Derivation of Equal-Time Commutation Relations in a Self-Consistent Quantum Field Theory

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A self-consistent quantum field theory is proposed wherein the field equations for the Heisenberg fields are postulated. The Heisenberg fields are required to be linear combinations of normal products of the physical free fields, and the coefficients of the expansion are determined self-consistently. No *a priori* knowledge of the equal-time commutation relations among the Heisenberg fields is assumed. Using a solvable model, it is shown that local microscopic causality requires the existence of a bound state.

I. INTRODUCTION

THIS paper aims to illustrate the main computational steps in a self-consistent formulation of quantum field theory by means of a solvable model. Special emphasis is placed on the self-consistent identification of the "composite particles" and on the self-consistent determination of the equal-time commutators among the Heisenberg fields. This situation is different from the traditional formulation of quantum field theory where canonical commutation relations are assumed for Heisenberg fields.

Let us briefly discuss how self-consistency plays an important role in formulating the quantum field theory.[1] In a traditional form of quantum field theory one begins with field equations (or with a Lagrangian) for Heisenberg operators which are assumed to satisfy equal-time canonical commutators. One reason for the assumption of canonical commutators was that it supplied us with a convenient way of introducing a particle interpretation of quantized fields. Such an interpretation emerged when the Fock space was adapted as a realization of the canonical commutators of Heisenberg fields. In the Fock space one can choose a basis in which each state vector is represented by a series of numbers, each number signifying the totality of particles in respective particle states. However, this reason for canonical commutators of Heisenberg fields turns out to be less impressive when one observes that the particle numbers concerned in physics are not those of the "bare particles" but of the experimentally observable (physical) particles. Also one finds that the canonical commutation relations usually possess a host of unitarily inequivalent realizations[2,3]—i.e., frequently the Fock space of the bare particles is not unitarily equivalent to that of the physical particles.[1,3] One may then pursue either of the three following possibilities: (a) Forget the Fock space of the bare particles and use only the Fock space of the physical particles. (b) Try to formulate a new theory in a nonseparable Hilbert space which includes all the realizations of the canonical commutation relations. (c) Try to use only the algebraic aspects and forget the realizations. In this paper we shall work with the first possibility because of the facility in performing practical computations.

Our task then is to formulate a quantum field theory where the field equations for the Heisenberg operators are described in terms of the Fock space of the physical particles. Since the physical particles are described by field operators satisfying free-field equations with observable masses and spins, the Fock space of the physical particles can be built as a realization of the canonical commutators of these free fields. Thus there is no need to assume the canonical commutators of the Heisenberg operators.

Our point of departure is the formal structure of the field equations for Heisenberg operators $\psi^{(i)}$ $(i=1,\ldots,n)$,

$$\Lambda^{(i)}(\partial)\psi^{(i)}(x)=j^{(i)}(\psi)\,, \tag{1}$$

which we regard as given. Here $\Lambda^{(i)}(\partial)$ are differential operators and $j^{(i)}$ are functions of Heisenberg fields. Let $\phi^{(i)}$ $(i=1,\ldots,m)$ stand for the free fields of physical particles.[4] We treat $\phi^{(i)}$ according to the well-known formulation of the quantized theory of free fields. Thus $\phi^{(i)}$ is a linear combination of creation and annihilation operators and the Fock space is constructed by applying the creation operators cyclically on the vacuum state. We require that the Heisenberg fields should be linear combinations of normal products of the physical free fields:

$$\psi^{(i)}(x)=\chi^{(i)}+C_{ij}\phi^{(j)}(x)+\int d^4y_1\int d^4y_2 \times C_{i,jk}(x-y_1,x-y_2):\phi^{(j)}(y_1)\phi^{(k)}(y_2):+\cdots. \tag{2}$$

Here $\chi^{(i)}$ denotes a c-number constant and the dots stand for higher-order normal products.[5] Our problem

[1] L. Leplae, R. N. Sen, and H. Umezawa, Progr. Theoret. Phys. (Kyoto) Suppl. 637, (1965); H. Umezawa, Acta Phys. Hung. XIX, 9 (1965).

[2] L. Van Hove, Physica 18, 145 (1952); A. J. Wightman and S. S. Schweber, Phys. Rev. 98, 312 (1955); H. Araki, J. Math. Phys. 1, 492 (1960).

[3] R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 29, No. 12 (1955).

[4] There is no reason to assume that $n=m$.

[5] Here we assumed the translational invariance of the system. The expansion form should be modified when we consider cases like crystals. In the relativistically invariant theory without any spontaneous breakdown of Lorentz symmetry the c-number term $\chi^{(i)}$ does not appear unless $\psi^{(i)}$ is a scalar or pseudoscalar. The problem of convergence of the expansion does not arise simply because the normal-product terms are linearly independent of each other; each expansion coefficient is the respective matrix element of $\psi^{(i)}$.

is to find solutions of (1) in the form (2). To do this requires a self-consistent determination of the physical fields. This is because the physical fields can be identified only after we solve the field equations (1) while we need the physical fields to write down expansion (2).[6] To begin the self-consistent approach we prepare as a candidate for the set of physical fields a small set of free fields whose masses are unknown. Regarding these free fields as physical fields, we write the Heisenberg fields $\psi^{(i)}$ in the form (2) and feed this expression into the field equations (1). We then obtain a set of equations for the expansion coefficients ($\chi^{(i)},C_{ij},C_{i,jk}, \ldots$). By solving these equations, we determine expansion (2) together with the masses of the physical particles. When these equations do not admit any solution, we modify the initial set of free fields and repeat the computations. Such a modification of the initial set of free fields is frequently made by introducing more members of the free fields. This is the way many composite particles are successively brought into the set of physical fields.

