which completes the demonstration. Hence $\bar{M}_{\mu\nu}^{(2\pi)}(q;p)$ is finite. Let us take note of the last term of (B5), which increases by a factor of q^2 faster at large q^2 than any other contribution to $\bar{M}_{\mu\nu}^{(2\pi)}$. The matrix elements of $\Theta_{\mu\nu}$ may be "softened"[6] by adding a finite term,

$$\frac{g}{24\pi^2}(\partial_\mu\partial_\nu-g_{\mu\nu}\Box^2)(\hat{\sigma}^2+\hat{\pi}^2), \quad (B15)$$

to $\Theta_{\mu\nu}$ computed from (3.13) and (3.11).

The computation of matrix elements of Θ, which are also finite, follows the same lines as those of $\Theta_{\mu\nu}$. For example, Figs. 1(a)–1(c) contribute to the OPI matrix elements of $\Theta\to 2\pi$, with the result from Fig. 1(b) increasing by a factor of q^2 more than is desirable. The addition of

$$-\frac{g}{8\pi^2}\Box^2(\hat{\sigma}^2+\hat{\pi}^2) \quad (B16)$$

to (3.14) softens the matrix elements of Θ in this order. The results of these computations are given in (3.16)–(3.24), with Eq. (3.27) also reflecting the modification of Θ due to (B16).

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Conserved Currents, Reggeization, and Mandelstam Counting in Second-Order Perturbation Theory*

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We consider the spinor pole in second-order spinor-vector and spinor–axial-vector scattering for all possible couplings and show that, except for the vector γ^μ coupling, none of the amplitudes factor. Therefore, only in the case of γ^μ coupling does the spinor lie on a nondegenerate Regge trajectory for all values of the coupling constant. The degeneracy in the case of the axial-vector $\gamma^\mu\gamma_5$ coupling is less than for the remainder of the couplings. We also consider the axial-vector pole in pseudoscalar-vector scattering and show that for one particular pseudoscalar–vector–axial-vector coupling, the sense amplitudes factor. These results are reconciled with the Mandelstam counting procedure, and the effects of gauge invariance and isospin on factorization are investigated.

I. INTRODUCTION

Several years ago Gell-Mann, Goldberger, Low, Marx, Singh, and Zachariasen[1–5] studied the conditions necessary for an elementary particle to lie on a Regge trajectory independently of the value of the coupling constant. They found that two conditions are necessary: There must exist at least one nonsense channel, and the perturbation-theory approximations to the scattering amplitudes must factor. They showed, by explicitly calculating in second and fourth order, that the spinor in spinor-vector scattering with a γ^μ coupling does Reggeize, but that the scalar in scalar-vector scattering does not Reggeize. Subsequently, Mandelstam[6] gave a method for determining to all orders if a field theory must Reggeize, by counting the conditions placed on, and the free parameters in, the relevant partial-wave amplitudes. This counting method ostensibly is independent of the particular coupling used and depends only on the spins of the particles; it requires, however, that the field theory give scattering amplitudes that are consistent with unitarity. Mandelstam showed that indeed spinor-vector scattering must Reggeize, while scalar-vector need not Reggeize. Recently Abers, Keller, and Teplitz[7] showed that if isotopic spin is added to the γ^μ coupling considered by GGLMZ, and the vector-vector-vector interaction of the Yang-Mills field is included as demanded by gauge invariance, the second-order Born terms continue to factor.

All this leads to some questions. (1) What about vector couplings other than γ^μ in spinor-vector scattering? Mandelstam counting would seem to insist that

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[7] E. Abers, R. Keller, and V. Teplitz, Phys. Rev. D 2, 1757 (1970); R. Keller, thesis (unpublished).

these Reggeize also; we can check this by seeing if the second-order terms factor. These other couplings, however, are nonrenormalizable and may or may not yield perturbation-theory amplitudes consistent with the unitarity bound. If they do (do not) give real phase shifts in second order, the Mandelstam procedure predicts that the theory has (has not) a factorizing trajectory to second order. The second-order sense-sense amplitudes for γ^μ coupling go as $1/E^2$ for large E so that these other couplings may have worse behavior and yet still be consistent with unitarity. (To be consistent with unitarity the amplitudes must go no faster than a constant.) If unitarity is violated, then factorization does not necessarily imply Reggeization, but if the amplitudes do not factor, they certainly cannot Reggeize. (2) If the amplitudes violate the unitarity bound, what is the effect on Mandelstam counting; in particular, can the counting be generalized by adding an additional parameter for each subtraction constant that is necessary to make the amplitudes unitary? (3) What about various couplings in axial-vector–spinor scattering? The counting is the same as in vector-spinor scattering. In particular, what about the $\gamma^\mu\gamma^5$ coupling? Current prejudice is that $\gamma^\mu\gamma^5$ should be as good for axial vector as γ^μ is for vector. (4) Finally, in view of Ref. 7, what are the roles of gauge invariance or partial conservation of axial-vector current (PCAC) in determining whether a particle Reggeizes? What is the effect of isospin on factorization of the amplitudes?

We will try to answer these questions by considering all possible vector and axial-vector couplings and showing that all of them except γ^μ violate the unitarity bound and that none of them, except γ^μ, factor. As might be expected, the $\gamma^\mu\gamma^5$ coupling comes closest to satisfying unitarity and we show that only two Regge poles exist in the coupled three-channel program, one sense-choosing and one nonsense-choosing. We also consider the axial-vector pole in pseudoscalar-vector scattering (or equivalently the vector pole in scalar-vector scattering) and show that, while none of the possible couplings give amplitudes which are consistent with unitarity, one of the couplings does give amplitudes which factor. We then use these results to discuss Mandelstam counting and the effect of gauge invariance.

In particular we shall see that gauge invariance (or PCAC) is related to the presence of ancestors. If the amplitudes are not gauge invariant then in most cases the partial-wave amplitudes will have terms which go as $\delta_{l,1}$ (or $\delta_{l,2}$, etc.), whereas if they are gauge invariant (or satisfy PCAC), the sense amplitudes will go as $\delta_{l,0}$. Then if the conditions of GGLMZ are satisfied, the higher-order corrections must replace the $\delta_{l,0}$ by $-\alpha/(l-\alpha)$ and the elementary particle lies on a Regge-pole trajectory. It should, however, be emphasized that, as will become apparent, the requirement of factorization is already nontrivial in second order.

There are five ways we can couple a vector to two spinors of momentum p_1 and p_2; namely, the couplings $\Gamma^\mu=\gamma^\mu$, $\sigma^{\mu\nu}k_\nu$, P^μ, k^μ, and $\sigma^{\mu\nu}P_\nu$, where $k^\mu=(p_2-p_1)^\mu$ and $P^\mu=(p_1+p_2)^\mu$. If the spinors are on the mass shell, then the five couplings are not independent. In particular,

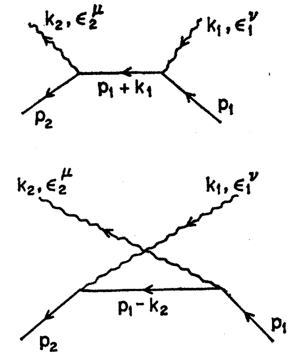

FIG. 1. s and u channels for spinor-vector scattering.

$$\bar{u}(p_2)k^\mu u(p_1)=-i\bar{u}(p_2)\sigma^{\mu\nu}P_\nu u(p_1) \quad (1)$$

and

$$\bar{u}(p_2)P^\mu u(p_1) =-i\bar{u}(p_2)\sigma^{\mu\nu}k_\nu u(p_1)+2m\bar{u}(p_2)\gamma^\mu u(p_1). \quad (2)$$

If we use the different couplings to calculate amplitudes as shown in Fig. 1,

$$T(\Gamma^\mu)\equiv \epsilon_2{}^\mu\epsilon_1{}^\nu \times[(s-m^2)^{-1}\bar{u}(p_2)\Gamma_\mu(\gamma\cdot p_2+\gamma\cdot k_2+m)\Gamma_\nu u(p_1) +(u-m^2)^{-1}\bar{u}(p_2)\Gamma_\nu(\gamma\cdot p_2-\gamma\cdot k_1+m)\Gamma_\mu u(p_1)] \quad (3)$$

($\epsilon_2{}^\mu$ and $\epsilon_1{}^\nu$ are defined in the Appendix), then, for example, by (2) the amplitude with $\Gamma^\mu=P^\mu$ need not be equal to the amplitude with $-i\sigma^{\mu\nu}k_\nu+2m\gamma^\mu$ at each vertex but it must not differ from that amplitude by more than a polynomial in k and P (a seagull). The amplitude with γ^μ coupling and the amplitude with $\sigma^{\mu\nu}k_\nu$ coupling are each gauge invariant. The amplitude with P^μ is not gauge invariant and if we add a seagull to make it gauge invariant then, of course, we may choose to add precisely the seagull which makes the P^μ amplitude equal to the amplitude with the sum of the other two couplings. In fact, no other reasonable seagull exists.

The amplitude with k^μ is zero since $\epsilon\cdot k=0$ so the amplitude with $\sigma^{\mu\nu}P_\nu$ must be only a polynomial. These two couplings violate G parity and have been named second class by Weinberg.[8]

There are also five axial-vector couplings which are obtained by multiplying the vector couplings by γ^5. Here we have the relations

$$-i\bar{u}(p_2)\sigma^{\mu\nu}P_\nu\gamma^5u(p_1) =\bar{u}(p_2)k^\mu\gamma^5u(p_1)-2m\bar{u}(p_2)\gamma^\mu\gamma^5u(p_1) \quad (4)$$

and

$$\bar{u}(p_2)P^\mu\gamma^5u(p_1)=-i\bar{u}(p_2)\sigma^{\mu\nu}k_\nu\gamma^5u(p_1). \quad (5)$$

Now, if we use these couplings to calculate the diagrams

[8] S. Weinberg, Phys. Rev. 112, 1375 (1958).

in Fig. 1, again the relations between the currents must hold for the amplitudes except for polynomials. The amplitude with $\gamma^\mu\gamma^5$ satisfies PCAC in that if we replace ϵ^μ by k^μ at one vertex the resulting amplitude is the same as the amplitude for a $2m\gamma^5$ coupling at that vertex. To see this consider

$$\text{``}\gamma^\mu\gamma^5\text{'' amplitude} = \text{``}\gamma^\mu\text{'' amplitude} + \frac{2m}{s-m^2}\bar{u}\gamma\cdot\epsilon_2\gamma\cdot\epsilon_1 u + \frac{2m}{u-m^2}\bar{u}\gamma\cdot\epsilon_1\gamma\cdot\epsilon_2 u.$$

If we replace ϵ_2 by k_2, we get

$$\frac{2m}{s-m^2}\bar{u}\gamma\cdot k_2\gamma\cdot\epsilon_1 u + \frac{2m}{u-m^2}\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_2 u,$$

since the "γ^μ" amplitude is gauge invariant. The amplitude with $2m\gamma^5$ at one vertex is

$$\frac{2m}{s-m^2}\bar{u}\gamma^5(\gamma\cdot p_2+\gamma\cdot k_2+m)\gamma\cdot\epsilon_1\gamma^5 u + \frac{2m}{u-m^2}\bar{u}\gamma\cdot\epsilon_1\gamma^5(\gamma\cdot p_1-\gamma\cdot k_2+m)\gamma^5 u,$$

which equals the expression above.

