D_t^V(s,t,u)_{e^+e^-} &= \theta(t-4)\,\mathrm{Im}_t\{(4\pi\alpha/t)\\
&\qquad\times\Gamma_1(0)[F(t)-F_1(0)]\},\\
D_t^S(s,t,u)_{e^+e^-} &= -\tfrac{1}{4}\kappa(s-u)\theta(t-4)\,\mathrm{Im}_t\{(4\pi\alpha/t)\\
&\qquad\times F_1(0)[F_2(t)-F_2(0)]\},\\
D_t^P(s,t,u)_{e^+e^-} &= -\tfrac{1}{4}\kappa(s-u)\theta(t-4)\,\mathrm{Im}_t\{(4\pi\alpha/t)\\
&\qquad\times\Gamma_1(0)[F_2(t)-F_2(0)]\},\\
D_t^A(s,t,u)_{e^+e^-} &= 0,\\
D_t^T(s,t,u)_{e^+e^-} &= -\tfrac{1}{8}\kappa t\theta(t-4)\,\mathrm{Im}_t\{(4\pi\alpha/t)\\
&\qquad\times\Gamma_1(0)[F_2(t)-F_2(0)]\},
\end{aligned} \quad (16)$$

where $F_1(t)$ and $F_2(t)$ [$\Gamma_1(t)$ and $\Gamma_2(t)$] are the electron form factors defined previously, and $F(t)$ is the vertex function defined by $F(t)=\Gamma(t)-\kappa F_2(t)$. $\Gamma(t)$ satisfies a once-subtracted[10] dispersion relation of the form

$$\Gamma(t)=\Gamma_1(0)+\frac{t}{\pi}\int_4^\infty dx\frac{\mathrm{Im}_x\Gamma(x)}{x(x-t)}, \quad (17)$$

where

$$\mathrm{Im}_t\Gamma(t)=2\pi\rho_2\tfrac{1}{6}\{2[f_0-(4\pi\alpha/t)](t+2) + (3f_1+f_2)(t-4)\}. \quad (18)$$

In (18), $2\pi\rho_2(t)f_l(t)=(t-2)^{-1}[-2\eta(t)\psi(l+1)]$, where $f_l(t)$ is the lowest-order term (in α) in the expansion of the Legendre transform of (15) (see Appendix A). $\psi(z)$ is the digamma function. We will find that the integral (17) is convergent for the particular value of $\eta(t)$ which makes our amplitude self-consistent. Note that no

[10] Note that a subtraction is necessary since the electron charge remains an arbitrary parameter in our theory as well as in perturbation theory.

infrared cutoff is necessary to define (17), in distinction to the perturbation-theory results. We see immediately that (16) does not contribute to the double-spectral functions. However, since it will be necessary to include the effects of the vertex corrections in our *ansatz*, results (16) will be convenient. With regard to later application, it is of interest to compare (16) with the Born term (5).

If we now consider those terms resulting from the evaluation of the elastic unitarity integral (14) which are proportional to the exchange spinor basis functions, we find that the discontinuity functions for each of the five invariant amplitudes can be written in the form

$$\tilde{D}_t^i(s,t,u)_{e^+e^-}=\theta(t-4)\frac{\alpha(t-2)}{2q_tW_t\eta(t)}\Big[\mathrm{Im}_t\tilde{A}_i^0(s,t,u)+\sum_{j=1}^{3}\tilde{p}_j^i(s,t,u)\,\mathrm{Im}_t\tilde{f}_j(s,t,u)\Big]. \quad (19)$$

In (19), $\tilde{A}_i^0(s,t,u)$ is our original lowest-order approximation to the invariant amplitude. The summation on the right-hand side of (19) can be obtained from the following:

$$\sum_{j=1}^{3}\tilde{p}_j^V(s,t,u)\tilde{f}_j(s,t,u)=\frac{1}{16}\bigg\{\bigg[-8+\frac{4(2s-t)}{s}+\frac{4(t-2)(t-4)}{s^2}-\frac{4(t-4)(u-4)(t-2)}{s^2t}\bigg]\tilde{f}_1(s,t,u)+\bigg[\frac{2(t-2)}{s^2}+\frac{2(t-4)(u-4)(t-2)}{s^2tu}\bigg]\tilde{f}_2(s,t,u)+\bigg[\frac{2(t-2)}{s^2}+\frac{2(s-u)(u-4)(t-2)}{s^2tu}\bigg]\tilde{f}_3(s,t,u)\bigg\},$$