As an example, suppose that we are given a field equation for the nucleon Heisenberg field. We then choose as the initial set of physical fields an isodoublet free Dirac field which is regarded as the physical nucleon. Writing the nucleon Heisenberg field in the form (2) by means of normal products of physical nucleons and then feeding this expression into the field equation for the nucleon Heisenberg field, we obtain a set of equations for the expansion coefficients. We may then find that these equations do not have a solution unless they are modified by introducing a new member in the initial set of free fields. This new member may turn out to be the physical deuteron.[7]

It is obvious that the success of the self-consistent procedure depends on the initial choice of free fields.

To be more precise, there are additional constraints to be imposed on expansion (2). One requirement is that if we use incoming fields to describe the physical particles then we require that effects of the in-fields on the Heisenberg fields come only from the past. This means[1] that the expansion coefficients in (2) are of a retarded nature.[8] Another requirement is the condition of microcausality, which states that two local operators on a spacelike surface commute with each other when their positions do not coincide. We thus require that equal-time commutators among Heisenberg operators are made up only of terms proportional to the δ function carrying powers of space derivatives of finite order:

$$[\psi^{(i)}(x),\psi^{(j)}(y)]_{\pm,t_x=t_y}=\sum_{n=0}^{l} a_n{}^{(i,j)}(\mathbf{x},\mathbf{y},t)\partial^n\delta(\mathbf{x}-\mathbf{y}). \quad (3)$$

Here the symbol $\pm$ signifies either anticommutator or commutator: $[A,B]_\pm=AB\pm BA$. This sign is determined according to the usual rule. In (3), ∂^n denotes the nth power of space derivatives; the highest power l is required to be finite, and the coefficients $a_n{}^{(i,j)}$ can be operators.

There are no *a priori* conditions on the $a_n{}^{(i,j)}$. As a matter of fact the coefficients $a_n{}^{(i,j)}$ in (3) can be computed when the expansion form (2) is determined. Thus in a self-consistent formalism of quantum field theory the equal-time commutators of Heisenberg fields are to be computed. This poses an interesting question: How general are the canonical commutators? The above argument also suggests that the self-consistent formalism may be able to treat cases where the canonical formalism is inconvenient. In this connection it would be an interesting problem to calculate the commutation relations of electromagnetic Heisenberg operators in quantum electrodynamics without the Gupta-Bleuler photons of negative probability.

In Sec. II we shall illustrate all the steps of the self-consistent procedure by means of a solvable model. We will find that the condition of microcausality plays an extremely important role: It is the condition which determines the renormalization constant (z factor) and which requires the existence of a composite particle.

Let us close this section with a comment on the Hamiltonian in the formalism under consideration. Denote by P_μ the translation operators:

$$\partial_\mu\psi^{(i)}(x)=i[\psi^{(i)}(x),P_\mu]. \quad (4)$$

Then expansion (2) shows that P_μ is equal to the translation operators of the free fields $\phi^{(i)}$:

$$P_\mu=P_\mu(\phi^{(i)}). \quad (5)$$

The Hamiltonian H, satisfying

$$i\frac{\partial\psi^{(i)}(x)}{\partial t}=[\psi^{(i)}(x),H],$$

is that of the free fields, i.e., $H=H(\phi^{(i)})=-iP_4$. This is an example of the general feature that when we consider any symmetry transformation the generator of the transformation should be prepared not in terms of the Heisenberg operators, but in terms of the free fields. We will see an example of this feature in Sec. II.

II. SELF-CONSISTENT CALCULATION

In this section we will exhibit the main steps of the self-consistent procedure by means of a solvable model.

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[6] A similar situation exists also in the LSZ (Lehmann, Symanzik, and Zimmermann) formalism. In the LSZ formalism the physical fields are determined by considering the weak limit ($t\to\pm\infty$) of the Heisenberg fields. However, in order to perform the weak limit, a preknowledge of the Fock space of the physical particles is required.

[7] This definition of composite particles is based on the irreducibility of the physical field operator ring. See H. Ezawa, K. Kikkawa, and H. Umezawa, Nuovo Cimento **25**, 1141 (1962); H. Ezawa, T. Muta, and H. Umezawa, Progr. Theoret. Phys. (Kyoto) **29**, 877 (1963).

[8] When the ϕ's are outgoing fields, the expansion coefficients in (2) are of an advanced nature.

The model is defined by the equations for the Heisenberg fields ψ and θ:

$$\left(\frac{\partial}{\partial t}+im\right)\psi(x)=-i\lambda\int d^4y\int d^4z\,\bar{\alpha}(x-y)\bar{\alpha}(x-z)\theta^\dagger(y)\psi(x)\theta(z)\,, \quad (6)$$

$$\left(\frac{\partial}{\partial t}+i(\mu^2-\nabla^2)^{1/2}\right)\theta(x)=-i\lambda\int d^4y\int d^4z\,\bar{\alpha}(y-x)\bar{\alpha}(y-z)\psi^\dagger(y)\psi(y)\theta(z)\,. \quad (7)$$

Here m and μ are constants with dimensions of mass and $\bar{\alpha}(x-y)$ is of the form

$$\bar{\alpha}(x-y)=\alpha(|\mathbf{x}-\mathbf{y}|)\delta(t_x-t_y)\,. \quad (8)$$

The function $\bar{\alpha}(|\mathbf{x}-\mathbf{y}|)$ is introduced in (6) and (7) to avoid infinities in the following computations.