The coupling $\sigma^{\mu\nu}k_\nu\gamma^5$ does not give PCAC since it is manifestly gauge invariant. Further, it cannot be made to satisfy PCAC by adding a polynomial since the PCAC statement involves obtaining an amplitude with poles from the substitution of k^μ for ϵ^μ. The same is true for the amplitudes with $P^\nu\gamma^5$ because, by (4), they are gauge invariant except for polynomials. The couplings $\sigma^{\mu\nu}k_\nu\gamma^5$ and $P^\mu\gamma^5$ violate G parity.

We now consider each of these couplings in detail. The calculations are elementary and will not be shown; the information necessary to check our results is given in the Appendix. Unless we specify otherwise, we will consider all the particles to have the same mass.

II. VECTOR WITH γ^μ COUPLING

This is the case considered by GGLMZ. The relevant second-order diagrams given in Fig. 1 serve to define the momenta. The result is

$$T(\gamma^\mu) = \frac{g_V^2}{(p_1+k_1)^2-m^2}(2\epsilon_2\cdot p_2 T_2 + T_{12}) + \frac{g_V^2}{(p_1-k_2)^2-m^2}(2\epsilon_1\cdot p_2 T_3 - T_8), \quad (6)$$

where the ϵ^μ and the T_i are given in the Appendix. g_V is the spinor-vector coupling constant. Using the Appendix, the helicity amplitudes are given by

$$\begin{aligned}
T_{1,1}{}^+ &= \frac{g_V^2}{s-m^2}\frac{1}{\sqrt{2}}\left(\frac{4E^2}{m}-2m-2E\right)+\frac{g_V^2}{u-m^2}\frac{1}{\sqrt{2}} \times\left[\left(\frac{2E^2}{m}-3m\right)(z-1)-E(1+z)\right],\\
T_{1,0}{}^+ &= \frac{g_V^2}{s-m^2}\left(\frac{4E^3}{m^2}-3E-m\right) + \frac{g_V^2}{u-m^2}\left[\frac{2E^3}{m^2}(1+z)-E(2+3z)-mz\right],\\
T_{0,0}{}^+ &= \frac{g_V^2}{s-m^2}\frac{1}{\sqrt{2}}\left(\frac{8E^4}{m^3}+\frac{4E^3}{m^2}-\frac{6E^2}{m}-5E-m\right) + \frac{g_V^2}{u-m^2}\frac{1}{\sqrt{2}}\left[\frac{4E^4}{m^3}(1+z)+\frac{2E^3}{m^2}(1+z) - \frac{2E^2}{m}(2+3z)-E(3+4z)+m\right],\\
T_{-1,0}{}^+ &= \frac{-g_V^2}{u-m^2}(E+m),\\
T_{-1,1}{}^+ &= \frac{-g_V^2}{u-m^2}\frac{1}{\sqrt{2}}(E+m),\\
T_{-1,-1}{}^+ &= \frac{-g_V^2}{u-m^2}\frac{1}{\sqrt{2}}\left(\frac{2E^2}{m}+E-m\right).
\end{aligned} \quad (7)$$

One may find the T^- amplitudes from the T^+ by changing the sign of E everywhere. This is MacDowell symmetry.

For large $z=\cos\theta$, the amplitudes in (7) become

$$\begin{aligned}
T_{1,1}{}^+ &= \frac{g_V^2}{2\sqrt{2}}\frac{1}{k^2}\frac{m}{2E-m}(E+m)+O(z^{-1}),\\
T_{1,0}{}^+ &= \frac{g_V^2}{2}\frac{1}{k^2}\frac{m}{2E-m}(E+m)+O(z^{-1}),\\
T_{0,0}{}^+ &= \frac{g_V^2}{\sqrt{2}}\frac{1}{k^2}\frac{m}{2E-m}(E+m)+O(z^{-1}),\\
T_{-1,0}{}^+ &= \frac{g_V^2}{2}\frac{1}{k^2}(E+m)\frac{1}{z}+O(z^{-2}),\\
T_{-1,1}{}^+ &= \frac{g_V^2}{2\sqrt{2}}\frac{1}{k^2}(E+m)\frac{1}{z}+O(z^{-2}),\\
T_{-1,-1}{}^+ &= \frac{g_V^2}{2\sqrt{2}}\frac{1}{k^2}\frac{2E-m}{m}(E+m)\frac{1}{z}+O(z^{-2}).
\end{aligned} \quad (8)$$

The amplitudes in (8) check with the results of GGLMZ and obviously factor for all E. That is, the matrix of coefficients of z^0 for sense amplitudes and of z^{-1} for sense-nonsense and nonsense-nonsense amplitudes has vanishing determinants and its minors have a vanishing determinant; GGLMZ explain in detail this requirement and the sense-nonsense choice. Notice that the sense-sense amplitudes each go as $1/E^2$ as E gets large. This is because of a delicate cancellation of the large powers of E between the s- and u-channel contributions; the s- and u-channel contributions separately go as E^0 for $T_{1,1}{}^+$, as E^1 for $T_{1,0}{}^+$, and as E^2 for $T_{0,0}{}^+$. By considering the z^{-1} terms it is easily seen that all of the higher partial waves in second order grow no faster than a constant as E gets large and thus are consistent with the unitarity bound.

III. AXIAL VECTOR WITH $g_A\gamma^\mu\gamma^5$ COUPLING

We evaluate the diagrams in Fig. 1 where the wavy lines are axial vectors instead of vectors. The amplitude here is the same as that of vector γ^μ coupling plus additional terms proportional to $2m$,

$$T(\gamma^\mu\gamma^5)=\frac{g_A{}^2}{(p_1+k_1)^2-m^2}\times(2\epsilon_2\cdot p_2T_2+T_{12}-4m\epsilon_1\cdot\epsilon_2T_1+2mT_5)+\frac{g_A{}^2}{(p_1-k_2)^2-m^2}(2\epsilon_1\cdot p_2T_3-T_8-2mT_5)\,,\quad(9)$$

where g_A is the spinor–axial-vector coupling constant. Using the Appendix, we find

$$T_{1,1}{}^+=\frac{g_A{}^2}{s-m^2}\frac{1}{\sqrt{2}}\left(\frac{4E^2}{m}+6E+2m\right)+\frac{g_A{}^2}{u-m^2}\frac{1}{\sqrt{2}}\times\left[\left(\frac{2E^2}{m}-m\right)(z-1)-E(1+z)\right],$$

$$T_{1,0}{}^+=\frac{g_A{}^2}{s-m^2}\left(\frac{4E^3}{m^2}+\frac{4E^2}{m}-E-m\right)+\frac{g_A{}^2}{u-m^2}\left[\frac{2E^3}{m^2}(1+z)-\frac{2E^2}{m}(1+z)-E(2+z)+m(2+z)\right],$$

$$T_{0,0}{}^+=\frac{g_A{}^2}{s-m^2}\frac{1}{\sqrt{2}}\left(\frac{8E^4}{m^3}+\frac{4E^3}{m^2}-\frac{6E^2}{m}-E+m\right)+\frac{g_A{}^2}{u-m^2}\frac{1}{\sqrt{2}}\left[\frac{4E^4}{m^3}(1+z)-\frac{6E^3}{m^2}(1+z)-\frac{2E^2}{m}(z+4)+E(5+4z)+3m\right],\quad(10)$$

$$T_{-1,0}{}^+=\frac{-g_A{}^2}{u-m^2}\left(\frac{2E^2}{m}-E-m\right),$$

$$T_{-1,1}{}^+=\frac{-g_A{}^2}{u-m^2}\frac{1}{\sqrt{2}}(E-m)\,,$$

$$T_{-1,-1}{}^+=\frac{-g_A{}^2}{u-m^2}\frac{1}{\sqrt{2}}\left(\frac{2E^2}{m}+E-3m\right).$$

Expanding (10) for large z,

$$T_{1,1}{}^+=\frac{-g_A{}^2}{2\sqrt{2}k^2}\frac{1}{2E-m}(-8E^2+3Em+5m^2)+O(z^{-1})\,,$$

$$T_{1,0}{}^+=\frac{-g_A{}^2}{2k^2}\frac{1}{m}(-4E^2+Em+3m^2)+O(z^{-1})\,,$$

$$T_{0,0}{}^+=\frac{-g_A{}^2}{2\sqrt{2}k^2}\frac{1}{m^2}(-8E^3+4E^2m+6Em^2-2m^3)+O(z^{-1})\,,\quad(11)$$

$$T_{-1,0}{}^+=\frac{g_A{}^2}{2k^2}\frac{1}{m}(2E^2-Em-m^2)\frac{1}{z}+O(z^{-2})\,,$$

$$T_{-1,1}{}^+=\frac{g_A{}^2}{2\sqrt{2}k^2}(E-m)\frac{1}{z}+O(z^{-2})\,,$$

$$T_{-1,-1}{}^+=\frac{g_A{}^2}{2\sqrt{2}k^2}\frac{1}{m}(2E^2+Em-3m^2)\frac{1}{z}+O(z^{-2})\,.$$

These do not factor and further $T_{0,0}{}^+$ goes as E for large E, and thus $(t_{0,0}{}^+)^{j=1/2}$ violates the unitarity bound. The cancellation of the high powers of E between the s- and u-channel contributions that occurred in the vector γ^μ case only partially occurs in the $2m$ terms and unitarity is violated. The higher partial waves ($j\geq\frac{1}{2}$), however, are consistent with unitarity; they come only from the u channel and therefore do not depend on a cancellation.