$$\sum_{j=1}^{3}\tilde{p}_j^S(s,t,u)\tilde{f}_j(s,t,u)=\frac{1}{16}\bigg\{\bigg[\frac{2(t-s-12)}{s}+\frac{4(t-s-2)(t-4)}{s^2}-\frac{4(s-t)(t-2)(t-4)}{s^2t}\bigg]\tilde{f}_1(s,t,u)+\bigg[\frac{2(t-s-2)}{s^2}+\frac{2(s-t)(t-2)(t-4)}{s^2tu}\bigg]\tilde{f}_2(s,t,u)+\bigg[\frac{2(t-s-2)}{s^2}+\frac{2(s-t)(t-2)(s-u)}{s^2tu}\bigg]\tilde{f}_3(s,t,u)\bigg\},$$

$$\sum_{j=1}^{3}\tilde{p}_j^P(s,t,u)\tilde{f}_j(s,t,u)=\frac{1}{16}\bigg\{\bigg[\frac{2(12-s+t)}{s}+\frac{32(t-4)}{su}-\frac{4(2+s-t)(t-4)}{s^2}-\frac{4(s-t)(t-2)(t-4)}{s^2t}\bigg]\tilde{f}_1(s,t,u)+\bigg[\frac{2(t-s-2)}{s^2}+\frac{16(s-t+4)}{su^2}+\frac{2(s-t)(t-2)(t-4)}{s^2tu}\bigg]\tilde{f}_2(s,t,u)+\bigg[\frac{-16}{u^2}+\frac{2(t-s-2)}{s^2}+\frac{16(u-s)}{su}+\frac{2(s-t)(t-2)(s-u)}{s^2tu}\bigg]\tilde{f}_3(s,t,u)\bigg\}, \quad (20)$$

$$\sum_{j=1}^{3}\tilde{p}_j^A(s,t,u)\tilde{f}_j(s,t,u)=\frac{1}{16}\bigg\{\bigg[8+\frac{4(2-s+u)}{s}+\frac{4(t-4)(t-u-2)}{s^2}-\frac{8(u-4)(t-4)}{s^2t}\bigg]\tilde{f}_1(s,t,u)+\bigg[\frac{4(t-3)}{s^2}+\frac{4(t-4)(u-4)}{s^2tu}\bigg]\tilde{f}_2(s,t,u)+\bigg[\frac{4(1-u)}{s^2}-\frac{4(u-4)(u-s)}{s^2tu}\bigg]\tilde{f}_3(s,t,u)\bigg\},$$

$$\sum_{j=1}^{3}\tilde{p}_j^T(s,t,u)\tilde{f}_j(s,t,u)=\frac{1}{16}\bigg\{\bigg[\frac{4-u}{s}-\frac{2(u-2)(t-4)}{s^2}-\frac{2(u-4)(t-2)(t-4)}{s^2t}\bigg]\tilde{f}_1(s,t,u)+\bigg[\frac{2-u}{s^2}+\frac{(u-4)(t-2)(t-4)}{s^2tu}\bigg]\tilde{f}_2(s,t,u)+\bigg[\frac{2-u}{s^2}+\frac{(u-4)(t-2)(s-u)}{s^2tu}\bigg]\tilde{f}_3(s,t,u)\bigg\}.$$