As for the microcausality condition (3), we assume the anticommutator for the ψ field and the commutator for the θ field. Accordingly, ψ and θ will be called fermion field and boson field, respectively. The coefficients a_n in (3) are still unknown and will be determined through the course of computation. We also assume the commutator in (3) when $\psi^{(i)}=\psi$ (or $\psi^\dagger$) and $\psi^{(j)}=\theta$ (or $\theta^\dagger$). Throughout the following computations, it is important to maintain the order among the Heisenberg fields on the right-hand sides of (6) and (7). This is because their commutation relations are still unknown. The field equations (6) and (7) together with the microcausuality condition are the basis of our considerations.

Note that the field equations (6) and (7) are invariant under the phase transformations

$$\psi(x)\to\psi(x)e^{i\alpha_1}\,,\quad \theta(x)\to\theta(x)e^{i\alpha_2}\,. \quad (9)$$

Since we do not assume any knowledge of the equal-time commutation relations of ψ and θ (except the microcausality), we cannot write down the generator for the above transformation.

To start the self-consistent calculation we need an initial set of free fields as a candidate for the set of physical incoming fields. Let us try the following choice: one physical fermion ψ^{in} and one physical boson θ^{in}. Their energies, which are still unknown, will be denoted by m_p and w_p, respectively:

$$\psi^{\text{in}}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\,N_{\mathbf{p}}{}^{\text{in}}e^{i(\mathbf{p}\cdot\mathbf{x}-m_{\mathbf{p}}t)}\,, \quad (10)$$

$$\theta^{\text{in}}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\,\theta_{\mathbf{p}}{}^{\text{in}}e^{i(\mathbf{p}\cdot\mathbf{x}-w_{\mathbf{p}}t)}\,. \quad (11)$$

Here $N_{\mathbf{p}}{}^{\text{in}}$ and $\theta_{\mathbf{p}}{}^{\text{in}}$ are the annihilation operators of the $N_{\mathbf{p}}{}^{\text{in}}$ and $\theta_{\mathbf{p}}{}^{\text{in}}$ particles, respectively:

$$[N_{\mathbf{p}}{}^{\text{in}},N_{\mathbf{q}}{}^{\text{in}\dagger}]_+=[\theta_{\mathbf{p}}{}^{\text{in}},\theta_{\mathbf{q}}{}^{\text{in}\dagger}]=\delta(\mathbf{p}-\mathbf{q})\,. \quad (12)$$

We shall write the Fourier form of $\bar{\alpha}(|\mathbf{x}-\mathbf{y}|)$ as

$$\bar{\alpha}(|\mathbf{x}-\mathbf{y}|)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{(2w_{\mathbf{p}})^{1/2}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\alpha(w_{\mathbf{p}})\,. \quad (13)$$

We assume that the Heisenberg fields ψ and θ are expanded in the form (2) in terms of normal products of ψ^{in} and θ^{in}. Let us first consider a special solution where the transformations in (9) are induced by the following in-field transformations:

$$\psi^{\text{in}}(x)\to\psi^{\text{in}}(x)e^{i\alpha_1}\,,\quad \theta^{\text{in}}(x)\to\theta^{\text{in}}(x)e^{i\alpha_2}\,. \quad (14)$$

Expansion (2) then reads

$$\theta(x)=z_1{}^{1/2}\theta^{\text{in}}(x)+\int d^3pd^3qd^3r\,N_{\mathbf{q}}{}^{\text{in}\dagger}N_{\mathbf{p}+\mathbf{q}-\mathbf{r}}{}^{\text{in}}\theta_{\mathbf{r}}{}^{\text{in}}c_{\mathbf{p}}(\mathbf{q},\mathbf{r})e^{i\mathbf{p}\cdot\mathbf{x}-i(w_{\mathbf{r}}+m_{\mathbf{p}+\mathbf{q}-\mathbf{r}}-m_{\mathbf{q}})t}$$
$$+\int d^3pd^3qd^3r\,\theta_{-\mathbf{p}+\mathbf{q}+\mathbf{r}}{}^{\text{in}\dagger}\theta_{\mathbf{q}}{}^{\text{in}}\theta_{\mathbf{r}}{}^{\text{in}}a_{\mathbf{p}}(\mathbf{q},\mathbf{r})e^{i\mathbf{p}\cdot\mathbf{x}-i(w_{\mathbf{r}}+w_{\mathbf{q}}-w_{\mathbf{q}+\mathbf{r}-\mathbf{p}})t}+\cdots\,, \quad (15)$$

$$\psi(x)=z_2{}^{1/2}\psi^{\text{in}}(x)+\int d^3pd^3qd^3r\,\theta_{\mathbf{q}}{}^{\text{in}\dagger}N_{\mathbf{p}+\mathbf{q}-\mathbf{r}}{}^{\text{in}}\theta_{\mathbf{r}}{}^{\text{in}}d_{\mathbf{p}}(\mathbf{q},\mathbf{r})e^{i\mathbf{p}\cdot\mathbf{x}-i(w_{\mathbf{r}}+m_{\mathbf{p}+\mathbf{q}-\mathbf{r}}-w_{\mathbf{q}})t}$$
$$+\int d^3pd^3qd^3r\,N_{-\mathbf{p}+\mathbf{q}+\mathbf{r}}{}^{\text{in}\dagger}N_{\mathbf{q}}{}^{\text{in}}N_{\mathbf{r}}{}^{\text{in}}b_{\mathbf{p}}(\mathbf{q},\mathbf{r})e^{i\mathbf{p}\cdot\mathbf{x}-i(m_{\mathbf{r}}+m_{\mathbf{q}}-m_{\mathbf{q}+\mathbf{r}-\mathbf{p}})t}+\cdots\,. \quad (16)$$

Here three dots stand for higher-order normal products, each of which contains at least five in-field operators.