The amplitudes would be consistent with unitarity if we could get rid of the $2m$ terms in (9). We can accomplish this by considering the axial vector and the spinor to have different masses and putting the spinor mass equal to zero. Then (9) is the same as the γ^μ case found from GGLMZ and MacDowell symmetry,

$$T_{1,1}{}^{+}=\frac{g_A{}^2}{2\sqrt{2}k^2}\frac{E}{E+\omega}(E-\omega)^2+O(z^{-1})\,,$$

$$T_{1,0}{}^{+}=\frac{g_A{}^2}{2k^2}\frac{E}{E+\omega}\lambda(E-\omega)+O(z^{-1})\,,$$

$$T_{0,0}{}^{+}=\frac{g_A{}^2}{\sqrt{2}k^2}\frac{E}{E+\omega}\lambda^2+O(z^{-1})\,,\tag{12}$$

$$T_{-1,0}{}^{+}=\frac{-g_A{}^2}{2k^2}E\lambda\frac{1}{z}+O(z^{-2})\,,$$

$$T_{-1,1}{}^{+}=\frac{-g_A{}^2}{2\sqrt{2}k^2}E(E-\omega)\frac{1}{z}+O(z^{-2})\,,$$

$$T_{-1,-1}{}^{+}=\frac{g_A{}^2}{2\sqrt{2}k^2}E(E+\omega)\frac{1}{z}+O(z^{-2})\,,$$

where ω and λ are the axial-vector energy and mass. The $\gamma^\mu\gamma^5$ amplitudes do factor when $m=0$. It is not true, however, that this factorization is because the axial-vector current is conserved and the amplitude is gauge invariant. We have examples of gauge-invariant couplings where the amplitudes do not factor ($\sigma^{\mu\nu}k_\nu$) and of couplings where the amplitudes are not gauge invariant but do factor (see below).

If the spinor mass is *not* equal to zero, Eqs. (11) do not factor and thus the elementary particle does not turn into one Regge pole. The determinant of (11) is zero, however, indicating that the particle turns into two (not three) trajectories, one of which chooses sense at $\alpha=0$ and one of which chooses nonsense. We can write these two Regge poles as

$$t_{\kappa,\nu}{}^{l+}=\frac{\eta_\kappa(E)\eta_\nu(E)}{l-\alpha(E)}+\frac{2\alpha(E)\beta_\kappa(E)\beta_\nu(E)}{l-\alpha(E)}\,,$$

$$\frac{t_{-1,\nu}{}^{l+}}{[l(l+2)]^{1/2}}=\frac{\zeta_{-1}(E)\eta_\nu(E)}{l-\alpha(E)}+\frac{\psi_{-1}(E)\beta_\nu(E)}{l-\alpha(E)}\,,\tag{13}$$

$$t_{-1,-1}{}^{l+}=\frac{2\alpha(E)[\zeta_{-1}(E)]^2}{l-\alpha(E)}+\frac{[\psi_{-1}(E)]^2}{l-\alpha(E)}\,,$$

where κ and ν equal 1 or 0, and where $\eta_{k,\nu}$ and ζ_{-1} are the residues for the nonsense-choosing trajectory. Then from (11) we find[9]

$$\frac{\eta_1}{\alpha^{1/2}}=g_A\left(\frac{E-m}{2\sqrt{2}k^2}\right)^{1/2}\left(\frac{9m}{2E-m}\right)^{1/2},$$

$$\frac{\eta_0}{\alpha^{1/2}}=g_A\left(\frac{E-m}{2\sqrt{2}k^2}\right)^{1/2}\left(\frac{2}{m}(2E-m)\right)^{1/2},\tag{14}$$

$$\alpha^{1/2}\zeta_{-1}=-g_A\left(\frac{E-m}{2\sqrt{2}k^2}\right)^{1/2}\left(\frac{2E-m}{2m}\right)^{1/2},$$

and

$$\beta_1=g_A\left(\frac{E-m}{\sqrt{2}k^2}\right)^{1/2},$$

$$\beta_0=g_A\sqrt{2}\frac{E}{m}\left(\frac{E-m}{\sqrt{2}k^2}\right)^{1/2},\tag{15}$$

$$\psi_{-1}=g_A\sqrt{2}\left(\frac{E-m}{\sqrt{2}k^2}\right)^{1/2}.$$

Now we can try to add isotopic spin to this theory. If the axial vector has isospin 1 and the spinor has isospin $\frac{1}{2}$, then the s-channel contributions in (10) are multiplied by $\tau_b\tau_a$ and the u-channel contributions in (10) are multiplied by $\tau_a\tau_b$, where the τ are the usual $SU(2)$ generators and satisfy $\tau_a\tau_b+\tau_b\tau_a=2\delta_{ab}$. If we assume there is a vector in the theory (we assume it couples to the spinor by γ^μ), then there is an additional contribution to the amplitude from exchanging the vector in the t channel. The vector–axial-vector–axial-vector coupling must be

$$g_1\epsilon_1\cdot\epsilon_2(k_1+k_2)^\mu+g_2(k_1\cdot\epsilon_2\epsilon_1{}^\mu+k_2\cdot\epsilon_1\epsilon_2{}^\mu)\,,\tag{16}$$

where g_1 and g_2 are here arbitrary coupling constants.

The t-channel contribution is $(\tau_b\tau_a-\tau_a\tau_b)T^t$, where T^t is given by

$$T^t=(t-m^2)^{-1}[g_1g_V2\epsilon_1\cdot\epsilon_2T_4+g_2g_V(k_1\cdot\epsilon_2T_2+k_2\cdot\epsilon_1T_3)]\,.\tag{17}$$

The helicity amplitudes are

$$T_{1,1}{}^{t+}=\frac{g_V}{t-m^2}\frac{1}{\sqrt{2}}\left\{g_1\left[\left(m-\frac{2E^2}{m}\right)(1+z)-E(1+z)\right]+g_22\left(\frac{E^2}{m}-m\right)(1-z)\right\},$$

$$T_{1,0}{}^{t+}=\frac{g_V}{t-m^2}\left\{g_1\left[\left(-\frac{2E^3}{m^2}+E\right)(1+z)-\frac{E^2}{m}(1-z)\right]+g_2\left[\left(E-\frac{E^3}{m^2}\right)(1+z)+\left(\frac{E^3}{m^2}-\frac{E^2}{2m}-E+\frac{m}{2}\right)(1-z)\right]\right\},$$

[9] It is interesting to notice that this solution does not satisfy the conditions for nontrivial evasion. See E. Abers, M. Cassandro, I. Muzinich, and V. Teplitz, Phys. Rev. **170**, 1331 (1968).

$$T_{0,0}{}^{t+}=\frac{g_V}{t-m^2}\frac{1}{\sqrt{2}}\left\{g_1\left[\frac{4E^4}{m^3}(1-z)-\frac{2E^3}{m^2}(1-z)-\frac{2E^2}{m}(3-z)+2E\right]+g_2(1-z)\times\left[\frac{4E^4}{m^3}-\frac{2E^3}{m^2}-\frac{4E^2}{m}+2E\right]\right\},$$

$$T_{-1,0}{}^{t+}=\frac{g_V}{t-m^2}\left[g_1\left(-\frac{2E^3}{m^2}+\frac{E^2}{m}+E\right)+g_2\left(-\frac{E^3}{m^2}+\frac{E^2}{2m}+E-\frac{m}{2}\right)\right],$$

$$T_{-1,1}{}^{t+}=\frac{g_V}{t-m^2}\frac{1}{\sqrt{2}}\left\{g_1\left[-\frac{2E^2}{m}+E+m\right]+g_2\left[-\frac{E^2}{m}+m\right]\right\},$$

$$T_{-1,-1}{}^{t+}=\frac{g_V}{t-m^2}\frac{g_1}{\sqrt{2}}\left[-\frac{2E^2}{m}+E+m\right].\quad(18)$$

To make the higher partial waves consistent with unitarity, we need to take $g_2/g_1=-2$ (see $T_{1,0}$ and $T_{-1,0}$).

Further, by comparing (18) with (10), each multiplied by the proper isospin factors, we see that the first partial wave comes closest to being consistent with unitarity when $(g_1+g_2)g_V=g_A{}^2$. Thus we take $g_2g_V=2g_A{}^2$ and $g_1g_V=-g_A{}^2$. This is an interesting result because it means that the vector–axial-vector–axial-vector interaction derived by considering unitarity is the same, except for an over-all factor of g_A/g_V, as the vector-vector-vector interaction required by gauge invariance in spinor-vector scattering.[7]

Expanding (18) for large z, we have

$$T_{1,1}{}^{t+}=\frac{g_A{}^2}{2\sqrt{2}k^2}\left(-\frac{2E^2}{m}-E+3m\right)+O(z^{-1}),$$

$$T_{1,0}{}^{t+}=\frac{g_A{}^2}{2k^2}\left(-\frac{2E^3}{m^2}+3E-m\right)+O(z^{-1}),$$

$$T_{0,0}{}^{t+}=\frac{g_A{}^2}{\sqrt{2}k^2}\left(-\frac{2E^4}{m^3}+\frac{E^3}{m^2}+\frac{3E^2}{m}-2E\right)+O(z^{-1}),\quad(19)$$

$$T_{-1,0}{}^{t+}=\frac{g_A{}^2}{2k^2}(E-m)\frac{1}{z}+O(z^{-2}),$$

$$T_{-1,1}{}^{t+}=\frac{-g_A{}^2}{2\sqrt{2}k^2}(E-m)\frac{1}{z}+O(z^{-2}),$$

$$T_{-1,-1}{}^{t+}=\frac{g_A{}^2}{2\sqrt{2}k^2}\left(\frac{2E^2}{m}-E-m\right)\frac{1}{z}+O(z^{-2}).$$

The total amplitude has both $I=\frac{1}{2}$ and $I=\frac{3}{2}$ parts. If we call the s-channel contribution $\tau_b\tau_aT^s$ and the u-channel contribution $\tau_a\tau_bT^u$, then the $I=\frac{1}{2}$ part is

$$3T^s-T^u+4T^t$$

and the $I=\frac{3}{2}$ part is

$$2T^u-2T^t.$$

Then from (10) and (18) or (19) we find that the $I=\frac{3}{2}$ amplitudes are

$$T_{1,1}{}^{3/2}=g_A{}^2\frac{\sqrt{2}}{k^2}(E-m)+O(z^{-1}),$$

$$T_{1,0}{}^{3/2}=g_A{}^2\frac{2}{k^2}(E-m)\frac{E}{m}+O(z^{-1}),$$

$$T_{0,0}{}^{3/2}=g_A{}^2\frac{2\sqrt{2}}{k^2}(E-m)\frac{E^2}{m^2}+O(z^{-1}),\quad(20)$$

$$T_{-1,0}{}^{3/2}=g_A{}^2\frac{2}{k^2}(E-m)\frac{E}{m}\frac{1}{z}+O(z^{-2}),$$

$$T_{-1,1}{}^{3/2}=g_A{}^2\frac{\sqrt{2}}{k^2}(E-m)\frac{1}{z}+O(z^{-2}),$$

$$T_{-1,-1}{}^{3/2}=g_A{}^2\frac{\sqrt{2}}{k^2}(E-m)\frac{1}{z}+O(z^{-2}).$$

These amplitudes factor giving a nonsense-choosing trajectory with residues precisely equal to (15). The $I=\frac{1}{2}$ amplitudes are

$$T_{1,1}{}^{1/2}=\frac{g_A{}^2}{2\sqrt{2}k^2}\frac{1}{2E-m}(E-m)(8E+23m)+O(z^{-1}),$$

$$T_{1,0}{}^{1/2}=\frac{g_A{}^2}{2k^2}\frac{1}{m}(E-m)(4E+9m)+O(z^{-1}),$$

$$T_{0,0}{}^{1/2}=\frac{g_A{}^2}{2\sqrt{2}k^2}\frac{1}{m^2}(E-m)(8E^2+12Em-6m^2)+O(z^{-1}),\quad(21)$$

$$T_{-1,0}{}^{1/2}=\frac{g_A{}^2}{2k^2}\frac{1}{m}(E-m)(-2E+3m)\frac{1}{z}+O(z^{-2}),$$

$$T_{-1,1}{}^{1/2}=\frac{-g_A{}^2}{2\sqrt{2}k^2}5(E-m)\frac{1}{z}+O(z^{-2}),$$

$$T_{-1,-1}{}^{1/2}=\frac{g_A{}^2}{2\sqrt{2}k^2}\frac{1}{m}(E-m)(6E+m)\frac{1}{z}+O(z^{-2}).$$

These do not factor but, as in the case without isotopic spin, the determinant of the amplitudes is zero indicating that we again have two trajectories—not three. Here the nonsense-choosing trajectory has residues equal to (15) while the sense-choosing trajectory has residues equal to $\sqrt{3}$ times the residues given by (14).