In (20) the $\tilde{f}_j(s,t,u)\equiv\tilde{f}_j(t,z)$ are given by

$$\begin{aligned}\tilde{f}_1(t,z)&=4\pi\alpha(t-2)^{-1}|\Gamma(1-i\eta)|^2[P_{i\eta}(-z)-1],\\ \tilde{f}_2(t,z)&=-4\pi\alpha(4q^2)(t-2)^{-1}|\Gamma(1-i\eta)|^2\{\tfrac{1}{2}(1+i\eta)\\ &\qquad\times[P_{i\eta}(-z)+P_{1+i\eta}(-z)]-\tfrac{1}{2}(1-z)\},\\ \tilde{f}_3(t,z)&=4\pi\alpha(4q^2)(t-2)^{-1}|\Gamma(1-i\eta)|^2 i\eta P_{i\eta}(-z),\end{aligned}\tag{21}$$

where $P_\nu(z)$ is a Legendre function of the first kind (see Appendix B). There are three points which must be emphasized concerning (19): (1) Although each of the $\tilde{p}_j{}^i(s,t,u)$ has poles at $s=0$, the coefficient of each pole in the discontinuity function is zero, so that (19) does not have any spurious singularities. (2) We observe that if we set $\eta(t)=\alpha(t-2)/2qW$, our initial trial amplitude $\tilde{A}_i{}^0(s,t,u)$ will reproduce itself. (3) Finally, if we assume that $\eta(t)$ is proportional to α, we note that the leading terms of $\tilde{f}_j(t,z)$ are of order α^2. These three considerations will play an important role in the construction of our *ansatz* for the electron-positron scattering amplitude.

With a little labor, it is possible to put (19) in the form implied by Eq. (1), with $\alpha(t)=-1+i\eta(t)$. From this, it is a relatively simple matter to obtain the elastic unitarity contribution to the double-spectral functions. We find that to second order in the fine-structure constant

$$\begin{aligned}\tilde{\rho}_{tu}{}^i(s,t,u)_{e^+e^-}&=\tilde{\rho}_{ut}{}^i(s,t,u)_{2\gamma},\\ \tilde{\rho}_{ts}{}^i(s,t,u)_{e^+e^-}&=0,\end{aligned}\tag{22}$$

where result (22) is independent of the precise form of $\eta(t)$. This last circumstance arises from the fact that (19) exhibits a factor $1/\eta(t)$, while the discontinuity in u of $\mathrm{Im}_t\tilde{f}_0(s,t,u)$ is proportional to $\eta(t)$. Result (22) is of particular importance, since it will allow a relatively simple analytic form for the second-order electron-positron scattering amplitude. Finally, in order to complete the specification of the double-spectral functions, it is necessary to consider the u-channel elastic unitarity. As we have indicated previously, this will be equivalent to antisymmetrization with respect to the exchange $k_4\leftrightarrow -k_3$. Thus, the elastic double-spectral functions, to second order, which we obtain from the explicit evaluation of diagrams 1(b) and 1(c) of Fig. 1, and their exchange counterparts, can be written as follows:

$$\begin{aligned}\rho_{tu}{}^i(s,t,u)&=-\tilde{\rho}_{ut}{}^i(s,u,t)=2\rho_{tu}{}^i(s,t,u)_{2\gamma},\\ \rho_{ts}{}^i(s,t,u)&=-\tilde{\rho}_{us}{}^i(s,u,t)=(-1)^i\rho_{tu}{}^i(u,t,s)_{2\gamma},\end{aligned}\tag{23}$$

where $\rho_{tu}{}^i(s,t,u)_{2\gamma}$ is given by (12).

V. ELECTRON-POSITRON SCATTERING AMPLITUDE

At this point it is possible to introduce an *ansatz* for the construction of the elastic electron-positron scattering amplitude. Referring to Eq. (2), we set

$$\begin{aligned}A_i(s,t,u)&=2F_i(s,t,u)+(-1)^iF_i(u,t,s),\\ \tilde{A}_i(s,t,u)&=-A_i(s,u,t),\end{aligned}\tag{24}$$

where $(-1)^i$ is defined by Eq. (9). $F_i(s,t,u)$ can be written

$$F_i(s,t,u)=F_i{}^0(s,t,u)+\sum_{j=1}^{3}p_j{}^i(s,t,u)f_j(s,t,u),\tag{25}$$

where the $p_j{}^i(s,t,u)$ and $f_j(s,t,u)$ are defined by (20) and (21), with $p_j{}^i(s,t,u)=-\tilde{p}_j{}^i(s,u,t)$, $f_j(s,t,u)=-\tilde{f}_j(s,u,t)$. $F_i{}^0(s,t,u)$ is just our original trial amplitude $A_i{}^0(s,t,u)$, with the vertex corrections indicated by (16). Explicitly,