Let us now feed (15) and (16) into (7) and consider the matrix element $\langle 0|\text{Eq. (7)}|\theta_{\mathbf{k}}{}^{\text{in}}\rangle$. We then find

$$\left[\frac{\partial}{\partial t}+i(\mu^2-\nabla^2)^{1/2}\right]\langle 0|\theta^{\text{in}}(x)|\theta_{\mathbf{k}}{}^{\text{in}}\rangle=0\,,$$

which leads to

$$\left[\frac{\partial}{\partial t}+i(\mu^2-\nabla^2)^{1/2}\right]\theta^{\rm in}(x)=0. \tag{17}$$

This shows that $w_{\mathbf{k}}=(k^2+\mu^2)^{1/2}$. In a similar way the matrix element $\langle 0|\text{Eq. (6)}|N_{\mathbf{p}}^{\rm in}\rangle$ leads to

$$\left(\frac{\partial}{\partial t}+im\right)\psi^{\rm in}(x)=0. \tag{18}$$

This means that $m_{\mathbf{p}}=m$, which is independent of $\mathbf{p}$. We have thus determined the energies of the physical particles.

By considering the matrix element $\langle\theta^{\rm in}|\text{Eq. (7)}|\theta^{\rm in}\theta^{\rm in}\rangle$ and using (15) and (16), we find that

$$a_{\mathbf{p}}(\mathbf{q},\mathbf{r})=0. \tag{19}$$

In a similar way, the matrix element $\langle N^{\rm in}|\text{Eq. (6)}|N^{\rm in}N^{\rm in}\rangle$ gives

$$b_{\mathbf{p}}(\mathbf{q},\mathbf{r})=0. \tag{20}$$

Note that a change of normalization of ψ and θ by a common factor (i.e., $\psi\to a\psi$, $\theta\to a\theta$) modifies only the coupling constant ($\lambda\to a^2\lambda$) in the field equations (6) and (7). We are thus free to fix either z_1 or z_2. We shall set $z_2=1$.

To determine the coefficients $c_{\mathbf{p}}(\mathbf{q},\mathbf{r})$, we substitute (15) and (16) into both sides of Eq. (7). Taking the matrix element $\langle N_{\mathbf{q}}^{\rm in}|\text{Eq. (7)}|N_{\mathbf{p}+\mathbf{q}-\mathbf{r}}^{\rm in}\theta_{\mathbf{r}}^{\rm in}\rangle$, we obtain

$$c_{\mathbf{p}}(\mathbf{q},\mathbf{r})=\frac{\lambda z_1^{1/2}}{(2\pi)^{3/2}}\frac{\alpha(w_{\mathbf{p}})\alpha(w_{\mathbf{r}})}{(4w_{\mathbf{p}}w_{\mathbf{r}})^{1/2}}\frac{1}{w_{\mathbf{r}}-w_{\mathbf{p}}+i\epsilon}+\frac{\lambda\alpha(w_{\mathbf{p}})}{(2w_{\mathbf{p}})^{1/2}}\frac{1}{w_{\mathbf{r}}-w_{\mathbf{p}}+i\epsilon}\times\int d^3l\frac{\alpha(w_{\mathbf{l}})}{(2w_{\mathbf{l}})^{1/2}}c_{\mathbf{l}}(\mathbf{p}+\mathbf{q}-\mathbf{l},\mathbf{r}). \tag{21}$$

The solution of (21) is

$$c_{\mathbf{p}}(q,r)=\frac{\lambda z_1^{1/2}}{(2\pi)^{3/2}}\frac{\alpha(w_{\mathbf{p}})\alpha(w_{\mathbf{r}})}{(4w_{\mathbf{p}}w_{\mathbf{r}})^{1/2}}\frac{1}{w_{\mathbf{r}}-w_{\mathbf{p}}+i\epsilon}\frac{1}{1-\lambda I(w_{\mathbf{r}})}, \tag{22}$$

where

$$I(w_{\mathbf{r}})=\int d^3l\frac{\alpha^2(w_{\mathbf{l}})}{2w_{\mathbf{l}}}\frac{1}{w_{\mathbf{r}}-w_{\mathbf{l}}+i\epsilon}. \tag{23}$$