This, together with Ref. 7, leads us to speculate that the addition of isospin in a Yang-Mills theory does not change the factorization properties of the theory.

If factorization is used as a basis for choosing couplings, then Ref. 7 shows that the vector-vector-vector interaction (in the presence of isospin) should be

$$-i\epsilon_{ijk}g_V[\epsilon_1\cdot\epsilon_2(k_1+k_2)\cdot\epsilon_3 -2\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_3-2\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_3], \quad (22)$$

where $g_V\gamma^\mu\tau_k/2$ is the vector-spinor coupling, and k_1 and k_2 are the momenta of vector No. 1 incoming and vector No. 2 outgoing, respectively. The indices i, j, and k give the isospin of particles 1, 2, and 3. Similarly we have shown here that with isospin the vector–axial-vector–axial-vector coupling should be

$$-i\epsilon_{ijk}(g_A{}^2/g_V)[\epsilon_1\cdot\epsilon_2(k_1+k_2)\cdot\epsilon_V -2\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_V-2\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_V] \quad (23)$$

if $g_A\gamma^\mu\gamma^5\tau_k/2$ is the axial-vector–spinor coupling. This is the same as the vector-vector-vector coupling except for the over-all factor g_A/g_V.

IV. OTHER VECTOR AND AXIAL-VECTOR COUPLINGS

A. Vector γ^μ Coupling with $m_1\neq m_2$

Consider the diagrams of Fig. 1 but where all the external particles have mass m_1 while the intermediate particle has mass m_2. This case is the same as the axial-vector $\gamma^\mu\gamma^5$ coupling except that the pole is at m_2 and the $2m$ in (9) is replaced by m_1-m_2. It does not factor and $T_{0,0}$ violates unitarity. Of course it is not gauge invariant.

B. Vector with $g_V(i\sigma^{\mu\nu}/m)k_\nu$ Coupling

The scattering amplitude is

$$T(\sigma^{\mu\nu}k_\nu)=\frac{g_V{}^2}{s-m^2}\left(\frac{2\epsilon_1\cdot p_1}{m^2}T_{14}+\frac{2}{m}T_{15}\right) +\frac{g_V{}^2}{u-m^2}\left(\frac{2\epsilon_2\cdot p_1}{m^2}T_{13}+\frac{2}{m}T_{16}\right)+\frac{g_V{}^2}{m^2}(T_8-T_{12}), \quad (24)$$

which gives the helicity amplitudes

$$T_{1,1}{}^+=\frac{g_V{}^2}{2\sqrt{2}k^2}\frac{1}{m^3}\frac{1}{2E-m}(-16E^5-8E^4m+36E^3m^2 +8E^2m^3-21Em^4-m^5)+O(z^{-1}),$$

$$T_{1,0}{}^+=\frac{-g_V{}^2}{2k^2}\frac{1}{m^2}\frac{1}{2E-m}\times(4E^4-5E^2m^2+Em^3+2m^4)+O(z^{-1}),$$

$$T_{0,0}{}^+=\frac{-g_V{}^2}{\sqrt{2}k^2}\frac{1}{m}\frac{1}{2E-m}E^2(E-m)+O(z^{-1}),$$

$$T_{-1,0}{}^+=\frac{-g_V{}^2}{2k^2}\frac{1}{m^2}\times(2E^3-E^2m-3Em^2+2m^3)\frac{1}{z}+O(z^{-2}),$$

$$T_{-1,1}{}^+=\frac{-g_V{}^2}{2\sqrt{2}k^2}\frac{1}{m^2}\times(4E^3+2E^2m-5Em^2-m^3)\frac{1}{z}+O(z^{-2}),$$

$$T_{-1,-1}{}^+=\frac{-g_V{}^2}{2\sqrt{2}k^2}\frac{1}{m^3}\times(-8E^4+10E^2m^2-Em^3-m^4)\frac{1}{z}+O(z^{-2}). \quad (25)$$

The sense-sense amplitudes factor at the pole $2E=m$ as they must but do not factor along the trajectory as we can see by setting $E=0$. In addition there is no cancellation of large powers of E of the type we saw in the vector γ^μ case. Thus none of the helicity amplitudes, except $T_{0,0}$, are consistent with unitarity. The higher-order partial waves are only consistent with unitarity for the sense-sense amplitudes.

The determinant of the amplitudes in (25) is not zero (again this can be easily checked by putting $E=0$). Thus this coupling does not satisfy a two-trajectory theory of the type we found for $\gamma^\mu\gamma^5$.

C. Axial Vector with $(-ig_A\sigma^{\mu\nu}/m)k_\nu\gamma^5$

This coupling is manifestly gauge invariant and so cannot be made to satisfy PCAC. It differs from the vector $\sigma^{\mu\nu}k_\nu$ by terms proportional to $2m$,

$$T(\sigma^{\mu\nu}k_\nu\gamma^5)=-T(\sigma^{\mu\nu}k_\nu) +\frac{2}{m}\frac{g_A{}^2}{s-m^2}T_{15}+\frac{2}{m}\frac{g_A{}^2}{u-m^2}T_{16}. \quad (26)$$

As was the case for $T(\sigma^{\mu\nu}k_\nu)$, the amplitudes are not consistent with unitarity and do not factor. Further, the higher partial waves are not consistent with unitarity for the sense-nonsense and nonsense-nonsense amplitudes. Also the determinant of the amplitudes is not zero.

D. Vector Coupling $g_V P^\mu$

The amplitude is

$$T(P^\mu)=g_V^2\frac{\epsilon_1\cdot(2p_1+k_1)\epsilon_2\cdot(2p_2+k_2)}{s-m^2}(2mT_1+T_4)+g_V^2\frac{\epsilon_1\cdot(2p_2-k_1)\epsilon_2\cdot(2p_1-k_2)}{u-m^2}(2mT_1-T_4). \quad (27)$$

A new feature enters here. The helicity amplitudes, expanded for large z, have terms linear in z in the sense-sense amplitudes and terms which go as a constant in the sense-nonsense and nonsense-nonsense amplitudes. These indicate the presence of an ancestor at $l\simeq 1$. [$\delta_{l,1}\approx(1-\alpha_1)/(l-\alpha_1)$, and $\delta_{l,0}\approx-\alpha_0/(l-\alpha_0)$.] Using Eq. (A5) of the Appendix for the partial-wave projections,[10] we find

$$t_{1,1}{}^{l+}=\frac{g_V^2}{6}\left(\frac{2E^2}{m}-E-3m\right)\delta_{l,1}+\frac{g_V^2}{12}\frac{1}{m}\frac{1}{E-m}\times(-8E^3+16E^2m-2Em^2-3m^3)\delta_{l,0},$$

$$t_{1,0}{}^{l+}=\frac{-g_V^2}{3\sqrt{2}}\frac{E}{m}\left(\frac{2E^2}{m}-E-3m\right)\delta_{l,1}+\frac{g_V^2}{6\sqrt{2}}\frac{E}{m^2}\frac{1}{E-m}\times(-8E^3+4E^2m+22Em^2-21m^3)\delta_{l,0},$$

$$t_{0,0}{}^{l+}=\frac{g_V^2}{3}\frac{E^2}{m^2}\left(\frac{2E^2}{m}-E-3m\right)\delta_{l,1}+\frac{g_V^2}{3}\frac{1}{2E-m}\frac{E^2}{m^3}\frac{\delta_{l,0}}{E-m}(52E^4-52E^3m-73E^2m^2+82Em^3-6m^4), \quad (28)$$

$$\frac{t_{-1,0}{}^{l+}}{l^{1/2}}=-\frac{g_V^2}{\sqrt{6}}\frac{E}{m}\left(\frac{2E^2}{m}-E-3m\right)\delta_{l,1}-\frac{g_V^2}{8}\frac{E}{E-m}(2E-3m)Q_{l-1},$$

$$\frac{t_{-1,1}{}^{l+}}{l^{1/2}}=\frac{g_V^2}{2\sqrt{3}}\left(\frac{2E^2}{m}-E-3m\right)\delta_{l,1}-\frac{g_V^2}{8\sqrt{2}}\frac{1}{E-m}(4E^2-6Em+3m^2)Q_{l-1},$$

$$t_{-1,-1}{}^{l+}=\frac{g_V^2}{2}\left(\frac{2E^2}{m}-E-3m\right)\delta_{l,1}+\frac{g_V^2}{8}\frac{1}{m}\frac{1}{E-m}\times(8E^3-8E^2m-10Em^2+9m^3)Q_{l-1},$$

where

$$Q_{l-1}(z_0)=-\tfrac{1}{2}\int_{-1}^{1}dz\frac{P_{l-1}(z)}{z-z_0}. \quad (29)$$

These partial waves and all the higher partial waves violate the unitarity bound badly. Nevertheless we see from (28) the $l=1$ ancestor does factor. The $l\approx 0$ amplitudes do not factor nor is their determinant zero.