$$\begin{aligned}F_V{}^0(s,t,u)&=\Gamma_1(0)F(t)f_0(s,t,u),\\ F_S{}^0(s,t,u)&=-\tfrac{1}{4}\kappa(s-u)F_1(0)F_2(t)f_0(s,t,u),\\ F_P{}^0(s,t,u)&=-\tfrac{1}{4}\kappa(s-u)\Gamma_1(0)F_2(t)f_0(s,t,u),\\ F_A{}^0(s,t,u)&=0,\\ F_T{}^0(s,t,u)&=-\tfrac{1}{8}\kappa t\Gamma_1(0)F_2(t)f_0(s,t,u),\end{aligned}\tag{26}$$

where $f_0(s,t,u)$ is defined by (15) and $F(t)$ is the vertex function discussed above Eq. (17). Note that the amplitude (24) is not the result of a dispersion integration over the discontinuity functions given in (16) and (19), although (24) will have these elastic unitarity cuts. A dispersion integral involving (16) and (19) would eliminate the left-hand cut in t which was found to obtain in the double-spectral functions. This left-hand cut, which is contained in the trajectory function, seems to be an inescapable feature of relativistic electromagnetic scattering; both the Klein-Gordon and Dirac Coulomb amplitudes exhibit this particular characteristic.

We now consider the properties which our electron-positron elastic scattering amplitude must exhibit, and verify that (24) represents a suitable solution to the problem. By construction, the amplitude defined in Eq. (24) is crossing symmetric. Moreover, each of the invariant amplitudes is cutoff independent. Comparison with (23), using (22), reveals that (24) has the correct double-spectral functions through second order in the fine-structure constant. Except for the δ-function terms, (24) is an analytic function of s, t, and u, with no singularities except those implied by unitarity and crossing. As we explained earlier, the δ functions are due to the fact that, for electromagnetic processes, it is the S matrix which is explicitly analytic, rather than the transition amplitude $iT=S-I$. It is possible to show that the δ-function terms in (24) combine to yield exactly $-i$ times the identity matrix in each channel (see Appendix C), so that the S matrix obtained from (24) will be a maximally analytic function of s, t, and u. Finally, we note that the amplitude (24) exhibits Regge asymptotic behavior, with the leading trajectory given by $\alpha_0(t)=-1+i\eta(t)$.[11] If we set

$$\eta(t)=\alpha(t-2)/2qW\,, \tag{27}$$

then the positronium poles will appear at the correct position, including reduced mass and recoil corrections,[12] and with the proper residue. The trajectory defined by (27) has often been suggested as an appropriate form for the description of bound states in electromagnetic processes.[13] Moreover, with the trajectory function (27), each of the invariant amplitudes defined by (24) will reduce to the correct Born term in lowest order. In fact, since $(t-2)/2W$ is equal to the reduced mass at threshold, (24) actually reproduces the Coulomb scattering amplitude in the low-energy limit. All that remains to be established is that (24) is self-consistent.

The amplitude defined by Eq. (24) will be self-consistent if its introduction into the right-hand side of the elastic unitarity integral (14) does not produce any second-order contributions to the double-spectral functions other than those given in Eq. (22). We have already noted that, with $\eta(t)$ defined by (27), the $F_i^0(s,t,u)$ will be reproduced in the evaluation of the elastic unitarity statement. It is only necessary to show that the remaining terms in (24) do not introduce any additional second-order terms into the double-spectral functions. However, this condition is obviously satisfied. Inspection of (24) reveals that with the exception of $F_i^0(s,t,u)$ the remainder of (24) is of second order in α. Insertion of these terms into the unitarity integral (14) will only produce terms of order three or greater, which can make no contribution to the second-order double-spectral functions. We remark that (24) has a well-defined Jacob-Wick expansion,[14] so that its reintroduction into the elastic unitarity can be effected without divergence. This last, aside from its implications concerning the practical applicability of (24) to dispersion calculations of electromagnetic quantities, ensures that our amplitude provides a suitable basis for an iterative calculation of the higher-order terms.