Likewise substituting (15) and (16) (with $z_2=1$) into both sides of Eq. (6) and taking the matrix element $\langle\theta_{\mathbf{q}}^{\rm in}|\text{Eq. (6)}|N_{\mathbf{p}+\mathbf{q}-\mathbf{r}}^{\rm in}\theta_{\mathbf{r}}^{\rm in}\rangle$ yields the relation

$$d_{\mathbf{p}}(\mathbf{q},\mathbf{r})=\frac{\lambda z_1}{(2\pi)^{3/2}}\frac{\alpha(w_{\mathbf{q}})\alpha(w_{\mathbf{r}})}{(4w_{\mathbf{q}}w_{\mathbf{r}})^{1/2}}\frac{1}{w_{\mathbf{r}}-w_{\mathbf{q}}+i\epsilon}+\lambda z_1^{1/2}\frac{\alpha(w_{\mathbf{q}})}{(2w_{\mathbf{q}})^{1/2}}\frac{1}{w_{\mathbf{r}}-w_{\mathbf{q}}+i\epsilon}\times\int d^3l\frac{\alpha(w_{\mathbf{l}})}{(2w_{\mathbf{l}})^{1/2}}c_{\mathbf{l}}(\mathbf{p}+\mathbf{q}-\mathbf{l},\mathbf{r}). \tag{24}$$

This, together with (22), leads to

$$d_{\mathbf{p}}(\mathbf{q},\mathbf{r})=\frac{\lambda z_1}{(2\pi)^{3/2}}\frac{\alpha(w_{\mathbf{q}})\alpha(w_{\mathbf{r}})}{(4w_{\mathbf{q}}w_{\mathbf{r}})^{1/2}}\frac{1}{w_{\mathbf{r}}-w_{\mathbf{q}}+i\epsilon}\frac{1}{1-\lambda I(w_{\mathbf{r}})}. \tag{25}$$

In (21) and (24) we have defined $1/(w_{\mathbf{r}}-w_{\mathbf{q}})$ by $1/(w_{\mathbf{r}}-w_{\mathbf{q}}+i\epsilon)$, so that $c_{\mathbf{p}}(\mathbf{q},\mathbf{r})$ and $d_{\mathbf{p}}(\mathbf{q},\mathbf{r})$ would be of a retarded nature.

We will now show that the microcausality condition (3) requires $z_1=1$. Using (15) and (16) (with $z_2=1$), we find

$$\langle 0|[\psi(x),\theta(y)]_{t_x=t_y}|N_{\mathbf{l}-\mathbf{k}}^{\rm in}\theta_{\mathbf{k}}^{\rm in}\rangle=e^{i\mathbf{l}\cdot\mathbf{y}}\int d^3p\, e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{p}}[c_{\mathbf{l}-\mathbf{p}}(\mathbf{p},\mathbf{k})-z_1^{1/2}d_{\mathbf{p}}(\mathbf{l}-\mathbf{k},\mathbf{k})].$$

This should have the form of the right-hand side of (3). This means that

$$c_{\mathbf{l}-\mathbf{p}}(\mathbf{p},\mathbf{k})-z_1^{1/2}d_{\mathbf{p}}(\mathbf{l}-\mathbf{p},\mathbf{k})=\text{finite polynomial in } \mathbf{p}.$$

Using (22) and (25), we find that $z_1^{1/2}=1$. This leads to

$$\langle 0|[\psi(x),\theta(y)]_{t_x=t_y}|N^{\rm in},\theta^{\rm in}\rangle=0, \tag{26}$$

even when $\mathbf{x}=\mathbf{y}$.

In a similar way we can show by considering

$$\langle N_{\mathbf{l}+\mathbf{k}}^{\rm in}|[\psi^\dagger(x),\theta(y)]_{t_{\mathbf{x}}=t_{\mathbf{y}}}|\theta_{\mathbf{k}}^{\rm in}\rangle \tag{27}$$

that

$$\frac{1}{(2\pi)^{3/2}}[c_{\mathbf{p}}(\mathbf{l}+\mathbf{k},\mathbf{k})+d_{\mathbf{p}+\mathbf{l}}(\mathbf{k},\mathbf{p})]+\int d^3r\, c_{\mathbf{p}}(\mathbf{l}+\mathbf{k},\mathbf{r})d_{\mathbf{p}+\mathbf{l}}^*(\mathbf{k},\mathbf{r})=\text{finite polynomial in } \mathbf{p}. \tag{28}$$

Evaluating

$$I=\int d^3r\, c_{\mathbf{p}}(\mathbf{l}+\mathbf{k},\mathbf{r})d_{\mathbf{p}+\mathbf{l}}^*(\mathbf{k},\mathbf{r}), \tag{29}$$

we find using (22) and (25) (with $z_1=1$) that

$$I=-\frac{1}{(2\pi)^3}\frac{2}{8\pi^2 i}\frac{\alpha(w_p)\alpha(w_k)}{(4w_pw_k)^{1/2}}\times\int d^3r[f(w_r)-f^*(w_r)]\frac{1}{\alpha^2(w_r)}\times\left[\frac{1}{w_r-w_p+i\epsilon}\frac{1}{w_r-w_k-i\epsilon}\frac{1}{(w_r{}^2-\mu^2)^{1/2}}\right]. \quad (30)$$

Here we have defined

$$f(w)=\frac{\lambda\alpha^2(w)}{2w}\frac{1}{1-\lambda I(w)},$$

which satisfies the relation

$$w(w^2-\mu^2)^{1/2}|f(w)|^2=-[f(w)-f^*(w)]/8\pi^2 i. \quad (31)$$

We can rewrite Eq. (30) as

$$I=A(\mathbf{p},\mathbf{k})\int_\mu^\infty dw[f(w)-f^*(w)]\times\left(\frac{1}{w-w_k-i\epsilon}-\frac{1}{w-w_p+i\epsilon}\right)\frac{w}{\alpha^2(w)},$$

where

$$A(\mathbf{p},\mathbf{k})=-\frac{1}{(2\pi)^3}\frac{1}{\pi i}\frac{\alpha(w_p)\alpha(w_k)}{(4w_pw_k)^{1/2}}\frac{1}{w_k-w_p+i\epsilon}. \quad (32)$$