Of course the P^μ coupling is not gauge invariant. If we add a seagull to make it gauge invariant then, as we discussed before, we have simply a $-i\sigma^{\mu\nu}k_\nu+2m\gamma^\mu$ coupling. The seagull can be written in several equivalent ways,

$$\begin{aligned}&g_V^2[-4m\epsilon_1\cdot\epsilon_2\bar{u}u-2\epsilon_2\cdot(p_1+p_2)\bar{u}\gamma\cdot\epsilon_1u+\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_1\gamma\cdot\epsilon_1u-\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_1\gamma\cdot\epsilon_2u]\\&=g_V^2[-4m\epsilon_1\cdot\epsilon_2\bar{u}u-2\epsilon_1\cdot(p_1+p_2)\bar{u}\gamma\cdot\epsilon_2u+\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_2\gamma\cdot\epsilon_1u-\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_2\gamma\cdot\epsilon_2u]\\&=g_V^2[-4m\epsilon_1\cdot\epsilon_2\bar{u}u-2\epsilon_1\cdot p_1\bar{u}\gamma\cdot\epsilon_2u-2\epsilon_2\cdot p_1\bar{u}\gamma\cdot\epsilon_1u+\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_2\gamma\cdot\epsilon_1u-\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_1\gamma\cdot\epsilon_2u].\end{aligned} \quad (30)$$

The first form is convenient to use when we are checking gauge invariance by replacing $\epsilon_1{}^\nu$ by $k_1{}^\nu$ and the second form is convenient when we replace $\epsilon_2{}^\mu$ by $k_2{}^\mu$. The third form is most convenient for evaluating the helicity amplitudes. Writing it in terms of the T_i,

$$g_V^2[-4m\epsilon_1\cdot\epsilon_2T_1-2\epsilon_1\cdot p_1T_3-2\epsilon_2\cdot p_1T_2+T_{12}-T_8]. \quad (31)$$

The total gauge-invariant amplitude is (27) plus (31). This does not have an $l=1$ ancestor. The $l\approx 0$ terms are very messy and do not factor.

At this point, one may conjecture that the role of gauge invariance is in eliminating ancestors from the amplitudes. This turns out to be true in all the cases, without zero-mass particles, that we have checked. With the freedom to add zero-mass particles, any theory may be made gauge invariant (in the sense of the substitution $\epsilon_\mu\to k_\mu$ yielding zero) without altering the original scattering amplitude.

E. Axial Vector with $g_A P^\mu\gamma^5$ Coupling

The amplitude is

$$T(P^\mu\gamma^5)=-g_A^2\frac{\epsilon_1\cdot(2p_1+k_1)\epsilon_2\cdot(2p_2+k_2)}{s-m^2}T_4+g_A^2\frac{\epsilon_1\cdot(2p_2-k_1)\epsilon_2\cdot(2p_1-k_2)}{u-m^2}T_4. \quad (32)$$

Like the vector P^μ case the helicity amplitudes have an $l\sim 1$ ancestor which factors. The $l\sim 0$ partial wave does not factor and neither the $l\sim 1$ nor the $l\sim 0$ amplitudes are consistent with unitarity.

This theory does not satisfy PCAC nor is it gauge invariant. It can be made gauge invariant by adding the seagull which makes it equal to the amplitude with

[10] The equations for the partial-wave projections in GGLMZ have two crucial misprints. In Eq. (2.16) the coefficient of P_{l-1} should be $l+2$ instead of $l+1$ and in Eq. (A13) the coefficient of the operator Δ_- should be $L(L+1+2\lambda)^2/(2L+1)$.

$\sigma^{\mu\nu}k_\nu\gamma^5$ coupling. This again shows that if two sets of amplitudes differ by a seagull, then the set which has PCAC or gauge invariance is the one without ancestors.

The amplitudes could also be made gauge invariant and thus the ancestor could be removed by adding to (32) the interaction

$$2g_A{}^2\frac{\epsilon_1\cdot(p_1+p_2)\epsilon_2\cdot(p_1+p_2)}{k_1\cdot(p_1+p_2)}T_4. \quad (33)$$

The resulting amplitudes at $l\sim0$ and all higher partial waves violate the unitarity bound, yet the amplitudes at $l\sim0$ do factor. This probably has no meaning. It is not clear what kind of an interaction (33) represents. (There is a pole at $s=u$.) Further, the term (33) plus an $\epsilon_1\cdot\epsilon_2$ seagull also makes the amplitudes with P^μ coupling gauge invariant, but they do not factor.

F. Vector with $g_V k^\mu$ or $-g_V(i/m)\sigma^{\mu\nu}P_\nu$ Coupling

Because $\epsilon_1\cdot k_1=\epsilon_2\cdot k_2=0$, the amplitudes with k^μ coupling is zero. Then, because of the relationship between k^μ and $\sigma^{\mu\nu}P_\nu$, the amplitude with $\sigma^{\mu\nu}P_\nu$ can only be a polynomial. We can explicitly check that it is given by

$$T\left(-\frac{i}{m}\sigma^{\mu\nu}P_\nu\right)=-\frac{g_V{}^2}{m^2}[2\epsilon_2\cdot p_2T_2+2\epsilon_1\cdot p_2T_3 -4m\epsilon_1\cdot\epsilon_2T_1+T_{12}-T_8]. \quad (34)$$

The amplitude has an $l\sim1$ part which does not factor and an $l\sim0$ part which does not factor either. Of course it is not gauge invariant.

G. Axial Vector with $(-i/m)\sigma^{\mu\nu}P_\nu\gamma^5$ Coupling

Since the $k^\mu\gamma^5$ coupling gives zero, the amplitude with $(-i/m)\sigma^{\mu\nu}P_\nu\gamma^5$ can only differ from the amplitude with $\gamma^\mu\gamma^5$ coupling by a polynomial. We find

$$\begin{aligned}T\left(\frac{-i}{m}\sigma^{\mu\nu}P_\nu\gamma^5\right)=&-\frac{12g_A{}^2}{m}\epsilon_1\cdot\epsilon_2T_1\\&+\frac{g_A{}^2}{m^2}[2\epsilon_2\cdot p_2T_2+2\epsilon_1\cdot p_2T_3+T_{12}-T_8]\\&+\frac{4g_A{}^2}{s-m^2}[-4m\epsilon_1\cdot\epsilon_2T_1+2\epsilon_2\cdot p_2T_2+T_{12}+2mT_5]\\&+\frac{4g_A{}^2}{u-m^2}[2\epsilon_1\cdot p_2T_3-T_8-2mT_5]. \quad (35)\end{aligned}$$

This has an ancestor at $l\sim1$ due to the polynomial. Neither the ancestor nor the $l\sim0$ part factors and both violate the unitarity bound. Also this does not satisfy PCAC. To obtain PCAC one would drop the polynomial and have the $\gamma^\mu\gamma^5$ amplitudes.

V. PSEUDOSCALAR-VECTOR SCATTERING

The pseudoscalar pole in pseudoscalar-vector scattering has been considered in detail by GGLSZ. They find that the amplitudes do not factor, indicating that the scalar does not Reggeize. Since the amplitudes are unitary, Mandelstam counting can be used and shows that the scalar need not Reggeize. We now want to consider the axial-vector pole in pseudoscalar-vector scattering (or equivalently the vector pole in scalar-vector scattering). The relevant diagrams are shown in Fig. 2. We can couple a pseudoscalar of momentum p, a vector of momentum k and polarization ϵ^V, and an axial vector of momentum $p+k$ and polarization ϵ^A in two independent ways; either $g_{AVP}\epsilon^V\cdot\epsilon^A$ or $h_{AVP}\epsilon^V\cdot p\epsilon^A\cdot k$. Using $g_{AVP}\epsilon^V\cdot\epsilon^A$ in Fig. 2, we get

$$\begin{aligned}T(\epsilon^V\cdot\epsilon^A)=&\frac{g_{AVP}{}^2}{s-m^2}\Big[\epsilon_1{}^V\cdot\epsilon_2{}^V-\frac{1}{m^2}\epsilon_1{}^V\cdot(p_1+k_1)\epsilon_2{}^V\cdot(p_2+k_2)\Big]\\&+\frac{g_{AVP}{}^2}{u-m^2}\Big[\epsilon_1{}^V\cdot\epsilon_2{}^V-\frac{1}{m^2}\epsilon_1{}^V\cdot(p_2-k_1)\epsilon_2{}^V\cdot(p_1-k_2)\Big]. \quad (36)\end{aligned}$$

This amplitude is not gauge invariant and further, since the gauge-dependent part involves poles, cannot be made gauge invariant by adding a seagull. The sense-sense helicity amplitudes do factor,[11]

$$\begin{aligned}T_{1,1}{}^{+j}&=\frac{g_{AVP}{}^2}{s-m^2}\frac{4E^2-3m^2}{2m^2}\tfrac{2}{3}\delta_{j,1},\\T_{1,0}{}^{+j}&=\frac{g_{AVP}{}^2}{s-m^2}\frac{4E^2-3m^2}{2m^2}\frac{E}{m}\tfrac{2}{3}\delta_{j,1},\\T_{0,0}{}^{+j}&=\frac{g_{AVP}{}^2}{s-m^2}\frac{4E^2-3m^2}{2m^2}\frac{E^2}{m^2}\tfrac{2}{3}\delta_{j,1}.\end{aligned} \quad (37)$$

Both $T_{1,0}$ and $T_{0,0}$ violate unitarity. If a nonsense channel can be found which maintains the factorization, then this is an excellent example of a case which is not consistent with the unitarity bound but still factors. We suggest that the nonsense channel is the channel with one axial vector and two vectors. The sense-nonsense and nonsense-nonsense amplitudes then correspond to diagrams like Fig. 3. We have not computed these.

A similar situation arises in considering the Reggeiza-

[11] This factorization does not depend on particles having equal mass. The amplitudes continue to factor when the pseudoscalar, vector, and axial-vector masses are unequal.

tion of the vector in spinor-antispinor scattering.[6] In this case the sense-sense residues are of order $\delta_{l,1}$ in the coupling constant, the sense-nonsense residues are of order g^3, but the nonsense-nonsense residues are of order g^6, not g^4, because no tree diagram exists. Thus the powers of the coupling constant preclude factorization.

If we use the other coupling, $\epsilon^V\cdot p\epsilon^A\cdot k$, in the diagrams of Fig. 2, we find

$$T(\epsilon^V\cdot p\epsilon^A\cdot k)$$
$$=h_{AVP}{}^2\frac{\epsilon_1{}^V\cdot p_1\epsilon_2{}^V\cdot p_2}{s-m^2}\left[k_1\cdot k_2-\frac{1}{m^2}k_1\cdot(k_1+p_1)k_2\cdot(k_2+p_2)\right]+h_{AVP}{}^2\frac{\epsilon_1{}^V\cdot p_2\epsilon_2{}^V\cdot p_1}{u-m^2}$$
$$\times\left[k_1\cdot k_2-\frac{1}{m^2}k_1\cdot(p_2-k_1)k_2\cdot(p_1-k_2)\right]. \quad (38)$$

This is not gauge invariant. The helicity amplitudes are not unitary and have ancestors at $l\sim3$ and $l\sim2$. The ancestor at $l\sim3$ factors but the $l\sim2$ and $l\sim1$ amplitudes do not factor.