VI. DISCUSSION

Using the electron-positron system as an example, we have outlined a procedure by which accurate dispersion-theoretic electromagnetic scattering amplitudes may be generated for processes in which bound states can appear. We have exhibited a Lorentz-covariant spinor amplitude which is analytic, cutoff independent, and crossing symmetric, which has the correct double-spectral functions through second order in α, and reduces to the usual Born approximation in lowest order. Moreover, the amplitude displays Regge asymptotic behavior and the positronium Regge poles. Self-consistency requires that the positronium poles appear at the correct position and with the proper residue, and that the amplitude possesses a well-defined Jacob-Wick expansion. The procedure can, thus, be extended to higher-order terms, although it is not known whether the higher-order contributions to the double-spectral functions, which we obtain from explicit evaluation of the unitarity diagrams to that order, will still represent the correct scattering amplitude. This is a moot point, however, since the exact amplitude is not available for comparison. We intend to introduce the electron-positron amplitude (24) into dispersion calculations of the electron anomalous magnetic moment and the Lamb shift in hydrogen. In fact, $T_0(k_f;k_i)$ has already been employed to this purpose with gratifying results.[5,15] It is interesting to speculate that our procedure for obtaining electromagnetic scattering amplitudes might also be applied directly to a calculation of the fine-structure and Lamb shift corrections to the positronium (or hydrogen atom) energy spectrum, since the quantity which occupies a central position in our calculation is the trajectory function. In that case, there would be no need for the bound-state perturbation theory, since the fine-structure and Lamb shift corrections would, presumably, be included in the higher-order terms of the trajectory functions. Unlike the perturbation calculation where the renormalization program and infrared difficulties necessitate an extremely cautious evaluation, our method can be employed without encountering any divergence difficulties. Moreover, it should be somewhat simpler, since only the double-spectral functions need be evaluated explicitly. However, this possibility must remain untested until the α^3 and higher-order corrections to the scattering amplitude can be evaluated. In the meantime, (24) can be employed in the usual manner in the dispersion calculations of the Lamb shift and electron anomalous magnetic moment. In any case, we feel that our dispersion-theoretic approach to electromagnetic interactions may represent a significant improvement over previous efforts.

APPENDIX A

In the evaluation of the elastic unitarity statement (14), we encounter integrals of the form

$$\tfrac{1}{2}\rho_2(t)\int d\Omega'\tilde{f}_0(t,z')\tilde{f}_0^*(t,z'')\times[(u')^m(u'')^n+(u')^n(u'')^m]\,, \tag{A1}$$

where $\tilde{f}_0(t,z)\equiv\tilde{f}_0(s,t,u)=-f_0(s,u,t)$ is defined by (15).

[11] Note that the asymptotic behavior of our amplitude is essentially the same as that obtained in the relativistic eikonal approximation. See, for example, M. Lévy and J. Sucher, Phys. Rev. D 2, 1716 (1970).

[12] G. Breit and G. E. Brown, Phys. Rev. 74, 1278 (1948).

[13] L. Durand, Phys. Rev. 154, 1538 (1967). Also, from a consideration of the infinite-dimensional representations of relativistic $O(4,2)$, A. O. Barut and A. Baiquni, *ibid.* 184, 1342 (1969). From the eikonal approximation, E. Brezin, C. Itzykson, and J. Zinn-Justin, Phys. Rev. D 1, 2349 (1970).

[14] The precise form of the partial-wave amplitudes which can be derived from (24) will be discussed in another place.

[15] J. McEnnan (unpublished).

$\tilde{f}_0(t,z)$ can be expanded in a Legendre series,

$$\tilde{f}_0(t,z)=\sum(2l+1)f_l(t)P_l(z)\,, \tag{A2}$$

where

$$f_l(t)=\left(\frac{\alpha}{qW\eta}\right)\left(\frac{1}{4\pi i\rho_2}\right)(e^{2i\delta_l}-1) \tag{A3}$$

and

$$e^{2i\delta_l}=\Gamma(l+1-i\eta)/\Gamma(l+1+i\eta)\,. \tag{A4}$$

Thus,

$$\tfrac{1}{2}\rho_2\int d\Omega'\tilde{f}_0(t,z')\tilde{f}_0{}^*(t,z'')=\left(\frac{\alpha}{qW\eta}\right)\mathrm{Im}_t\tilde{f}_0(t,z)\,. \tag{A5}$$