We can further rewrite I:

$$I=A(\mathbf{p},\mathbf{k})\int_\mu^\infty dw\times\left[\left(\frac{1}{w-w_k+i\epsilon}-\frac{1}{w-w_p+i\epsilon}\right)\frac{1}{1-\lambda I(w)}-\text{c.c.}\right]+2\pi iA(\mathbf{p},\mathbf{k})\left[\frac{w_k}{\alpha^2(w_k)}f(w_k)-\frac{w_p}{\alpha^2(w_p)}f^*(w_p)\right]$$

$$=(w_k-w_p)A(\mathbf{p},\mathbf{k})\int_\mu^\infty dw\times\left[\frac{1}{(w-w_k+i\epsilon)(w-w_p+i\epsilon)}\frac{1}{1-\lambda I(w)}-\text{c.c.}\right]-\frac{1}{(2\pi)^{3/2}}c_p(\mathbf{1}+\mathbf{k},\mathbf{k})-\frac{1}{(2\pi)^{3/2}}d_{p+1}{}^*(\mathbf{k},\mathbf{p}). \quad (33)$$

Here c.c. means the complex conjugate. We thus find that

$$\frac{1}{(2\pi)^{3/2}}[c_p(\mathbf{1}+\mathbf{k},\mathbf{k})+d_{p+1}{}^*(\mathbf{k},\mathbf{p})]+\int d^3r\,c_p(\mathbf{1}+\mathbf{k},\mathbf{r})d_{p+1}{}^*(\mathbf{k},\mathbf{r})=(w_k-w_p)A(\mathbf{p},\mathbf{k})\times\int_C dw\frac{1}{(w-w_k)(w-w_p)}\frac{1}{1-\lambda I(w)}. \quad (34)$$

The contour C for this integral is made up of C_1, C_2, and C_3: C_1 moves below the real axis from ∞ to μ, C_2 moves above the real axis from ∞ to μ, and C_3 is the infinite circle. The integral on the right-hand side of (34) vanishes unless there is a point w_0 inside of the contour such that

$$1-\lambda I(w_0)=0. \quad (35)$$

Relation (35) does not have more than one solution since $I(w)$ is a monotonically decreasing function.

Let us assume the case where the coupling constant λ is such that (35) has a solution. This solution w_0 resides on the real axis and $w_0<\mu$. Then (34) becomes

$$\frac{1}{(2\pi)^{3/2}}[c_p(\mathbf{1}+\mathbf{k},\mathbf{k})+d_{p+1}(\mathbf{k},\mathbf{p})]+\int d^3r\,c_p(\mathbf{1}+\mathbf{k},\mathbf{r})d_{p+1}{}^*(\mathbf{k},\mathbf{r})=-\frac{1}{(2\pi)^3}\frac{\alpha(w_p)\alpha(w_k)}{(4w_pw_k)^{1/2}}\frac{B(w_0)}{(w_0-w_k)(w_0-w_p)}. \quad (36)$$

Here $B(w_0)$ is defined by

$$\frac{1}{1-\lambda I(w)}=\frac{B(w_0)}{w_0-w}+\eta(w),$$

where $\eta(w)$ has no pole terms at $w=w_0$. Making use of (35) we obtain

$$B(w_0)=\left[\lambda\int d^3r\frac{\alpha^2(w_r)}{2w_r}\frac{1}{(w_r-w_0)^2}\right]^{-1}. \quad (37)$$

Comparing (36) with (28), we see that microcausality is violated because the right-hand side of (36) is not a finite polynomial in $\mathbf{p}$. Thus we need to modify the initial set of physical fields.

We can obtain a clue in this direction by noting that (29) was the contribution of the following term:

$$\sum_{p,r}\langle N_{1+k}{}^{\text{in}}|\theta(y)|N_{p+1+k-r}{}^{\text{in}}\theta_r{}^{\text{in}}\rangle\times\langle N_{p+1+k-r}{}^{\text{in}}\theta_r{}^{\text{in}}|\psi^\dagger(x)|\theta_k{}^{\text{in}}\rangle,$$

where the energies of the intermediate $N^{\text{in}}\theta^{\text{in}}$ states are $m+w_r$. We may thus expect that the microcausality condition can be secured by the effect of a single-particle state which has the energy $m+w_0$ and behaves like $N^{\text{in}}\theta^{\text{in}}$ under the phase transformation (9). We shall thus introduce a fermion B^{in} which transforms as

$$B^{\text{in}} \longrightarrow B^{\text{in}} e^{i(\alpha_1+\alpha_2)} \tag{38}$$

under the transformation (9). Due to this nature, B^{in} is called a bound state of N and θ. Our set of physical fields is now $(N^{\text{in}},\theta^{\text{in}},B^{\text{in}})$. Since (14) and (38) should induce the transformation (9), expansions (15) and (16) are modified as follows:

$$\theta(x)=\theta^{\text{in}}(x)+\int d^3p d^3q d^3r\, N_{\mathbf{q}}^{\text{in}\dagger} N_{\mathbf{p}+\mathbf{q}-\mathbf{r}}^{\text{in}} \theta_{\mathbf{r}}^{\text{in}} c_{\mathbf{p}}(\mathbf{q},\mathbf{r}) e^{i\mathbf{p}\cdot\mathbf{x}-iw_{\mathbf{r}}t}+\int d^3p d^3l\, g_{\mathbf{p}}(\mathbf{l}) N_{\mathbf{l}}^{\text{in}\dagger} B_{\mathbf{p}+\mathbf{l}}^{\text{in}} e^{i\mathbf{p}\cdot\mathbf{x}-i(M_{\mathbf{p}+\mathbf{l}}-m)t}+\cdots, \tag{39}$$

$$\psi(x)=\psi^{\text{in}}(x)+\int d^3p d^3q d^3r\, \theta_{\mathbf{q}}^{\text{in}\dagger} N_{\mathbf{p}+\mathbf{q}-\mathbf{r}}^{\text{in}} \theta_{\mathbf{r}}^{\text{in}} d_{\mathbf{p}}(\mathbf{q},\mathbf{r}) e^{i\mathbf{p}\cdot\mathbf{x}-i(w_{\mathbf{r}}-w_{\mathbf{q}}+m)t}$$
$$+\int d^3p d^3l\, h_{\mathbf{p}}(\mathbf{l}) \theta_{\mathbf{l}}^{\text{in}\dagger} B_{\mathbf{p}+\mathbf{l}}^{\text{in}} e^{i\mathbf{p}\cdot\mathbf{x}-i(M_{\mathbf{p}+\mathbf{l}}-w_{\mathbf{l}})t}+\cdots. \tag{40}$$

Here $M_{\mathbf{p}+\mathbf{l}}$ is the energy of the fermion B^{in} and $g_{\mathbf{p}}(\mathbf{l})$ and $h_{\mathbf{p}}(\mathbf{l})$ are coefficients to be determined. The reader should convince himself that the inclusion of the B^{in} in (39) and (40) in no way affects our arguments from the beginning of this section to (26).

By substituting (39) and (40) into the field equations (7) and (6) and taking the matrix elements $\langle N_{\mathbf{l}}^{\text{in}}| \times$Eq. (7)$|B_{\mathbf{p}+\mathbf{l}}^{\text{in}}\rangle$ and $\langle\theta_{\mathbf{l}}^{\text{in}}|$Eq. (6)$|B_{\mathbf{p}+\mathbf{l}}^{\text{in}}\rangle$, we obtain the relations

$$g_{\mathbf{p}}(\mathbf{l})=\frac{\lambda}{M_{\mathbf{p}+\mathbf{l}}-w_{\mathbf{p}}-m}\frac{\alpha(w_{\mathbf{p}})}{(2w_{\mathbf{p}})^{1/2}} \times\int d^3k \frac{\alpha(w_{\mathbf{k}})}{(2w_{\mathbf{k}})^{1/2}} g_{\mathbf{k}}(\mathbf{p}+\mathbf{l}-\mathbf{k}), \tag{41}$$

$$h_{\mathbf{p}}(\mathbf{l})=\frac{\lambda}{M_{\mathbf{p}+\mathbf{l}}-w_{\mathbf{l}}-m}\frac{\alpha(w_{\mathbf{l}})}{(2w_{\mathbf{l}})^{1/2}} \times\int d^3k \frac{\alpha(w_{\mathbf{k}})}{(2w_{\mathbf{k}})^{1/2}} g_{\mathbf{k}}(\mathbf{p}+\mathbf{l}-\mathbf{k}). \tag{42}$$

From these relations we deduce that

$$h_{\mathbf{l}}(\mathbf{p})=g_{\mathbf{p}}(\mathbf{l})\,. \tag{43}$$

Defining

$$\beta(\mathbf{p}+\mathbf{l})=\lambda\int d^3k \frac{\alpha(w_{\mathbf{k}})}{(2w_{\mathbf{k}})^{1/2}} g_{\mathbf{k}}(\mathbf{p}+\mathbf{l}-\mathbf{k}), \tag{44}$$

Eq. (41) can be written as

$$g_{\mathbf{p}}(\mathbf{l})=\frac{\beta(\mathbf{p}+\mathbf{l})}{M_{\mathbf{p}+\mathbf{l}}-w_{\mathbf{p}}-m}\frac{\alpha(w_{\mathbf{p}})}{(2w_{\mathbf{p}})^{1/2}}. \tag{45}$$

Substituting (45) into (44), we find that

$$\lambda I(M_{\mathbf{p}+\mathbf{l}}-m)=1\,. \tag{46}$$

Comparing this with (35) leads to

$$M_{\mathbf{p}+\mathbf{l}}=w_0+m\,, \tag{47}$$

as was expected.[9] Equation (47) also shows that $M_{\mathbf{p}+\mathbf{l}}$ is independent of $\mathbf{p}+\mathbf{l}$. $M_{\mathbf{p}+\mathbf{l}}$ will be denoted by M in the following argument.