We can combine the two couplings into a gauge-invariant coupling, $f(\epsilon^V\cdot\epsilon^A k\cdot p-\epsilon^V\cdot p\epsilon^A\cdot k)$. The amplitude is

$$T(\epsilon^V\cdot\epsilon^A k\cdot p-\epsilon^V\cdot p\epsilon^A\cdot k)$$
$$=[f^2/(s-m^2)][\epsilon_1{}^V\cdot\epsilon_2{}^V k_1\cdot p_1 k_2\cdot p_2$$
$$+\epsilon_1{}^V\cdot p_1\epsilon_2{}^V\cdot p_2 p_2\cdot k_2-\epsilon_1{}^V\cdot p_1\epsilon_2{}^V\cdot p_2 p_1\cdot k_2$$
$$-\epsilon_1{}^V\cdot k_2\epsilon_2{}^V\cdot p_2 k_1\cdot p_1-\epsilon_1{}^V\cdot p_1\epsilon_2{}^V\cdot k_1 p_2\cdot k_2]$$
$$+[f^2/(u-m^2)][\epsilon_1{}^V\cdot\epsilon_2{}^V k_1\cdot p_2 k_2\cdot p_1$$
$$+\epsilon_1{}^V\cdot p_2\epsilon_2{}^V\cdot p_1 k_1\cdot k_2-\epsilon_1{}^V\cdot p_2\epsilon_2{}^V\cdot k_1 p_1\cdot k_2$$
$$-\epsilon_1{}^V\cdot k_2\epsilon_2{}^V\cdot p_1 p_2\cdot k_1-m^2\epsilon_1{}^V\cdot p_2\epsilon_2{}^V\cdot p_1]. \quad (39)$$

The resulting helicity amplitudes do not factor and violate unitarity, but the ancestors present when we considered the coupling $\epsilon^V\cdot p\epsilon^A\cdot k$ alone have been canceled. This shows once again that the role of gauge invariance is to eliminate ancestors. This can also be seen by considering the scalar pole in scalar-vector scattering and working in the $\epsilon\cdot k=0$ gauge rather than the GGLSZ gauge. There, adding the seagull which makes the amplitude gauge invariant, cancels an ancestor which arises from the u channel, but the seagull does not help with factorization.

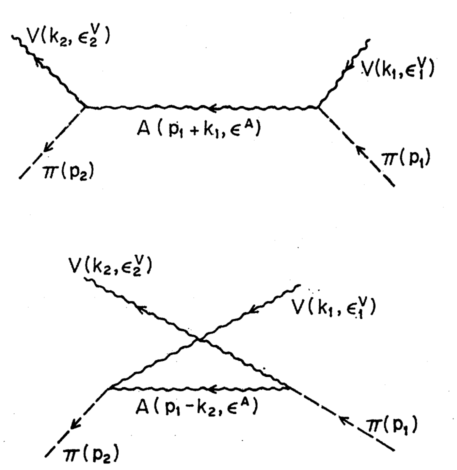

FIG. 2. Axial-vector pole in pseudoscalar-vector scattering.

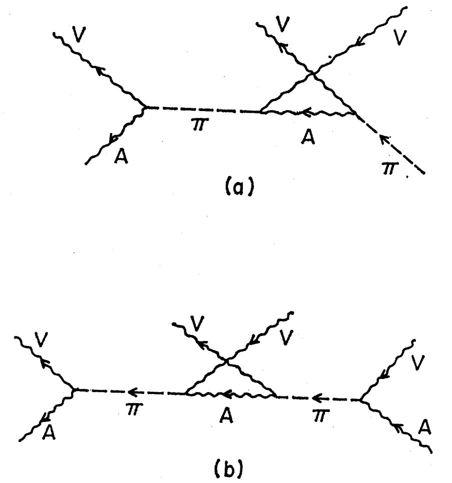

FIG. 3. (a) Example of a sense-nonsense amplitude with an axial-vector pole. (b) Example of a nonsense-nonsense amplitude with an axial-vector pole.

If we require the absence of nonsense-choosing, $l=1$, $I=0$ or 2 trajectories, factorization indicates the pseudoscalar–vector–axial-vector coupling is purely S wave,

$$g_{AVP}\epsilon^A\cdot\epsilon^V, \quad (40)$$

where g_{AVP} is arbitrary. The axial-vector-pole amplitudes in pseudoscalar-vector scattering with the coupling (40) continue to factor if the masses are unequal. Also if the pseudoscalar, vector, and axial vector are each isovectors and we add a diagram with vector exchange in the t channel, the amplitudes continue to factor as in (37). In this case g_{AVP} is no longer arbitrary but is given by

$$g_{AVP}{}^2=2m_A{}^2 f_{VPP}g_V, \quad (41)$$

where the pseudoscalar-pseudoscalar-vector coupling is $f_{VPP}\epsilon^V\cdot(p_1+p_2)i\epsilon_{ijk}$ and the vector-vector-vector vertex is given by (22). The mass of the axial-vector meson is m_A.

Thus we can use this coupling to discuss the decay $A_1\to\rho+\pi$. In A_1 decay the probability that the pion makes an angle θ with respect to the direction of the A_1 polarization is proportional to[12,13]

$$(m_V{}^2/m_A{}^2)g_T{}^2\sin^2\theta+g_L{}^2\cos^2\theta, \quad (42)$$

[12] S. Brown and G. West, Phys. Rev. **180**, 1613 (1969).
[13] J. Ballam, A. D. Brody, G. B. Chadwick, D. Fries, Z. G. T. Guiragossián, W. B. Johnson, R. R. Larsen, D. W. G. S. Leith, F. Martin, M. Perl, E. Pickup, and T. H. Tan, Phys. Rev. Letters **21**, 934 (1968).

where m_V is the vector (ρ) mass. For $m_A{}^2=2m_V$ and $m_\pi{}^2=0$, the coupling (40) predicts

$$|g_T/g_L|=\tfrac{4}{3}. \qquad (43)$$

This ratio has been measured to be 0.80 ± 0.15.[12,13] If we go further and use relation (41) with the value[14]

$$g_V g_{V\pi\pi}/4\pi=2.5\,,$$

we can calculate the total decay width for $A_1\to\rho+\pi$. We find $\Gamma(A_1\to\rho\pi)\approx1400$ MeV, which is obviously much too large; the experimental width is around 100 MeV, and even the naive current-algebra result is only 800 MeV.[15]

We do not mean to imply that the $\epsilon^V\cdot\epsilon^A$ coupling is the only one which factors. In particular, a combination of the $\epsilon^V\cdot\epsilon^A$ coupling and the $\epsilon^V\cdot\epsilon^A k\cdot p-\epsilon^V\cdot p\epsilon^A\cdot k$ coupling, namely, $\epsilon^V\cdot\epsilon^A k\cdot(p+k)-\epsilon^V\cdot(p+k)\epsilon^A\cdot k$, also factors in the absence of isospin. Both the s and u channel poles are separately gauge-invariant. If we add isospin and a vector particle in the t channel with a coupling given by (22), the resulting amplitude is not gauge invariant because the t channel pole is not gauge invariant by itself. We can recapture gauge invariance by adding a seagull—$\epsilon_1{}^V\cdot\epsilon_2{}^V$ with the same coefficient as the t channel pole. Then the seagull exactly cancels the largest power of z from the t channel pole and the $I=1$ amplitude is simply two times the s-channel pole amplitude plus the u-channel pole amplitude. This factors. The $I=0$ and $I=2$ amplitudes are proportional to the u-channel pole amplitude. This also factors, choosing nonsense. But now we cannot require that the $I=0$ and $I=2$ amplitudes be free of trajectories. Thus we have no method of determining g_{AVP} and we cannot calculate the $A_1\rho\pi$ decay width for this case.

VI. CONCLUSIONS

Except for γ^μ no vector-spinor or axial-vector–spinor coupling is consistent with unitarity. If we include all the higher partial waves, $\gamma^\mu\gamma^5$ is the only coupling which violates the unitarity bound a finite number of times. Thus it is not surprising that $\gamma^\mu\gamma^5$ is the only other coupling that comes close to Reggeizing, requiring two moving poles rather than one. Mandelstam counting for vector-spinor or axial-vector–spinor scattering counts an excess of three conditions over the number of parameters necessary to specify the theory. If the counting could be generalized to theories which are not consistent with unitarity by adding a parameter for each subtraction constant necessary to make the theory consistent with unitarity, then the $\gamma^\mu\gamma^5$ case would still have an excess of two conditions. (No other case would have an excess of conditions, even if we only count the violations in the lowest-order partial wave.) Since $\gamma^\mu\gamma^5$ coupling does not lead to amplitudes which factor, this generalization of Mandelstam counting cannot be correct. Any violation of the unitarity bound seems to invalidate the counting procedure.

The axial-vector pole in pseudoscalar-vector scattering is an example where the sense-sense amplitudes factor for one of the possible couplings but where the $j=1$ and all higher partial waves violate the unitarity bound. It would be interesting to know if factorization also holds when the sense-nonsense and nonsense-nonsense amplitudes are included.

As has been emphasized in Secs. IV and V, the role of gauge invariance seems to be to eliminate ancestors. There are many gauge-invariant couplings that do not factor and one coupling ($\epsilon^A\cdot\epsilon^V$ in the pseudoscalar–vector–axial-vector case) which factors but is not gauge invariant. The addition of isospin through a Yang-Mills theory does not seem to affect factorization if the couplings of the Yang-Mills fields are properly chosen.

ACKNOWLEDGMENTS

We would like to thank Ernest Abers, Kenneth Johnson, and Robert Keller for interesting discussions. Professor Abers calculated four of the T_i's listed in the Appendix and we thank him for sending us his results before we started this work.

APPENDIX

We study the elastic scattering of a vector or axial-vector particle of mass m with a spinor (or scalar as in

TABLE I. Values of the invariants that depend on the polarization vectors are listed for projections of the polarization vectors. The first of the two numbers at the top of each column refers to the projection of $\epsilon_2\,[\equiv\epsilon(k_2)]$ while the second number gives the projection of $\epsilon_1\,[\equiv\epsilon(k_1)]$.