For the case $m,\,n\neq 0$, we note that $u^k\tilde{f}_0(t,z)$ can also be expanded in a Legendre series, with coefficients $a_l(t)$ given by

$$a_l(t)=\frac{e^{2i\delta_l}}{4\pi i\rho_2}(4q^2)^k\left(\frac{\alpha}{qW\eta}\right) \times\prod_{m=1}^{k}\frac{[m-1+i\eta]^2}{[l-(m-1+i\eta)][l+1+(m-1+i\eta)]}. \tag{A6}$$

Equation (A6) can be rewritten

$$a_l(t)=\frac{e^{2i\delta_l}}{4\pi i\rho_2} \times\sum_{m=1}^{k}\frac{\alpha_m(k,\eta)}{[l-(m-1+i\eta)][l+1+(m-1+i\eta)]}, \tag{A7}$$

where the $\alpha_m(k,\eta)$ are independent of l. Given the Lengendre coefficients (A6) or (A7), there is no difficulty in performing the integration (A1). The result will be expressed in terms of another Legendre series, whose coefficients, we find, can always be written as a linear combination of the $a_l(t)$ and terms of the form

$$g_l(k,\eta)=\beta(k,\eta)[l-(k-m\pm i\eta)]^{-1} \times[l+1+(k-m\pm i\eta)]^{-1}, \tag{A8}$$

where the $\beta(k,\eta)$ are independent of l. A Legendre series with coefficients of the form (A8) can be explicitly summed in terms of the generalized Legendre functions of degree $k-m\pm i\eta$. These functions are discussed in Appendix B.

APPENDIX B

For convenience, we shall display here a few properties of the generalized Legendre functions of the first kind, $P_\nu(z)$.[16] These may be defined in terms of the hypergeometric function, where

$$P_\nu(z)=F\big(-\nu,\,\nu+1;\,1;\,\tfrac{1}{2}(1-z)\big)\,. \tag{B1}$$

[16] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U. S. Dept. of Commerce, Natl. Bur. Std., Washington, D. C., 1966).

$P_\nu(z)$ is an analytic function in the z plane cut along the negative real axis from $-\infty<z<-1$. The discontinuity across the cut is $-2i\sin\pi\nu P_\nu(-z)$, so that

$$P_\nu(z)=\frac{\sin\pi\nu}{\pi}\int_{-\infty}^{-1}\frac{P_\nu(-z')}{z'-z}dz'\,. \tag{B2}$$

For $\mathrm{Re}\nu>0$, $P_\nu(z)\to z^\nu$ for large z. When ν is a positive integer, $P_\nu(z)$ reduces to the usual Legendre polynomial. We also note that $P_\nu(z)=P_{-\nu-1}(z)$. In the complex ν plane, $P_\nu(z)$ is an entire function, with an essential singularity at infinity.

Of particular interest is the following integral:

$$\tfrac{1}{2}\int_{-1}^{1}dz\,P_l(z)P_\nu(z)=\frac{1}{\pi}(-1)^{l+1} \times\sin\pi\nu[(l-\nu)(l+1+\nu)]^{-1}\,. \tag{B3}$$

We note that the right-hand side of (B3) is essentially the same as that of (A8), with $\nu=k-m\pm i\eta$.

APPENDIX C

We wish to verify that the δ-function terms in (22) reduce to the appropriate identity matrix in each channel. Using our result for the trajectory function, we find the δ-function terms may be written

$$\begin{aligned}&-\left\{(2)\left[\frac{1}{2\pi i\rho_2(u)}\right]\left|\frac{1}{u-2}\right|\delta\!\left(\frac{-t}{u-4}\right)-\left[\frac{1}{2\pi i\rho_2(s)}\right]\left|\frac{1}{s-2}\right|\delta\!\left(\frac{-t}{s-4}\right)\right\}Y(s,t,u)\\&+\left\{(2)\left[\frac{1}{2\pi i\rho_2(t)}\right]\left|\frac{1}{t-2}\right|\delta\!\left(\frac{-u}{t-4}\right)-\left[\frac{1}{2\pi i\rho_2(s)}\right]\left|\frac{1}{s-2}\right|\delta\!\left(\frac{-u}{s-4}\right)\right\}\tilde{Y}(s,t,u)\,,\end{aligned} \tag{C1}$$