	1, 1	1, 0	0, 0	−1, 0	−1, 1	−1, −1
$\epsilon_1\cdot\epsilon_2$	$-\frac{1}{2}(1+\cos\theta)$	$-2^{-1/2}(E/m)\sin\theta$	$(1/m^2)(k^2-E^2\cos\theta)$	$2^{-1/2}(E/m)\sin\theta$	$\frac{1}{2}(\cos\theta-1)$	$-\frac{1}{2}(1+\cos\theta)$
$\epsilon_1\cdot p_1$	0	$2kE/m$	$2kE/m$	$2kE/m$	0	0
$\epsilon_2\cdot p_2$	0	0	$2kE/m$	0	0	0
$\epsilon_1\cdot p_2$	$-2^{-1/2}k\sin\theta$	$k(E/m)(1+\cos\theta)$	$k(E/m)(1+\cos\theta)$	$k(E/m)(1+\cos\theta)$	$-2^{-1/2}k\sin\theta$	$2^{-1/2}k\sin\theta$
$\epsilon_2\cdot p_1$	$2^{-1/2}k\sin\theta$	$2^{-1/2}k\sin\theta$	$k(E/m)(1+\cos\theta)$	$-2^{-1/2}k\sin\theta$	$-2^{-1/2}k\sin\theta$	$-2^{-1/2}k\sin\theta$
$\epsilon_1\cdot k_2$	$2^{-1/2}k\sin\theta$	$k(E/m)(1-\cos\theta)$	$k(E/m)(1-\cos\theta)$	$k(E/m)(1-\cos\theta)$	$2^{-1/2}k\sin\theta$	$-2^{-1/2}k\sin\theta$
$\epsilon_2\cdot k_1$	$-2^{-1/2}k\sin\theta$	$-2^{-1/2}k\sin\theta$	$k(E/m)(1-\cos\theta)$	$2^{-1/2}k\sin\theta$	$2^{-1/2}k\sin\theta$	$2^{-1/2}k\sin\theta$

[14] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966).
[15] D. Geffen, Phys. Rev. Letters **19**, 770 (1967); B. Renner, Phys. Letters **21**, 453 (1966).

TABLE II. Values of the quantities T_i for various helicities. The spinor u is a function of p_1. $\bar{u}$ is a function of p_2. The polarization vectors ϵ_1 and ϵ_2 are functions of the momenta k_1 and k_2, respectively.

i	T_i	$T_{1,\frac12;\,1,\frac12}$	$T_{-1,-\frac12;\,1,\frac12}$	$T_{1,\frac12;\,0,\frac12}$	$T_{-1,-\frac12;\,0,\frac12}$	$T_{0,\frac12;\,0,\frac12}$	$T_{0,-\frac12;\,0,\frac12}$
1	$\bar{u}1u$	$\cos\frac12\theta$	$(E/m)\sin\frac12\theta$	$\cos\frac12\theta$	$(E/m)\sin\frac12\theta$	$\cos\frac12\theta$	$(E/m)\sin\frac12\theta$
2	$\bar{u}\gamma\cdot\epsilon_1 u$	$-\sqrt{2}(k/m)\sin\frac12\theta$	0	$2k(E/m^2)\cos\frac12\theta$	$(k/m)]\sin\frac12\theta$	$2E(k/m^2)\cos\frac12\theta$	$(k/m)\sin\frac12\theta$
3	$\bar{u}\gamma\cdot\epsilon_2 u$	$\sqrt{2}(k/m)\sin\frac12\theta$	0	$\sqrt{2}(k/m)\sin\frac12\theta$	0	$2E(k/m^2)\cos\frac12\theta$	$(k/m)\sin\frac12\theta$
4	$\bar{u}\gamma\cdot k_1 u$	$[2(E^2/m)-m]\cos\frac12\theta$	$E\sin\frac12\theta$	$[2(E^2/m)-m]\cos\frac12\theta$	$E\sin\frac12\theta$	$[2(E^2/m)-m]\cos\frac12\theta$	$\sin\frac12\theta$
5	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot\epsilon_2 u$	$(1-\cos\theta)\cos\frac12\theta$	$(E/m)(1+\cos\theta)\sin\frac12\theta$	$-\sqrt{2}(E/m)\cos\theta\sin\frac12\theta$	$-\sqrt{2}(E^2/m^2)(1+\cos\theta)$ $\times\cos\frac12\theta+\sqrt{2}\cos\frac12\theta$	$2(E^2/m^2)(1-\cos\theta)$ $\times\cos\frac12\theta-\cos\frac12\theta$	$-2(E^3/m^3)(1+\cos\theta)$ $\times\sin\frac12\theta+(E/m)\sin\frac12\theta$
6	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_1 u$	$\sqrt{2}k\sin\frac12\theta$	$-\sqrt{2}2k(E/m)\cos\frac12\theta$	0	$-k\sin\frac12\theta$	0	$-k\sin\frac12\theta$
7	$\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_1 u$	$-\sqrt{2}k\sin\frac12\theta$	$-\sqrt{2}2k(E/m)\cos\frac12\theta$	$-\sqrt{2}k\sin\frac12\theta$	$\sqrt{2}2E(k/m)\cos\frac12\theta$	0	$k[4(E^2/m^2)-1]\sin\frac12\theta$
8	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_1\gamma\cdot\epsilon_2 u$	$-m(1-\cos\theta)\cos\frac12\theta$	$-E(1+\cos\theta)\sin\frac12\theta$	$\sqrt{2}E\cos\theta\sin\frac12\theta$	$\sqrt{2}m\cos\theta\cos\frac12\theta$	$[2(E^2/m)\cos\theta-m]$ $\times\cos\frac12\theta$	$E(1+2\cos\theta)\sin\frac12\theta$
9	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_2 u$	$\sqrt{2}k\cos\theta\sin\frac12\theta$	$-\sqrt{2}k(E/m)(1+\cos\theta)$ $\times\cos\frac12\theta$	$2k(E/m)\sin\theta\sin\frac12\theta$	$-2k(E^2/m^2)(1+\cos\theta)$ $\times\sin\frac12\theta+k\sin\frac12\theta$	$2k(E/m)\sin\theta\sin\frac12\theta$	$-2E^2(k/m)(1+\cos\theta)$ $\times\sin\frac12\theta+k\sin\frac12\theta$
10	$\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_2 u$	$\sqrt{2}k\sin\frac12\theta$	$\sqrt{2}2k(E/m)\cos\frac12\theta$	$\sqrt{2}k\sin\frac12\theta$	$\sqrt{2}2k(E/m)\cos\frac12\theta$	0	$k\sin\frac12\theta$
11	$\bar{u}\gamma\cdot k_1\gamma\cdot k_2 u$	$2k^2\sin\theta\sin\frac12\theta$ $+m^2\cos\frac12\theta$	$-2E(k^2/m)\sin\theta\cos\frac12\theta$ $+Em\sin\frac12\theta$	$2k^2\sin\theta\sin\frac12\theta$ $+m^2\cos\frac12\theta$	$-2E(k^2/m)\sin\theta\cos\frac12\theta$ $+mE\sin\frac12\theta$	$2k^2\sin\theta\sin\frac12\theta$ $+m^2\cos\frac12\theta$	$-2E(k^2/m)\sin\theta\cos\frac12\theta$ $+Em\sin\frac12\theta$
12	$\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_2\gamma\cdot\epsilon_1 u$	$[4(E^2/m)-2m]\cos\frac12\theta$	$2E\sin\frac12\theta$	$\sqrt{2}[4(E^3/m^2)-3E]$ $\times\sin\frac12\theta$	$-\sqrt{2}m\cos\frac12\theta$	$(2E^2/m-m)\cos\frac12\theta$	$-E\sin\frac12\theta$
13	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_1\gamma\cdot k_2 u$	$-\sqrt{2}km\cos\theta\sin\frac12\theta$	$-\sqrt{2}kE\sin\theta\sin\frac12\theta$	$2kE\cos\theta\cos\frac12\theta$	$km(1+2\cos\theta)\sin\frac12\theta$	$2Ek\cos\theta\cos\frac12\theta$	$km(1+2\cos\theta)\sin\frac12\theta$
14	$\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_2\gamma\cdot k_1 u$	$\sqrt{2}[4k(E^2/m)-km]$ $\times\sin\frac12\theta$	0	$\sqrt{2}k[4(E^2/m)-m]$ $\times\sin\frac12\theta$	0	$2Ek\cos\frac12\theta$	$-km\sin\frac12\theta$
15	$\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_2\gamma\cdot\epsilon_1\gamma\cdot k_1 u$	$2m^2\cos\frac12\theta$	$[8(E^3/m)-6Em]$ $\times\sin\frac12\theta$	$\sqrt{2}Em\sin\frac12\theta$	$-\sqrt{2}(2E^2-m^2)\cos\frac12\theta$	$m^2\cos\frac12\theta$	$-Em\sin\frac12\theta$
16	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_1\gamma\cdot\epsilon_2\gamma\cdot k_2 u$	$-m^2(1-\cos\theta)\cos\frac12\theta$	$[3Em-4(E^3/m)]$ $\times(1+\cos\theta)\sin\frac12\theta$	$\sqrt{2}Em\cos\theta\sin\frac12\theta$	$\sqrt{2}(2E^2-m^2)\cos\theta$ $\times\cos\frac12\theta$	$m^2(2\cos\theta-1)\cos\frac12\theta$	$mE(1+2\cos\theta)\sin\frac12\theta$

i	T_i	$T_{-1,\frac12;\,0,\frac12}$	$T_{1,-\frac12;\,0,\frac12}$	$T_{-1,\frac12;\,1,\frac12}$	$T_{1,-\frac12;\,1,\frac12}$	$T_{-1,\frac12;\,-1,\frac12}$	$T_{1,-\frac12;\,-1,\frac12}$
1	$\bar{u}1u$	$\cos\frac12\theta$	$(E/m)\sin\frac12\theta$	$\cos\frac12\theta$	$(E/m)\sin\frac12\theta$	$\cos\frac12\theta$	$(E/m)\sin\frac12\theta$
2	$\bar{u}\gamma\cdot\epsilon_1 u$	$2k(E/m^2)\cos\frac12\theta$	$(k/m)\sin\frac12\theta$	$-\sqrt{2}(k/m)\sin\frac12\theta$	0	0	0
3	$\bar{u}\gamma\cdot\epsilon_2 u$	0	0	0	0	0	0
4	$\bar{u}\gamma\cdot k_1 u$	$[2(E^2/m)-m]\cos\frac12\theta$	$E\sin\frac12\theta$	$[2(E^2/m)-m]\cos\frac12\theta$	$E\sin\frac12\theta$	$[2(E^2/m)-m]$ $\times\cos\frac12\theta$	$E\sin\frac12\theta$
5	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot\epsilon_2 u$	$\sqrt{2}(E/m)(1+\cos\theta)\sin\frac12\theta$	$-\sqrt{2}(E^2/m^2)(1-\cos\theta)$ $+\times\cos\frac12\theta$	$-(1-\cos\theta)\cos\frac12\theta$	$-(E/m)(1+\cos\theta)$ $\times\sin\frac12\theta$	$-(1+\cos\theta)\cos\frac12\theta$	$(-E/m)(1-\cos\theta)\sin\frac12\theta$
6	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_1 u$	0	$-k\sin\frac12\theta$	$\sqrt{2}k\sin\frac12\theta$	$-\sqrt{2}2k(E/m)\cos\frac12\theta$	0	0
7	$\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_1 u$	0	0	0	0	0	0
8	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_1\gamma\cdot\epsilon_2 u$	$-\sqrt{2}E(1+\cos\theta)\sin\frac12\theta$	$\sqrt{2}m(1-\cos\theta)\cos\frac12\theta$	$m(1-\cos\theta)\cos\frac12\theta$	$E(1+\cos\theta)\sin\frac12\theta$	$[2(E^2/m)-m]$ $\times(1+\cos\theta)\cos\frac12\theta$	$E(1-\cos\theta)\sin\frac12\theta$
9	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_2 u$	$2k(E/m)\sin\theta\sin\frac12\theta$	$-2k(E^2/m^2)(1+\cos\theta)$ $\times\sin\frac12\theta+k\sin\frac12\theta$	$\sqrt{2}k\cos\theta\sin\frac12\theta$	$-\sqrt{2}k(E/m)(1+\cos\theta)$ $\times\cos\frac12\theta$	$-\sqrt{2}k\sin\theta\cos\frac12\theta$	$-\sqrt{2}k(E/m)\sin\theta\sin\frac12\theta$
10	$\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_2 u$	0	0	0	0	0	0
11	$\bar{u}\gamma\cdot k_1\gamma\cdot k_2 u$	$2k^2\sin\theta\sin\frac12\theta+m^2\cos\frac12\theta$	$-2E(k^2/m)\sin\theta\cos\frac12\theta$ $+mE\sin\frac12\theta$	$2k^2\sin\theta\sin\frac12\theta$ $+m^2\cos\frac12\theta$	$-2E(k^2/m)\sin\theta\cos\frac12\theta$ $+mE\sin\frac12\theta$	$2k^2\sin\theta\sin\frac12\theta$ $+m^2\cos\frac12\theta$	$-2E(k^2/m)\sin\theta\cos\frac12\theta$ $+mE\sin\frac12\theta$
12	$\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_2\gamma\cdot\epsilon_1 u$	0	0	0	0	0	0
13	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_1\gamma\cdot k_2 u$	$2kE\cos\theta\cos\frac12\theta$	$mk(1+2\cos\theta)\sin\frac12\theta$	$-\sqrt{2}mk\cos\theta\sin\frac12\theta$	$-\sqrt{2}kE\sin\theta\sin\frac12\theta$	$\sqrt{2}k[2(E^2/m)-m]$ $\times\sin\theta\cos\frac12\theta$	$\sqrt{2}kE\sin\theta\sin\frac12\theta$
14	$\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_2\gamma\cdot k_1 u$	0	0	0	0	0	0
15	$\bar{u}\gamma\cdot\epsilon_2\gamma\cdot k_2\gamma\cdot\epsilon_1\gamma\cdot k_1 u$	0	0	0	0	0	0
16	$\bar{u}\gamma\cdot\epsilon_1\gamma\cdot k_1\gamma\cdot\epsilon_2\gamma\cdot k_2 u$	$-\sqrt{2}mE(1+\cos\theta)\sin\frac12\theta$	$\sqrt{2}m^2(1-\cos\theta)\cos\frac12\theta$	$(2E^2-m^2)(1-\cos\theta)$ $\times\cos\frac12\theta$	$mE(1+\cos\theta)\sin\frac12\theta$	$m^2(1+\cos\theta)\cos\frac12\theta$	$mE(1-\cos\theta)\sin\frac12\theta$

Sec. V) of the same mass. We work in the c.m. frame where the vector has initial 4-momentum $k_1(E,k\hat{n}_z)$ and initial helicity λ_1 and the spinor has initial 4-momentum $p_1=(E,-k\hat{n}_z)$ and helicity σ_1. The final 4-momenta are

$$k_2=(E,k\hat{n}_x\sin\theta+k\hat{n}_z\cos\theta)$$

and

$$p_2=(E,-k\hat{n}_x\sin\theta-k\hat{n}_z\cos\theta)$$

with helicities λ_2 and σ_2. Thus the scattering angle is

$$\hat{n}_z\cdot(\hat{n}_x\sin\theta+\hat{n}_z\cos\theta)=\cos\theta\equiv z$$

and, as usual,

$$s=(p_1+k_1)^2=4E^2$$

and

$$u=(p_1-k_2)^2=-2k^2(1+z).$$

Our metric is such that $k_1{}^2=E^2-k^2=m^2$ and our γ matrices make the Dirac equation $(\gamma\cdot p-m)u(p)=0$.

We denote the helicity amplitudes $T_{\lambda_2,\sigma_2;\lambda_1,\sigma_1}$, where our T is (minus) the quantity M of GGLMZ. In particular, if we define $f=m/8\pi E$ then, for each collection of helicities,

$$d\sigma/d\Omega=|f|^2.$$

For the 12 independent helicity amplitudes, we choose $T_{1,\frac{1}{2};1,\frac{1}{2}}$, $T_{-1,\frac{1}{2};1,\frac{1}{2}}$, $T_{-1,\frac{1}{2};-1,\frac{1}{2}}$, $T_{1,-\frac{1}{2};0,\frac{1}{2}}$, $T_{-1,-\frac{1}{2};0,\frac{1}{2}}$, and $T_{0,\frac{1}{2};0,\frac{1}{2}}$ for the ones which are even under parity and time reversal and $T_{-1,-\frac{1}{2};1,\frac{1}{2}}$, $T_{1,-\frac{1}{2};1,\frac{1}{2}}$, $T_{1,-\frac{1}{2};-1,\frac{1}{2}}$, $T_{-1,\frac{1}{2};0,\frac{1}{2}}$, $T_{1,\frac{1}{2};0,\frac{1}{2}}$, and $T_{0,-\frac{1}{2};0,\frac{1}{2}}$ for those which are odd under parity and time reversal. To calculate these amplitudes, we use the polarization vectors

$$\begin{aligned}
\epsilon_1{}^\mu&=-\lambda_1(1/\sqrt{2})(\hat{n}_x+i\lambda_1\hat{n}_y)\,, && \lambda_1=\pm1\\
\epsilon_1{}^\mu&=(1/m)(k,E\hat{n}_z)\,, && \lambda_1=0\\
\epsilon_2{}^\mu&=-(\lambda_2/\sqrt{2})(\hat{n}_x\cos\theta-\hat{n}_z\sin\theta-i\lambda_2\hat{n}_y)\,, && \lambda_2=\pm1\\
\epsilon_2{}^\mu&=(1/m)(k,E\hat{n}_x\sin\theta+E\hat{n}_z\cos\theta)\,, && \lambda_2=0
\end{aligned}\tag{A1}$$

such that $\epsilon_1\cdot k_1=\epsilon_2\cdot k_2=0$. (Other products of the polarization vectors with the various momenta are found in Table I.) We define our spinors as

$$u_{\sigma_1}(p_1)=\left(\frac{E+m}{2m}\right)^{1/2}\begin{pmatrix}1\\ 2\sigma_1k/(E+m)\end{pmatrix}\chi_{-\sigma_1}$$

and

$$\bar{u}_{\sigma_2}(p_2)=\left(\frac{E+m}{2m}\right)^{1/2}\chi_{-\sigma_2}{}^+e^{i\theta(\sigma_y/2)}\left(1,\frac{-2\sigma_2k}{E+m}\right),\tag{A2}$$

where

$$\chi_\sigma{}^+e^{i\theta(\sigma_y/2)}\chi_\tau=d_{\sigma\tau}{}^{1/2}(-\theta)=\begin{pmatrix}\cos\frac{1}{2}\theta & \sin\frac{1}{2}\theta\\ -\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta\end{pmatrix}.\tag{A3}$$

The results of the calculation of the helicity amplitudes for 16 different combinations of polarization vectors and momenta are given in Table II, which defines the T_i used in (6), (9), (17), (22), (24), (25), (29), and (30)–(33). Any possible combination of polarization vectors and momenta can be found by using Tables I and II and energy-momentum conservation $p_1+k_1=p_2+k_2$. (In fact, only the first eight combinations of Table II are necessary; the second eight are simply added for convenience.)

Once the helicity amplitudes have been found, we find the parity-conserving helicity amplitudes as defined in GGLMZ

$$\begin{aligned}
T_{\sigma,\nu}{}^\pm=T_{\sigma,\frac{1}{2};\nu,\frac{1}{2}}&(\sqrt{2}\cos\tfrac{1}{2}\theta)^{-|\lambda+\mu|}(\sqrt{2}\sin\tfrac{1}{2}\theta)^{-|\lambda-\mu|}\\
&\pm(-1)^{\lambda+\lambda_m+1}\eta T_{-\sigma,-\frac{1}{2};\nu,\frac{1}{2}}(\sqrt{2}\cos\tfrac{1}{2}\theta)^{-|\lambda-\mu|}\\
&\times(\sqrt{2}\sin\tfrac{1}{2}\theta)^{-|\lambda+\mu|}\,,
\end{aligned}\tag{A4}$$

where $\lambda=\nu-\frac{1}{2}$, $\mu=\sigma-\frac{1}{2}$, and $\lambda_m=\max(|\lambda|,|\mu|)$. η is the intrinsic parity of the vector, i.e., -1 for the vector and $+1$ for the axial vector. For *pseudo*scalar-vector scattering we also use (A4), where $T_{\sigma,\frac{1}{2};\nu,\frac{1}{2}}$ and $T_{-\sigma,-\frac{1}{2};\nu,\frac{1}{2}}$ are replaced by $T_{\sigma,0;\nu,0}$ and $T_{-\sigma,0;\nu,0}$ so that $\lambda=\nu$ and $\mu=\sigma$ and the factor $(-1)^{\lambda+\lambda_m+1}$ is replaced by $(-1)^{\lambda+\lambda_m}$.

Finally, the partial waves of definite parity are given by

$$t_{\sigma,\nu}{}^{j\pm}=\tfrac{1}{2}\int_{-1}^{1}dz[c_{\lambda\mu}{}^{j+}(z)T_{\sigma,\nu}{}^\pm+c_{\lambda\mu}{}^{j-}(z)T_{\sigma,\nu}{}^\mp],\tag{A5}$$

where μ and λ are defined as before and the c's are the polynomials defined in GGLMZ. The unitarity relation in the elastic region is then

$$\mathrm{Im}t^{j\pm}=\frac{m}{8\pi}\frac{k}{E}|t^{j\pm}|^2.\tag{A6}$$

Thus to be consistent with the unitarity bound, the $t^{j\pm}$ and the $T_{\sigma,\nu}{}^\pm$ must not go faster than a constant for large E.