where $\rho_2(t)$ is the two-body phase-space factor and $Y(s,t,u)$ is the appropriate combination of spinor basis functions. We will consider here the t channel, where $t\geq 4$ and $s=-2q^2(1+z)$, $u=-2q^2(1-z)$. Since our amplitude is explicitly crossing symmetric, it is sufficient to consider only one channel. Thus, the first terms in (C1) are zero, since the arguments of their δ functions are nowhere zero in the t-channel physical region. The remaining terms can be rewritten using the fact that $\delta(z)f(z)=\delta(z)f(0)$. We have

$$\left(\frac{1}{2\pi i\rho_2}\right)\left|\frac{1}{t-2}\right|\left[2\delta\!\left(\frac{1-z}{2}\right)-\left(\frac{t}{t-4}\right)\delta\!\left(\frac{-u}{s-4}\right)\right]\tilde{Y}(t,z)\,. \tag{C2}$$

We now use the identity $\delta[f(z)]=\delta(z-z_0)/|f'(z_0)|$, where z_0 is a zero of $f(z)$, to write (C2) in the form

$$\left(\frac{1}{2\pi i\rho_2}\right)\left|\frac{2}{t-2}\right|\delta(1-z)\tilde{Y}(t,z). \tag{C3}$$

Finally, it is a relatively simple matter, using the helicity representation, for example, to show that

$$\tilde{Y}(t,\,z=1)=\tfrac{1}{2}(t-2)\delta_{\lambda_4}{}^{\lambda_1}\delta_{\lambda_2}{}^{\lambda_3}. \tag{C4}$$

Thus, in the t channel, (C1) reduces to

$$\frac{1}{2\pi i\rho_2(t)}\delta(1-z)\delta_{\lambda_4}{}^{\lambda_1}\delta_{\lambda_2}{}^{\lambda_3}, \tag{C5}$$

which is just $-i$ times the identity, as promised.

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Dual-Resonance Model with Quark Spin*†

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We present a dual-resonance model with nontrivial quark spin factors. An amplitude is found which satisfies factorization and eliminates the parity-doubling ghosts. An application to $\pi\pi$ elastic scattering indicates that the positivity condition is not met on the first daughter trajectory if one assumes realistic values for the mass and intercept.

I. INTRODUCTION

RECENT developments of the dual-resonance model[1,2] have revealed a close connection of the duality concept with the quark model.[3–5] It is now possible to embark on the construction of a hadron model out of quarks in the manner represented by the Harari-Rosner quark diagram.[6,7]

A crucial step in this program is a proof of the factorization property of the dual-resonance model. The proof of factorization has been extended to all the daughter trajectories.[8,9] The resonance spectrum in the model has been greatly clarified using the simple device of the harmonic oscillator.[10–12] Roughly speaking, mesons appear to be bound states of the quark and antiquark with a relativistic string between them.

Previous attempts[4,5] to incorporate quark spin into the dual model have suffered from a serious defect. Consistent factorization of the spin factor considered in that approach demands the existence of ghosts associated with negative-parity quarks. On the other hand, a recent work of Carlitz and Kislinger[13] provided a new way to avoid parity doubling of the fermion trajectory in the Van Hove model. Motivated by this work, many people have proposed to dualize the projection operator to eliminate parity-doubling ghosts.[14–16] We will present in this paper a different, but closely related approach to a correct treatment of the quark spin.

Our guiding principle in selecting a spin factor is the simple over-all picture of the dual-resonance model of Refs. 10–12. After constructing a cyclically symmetric amplitude of mesons, we proceed to check factorization of the whole amplitude. A simple quark propagator considered in Sec. II turns out to eliminate the parity-doubling ghosts from the leading trajectory only. A generalization of the propagator is then suggested, and elimination of ghosts from all the trajectories, as well as complete factorization, is proved in Sec. III. The generalized amplitude resembles a recent model of Carlitz, Ellis, Freund, and Matsuda.[16] The main difference lies in our insistence on the original form of the quark projector; therefore, we factorize the meson

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