We have seen that the quantities $g_{\mathbf{p}}(\mathbf{l})$ and $h_{\mathbf{p}}(\mathbf{l})$ are determined by (45) and (43) up to the unknown normalization factor $\beta(\mathbf{p}+\mathbf{l})$. This means that the field equations (6) and (7), although not requiring the existence, are indeed consistent with the existence of the composite particle B^{in} of energy M. However, the existence of B^{in} is required by the microcausality condition. This leads to a nonvanishing value of $\beta(\mathbf{p}+\mathbf{l})$. To demonstrate this we shall study the matrix element (27). Using (39) and (40), the microcausality condition (28) is modified as

$$\frac{1}{(2\pi)^{3/2}}[c_{\mathbf{p}}(\mathbf{l}+\mathbf{k},\mathbf{k})+d_{\mathbf{p}+\mathbf{l}}^*(\mathbf{k},\mathbf{p})] +\int d^3r\, c_{\mathbf{p}}(\mathbf{l}+\mathbf{k},\mathbf{r}) d_{\mathbf{p}+\mathbf{l}}^*(\mathbf{k},\mathbf{r})+g_{\mathbf{p}}(\mathbf{l}+\mathbf{k}) h_{\mathbf{p}+\mathbf{l}}^*(\mathbf{k}) =\text{finite polynomial in }\mathbf{p}. \tag{48}$$

Making use of (36), (43), (45), and (47), we obtain from (48)

$$\beta^2=[1/(2\pi)^3]B(M-m)\,, \tag{49}$$

where β means $\beta(\mathbf{p}+\mathbf{l}+\mathbf{k})$. Equation (49) shows that $\beta(\mathbf{p}+\mathbf{l}+\mathbf{k})$ is a constant independent of $(\mathbf{p}+\mathbf{l}+\mathbf{k})$. The quantities $g_{\mathbf{p}}(\mathbf{l})$ and $h_{\mathbf{p}}(\mathbf{l})$ are now determined:

$$g_{\mathbf{p}}(\mathbf{l})=h_{\mathbf{l}}(\mathbf{p}) =\left[\frac{B(M-m)}{(2\pi)^3}\right]^{1/2}\frac{1}{M-m-w_{\mathbf{p}}}\frac{\alpha(w_{\mathbf{p}})}{(2w_{\mathbf{p}})^{1/2}}. \tag{50}$$

[9] The Bethe-Salpeter equation cannot be used to look for the bound state here since we do not *a priori* know the commutation relations for the Heisenberg fields.

Equations (39) and (40) show that

$$H=m\int d^3k\, N_{\mathbf{k}}^{\mathrm{in}\dagger}N_{\mathbf{k}}^{\mathrm{in}}+\int d^3k\, w_{\mathbf{k}}\theta_{\mathbf{k}}^{\mathrm{in}\dagger}\theta_{\mathbf{k}}^{\mathrm{in}} +M\int d^3k\, B_{\mathbf{k}}^{\mathrm{in}\dagger}B_{\mathbf{k}}^{\mathrm{in}} \quad (51)$$

acts as the Hamiltonian:

$$i\frac{\partial\psi(x)}{\partial t}=[\psi(x),H],\quad i\frac{\partial\theta(x)}{\partial t}=[\theta(x),H]. \quad (52)$$

According to (14) and (38) the transformation (9) can be generated as

$$\begin{aligned}\exp(-i\alpha_1\eta_N-i\alpha_2\eta_\theta)\psi\exp(i\alpha_1\eta_N+i\alpha_2\eta_\theta)&=e^{i\theta_1}\psi\,,\\ \exp(-i\alpha_1\eta_N-i\alpha_2\eta_\theta)\theta\exp(i\alpha_1\eta_N+i\alpha_2\eta_\theta)&=e^{i\theta_2}\psi\,,\end{aligned} \quad (53)$$

where the generators are

$$\eta_N=\int d^3k\,(N_{\mathbf{k}}^{\mathrm{in}\dagger}N_{\mathbf{k}}^{\mathrm{in}}+B_{\mathbf{k}}^{\mathrm{in}\dagger}B_{\mathbf{k}}^{\mathrm{in}})\,, \quad (54a)$$

$$\eta_\theta=\int d^3k\,(\theta_{\mathbf{k}}^{\mathrm{in}\dagger}\theta_{\mathbf{k}}^{\mathrm{in}}+B_{\mathbf{k}}^{\mathrm{in}\dagger}B_{\mathbf{k}}^{\mathrm{in}})\,. \quad (54b)$$

The eigenvalues of η_N and η_θ are called the N number and θ number, respectively. For example, $\eta_N=1$, $\eta_\theta=0$ for $|N^{\mathrm{in}}\rangle$, $\eta_N=0$, $\eta_\theta=1$ for $|\theta^{\mathrm{in}}\rangle$, and $\eta_N=\eta_\theta=1$ for $|B^{\mathrm{in}}\rangle$, etc. The terms denoted by three dots in (39) and (40) do not contribute to any calculation confined to the subspace made up of the states of ($\eta_N\leq 1$, $\eta_\theta\leq 1$). We have thus determined ψ and θ completely as far as the above subspace is concerned. In the Appendix we present calculations of higher-order terms in (39) and (40).

Knowing expansions (39) and (40), we can evaluate the equal-time commutators for the Heisenberg fields. Here we confine our attention to the above subspace. The result is the canonical equal-time commutators.

To look for solutions of other types, we need to modify (14). Since the transformation (9) keeps the Heisenberg field equations (6) and (7) invariant, corresponding transformations of the in-fields maintain the invariance of the in-field equations. Because the in-field equations are linear homogeneous differential equations, we may consider the following possibilities:

$$\psi^{\mathrm{in}}\to e^{ic_1\alpha_1}\psi^{\mathrm{in}}\,, \quad (55a)$$

$$\theta^{\mathrm{in}}\to e^{ic_2\alpha_2}\theta^{\mathrm{in}}+(d_1\alpha_1+d_2\alpha_2)\,. \quad (55b)$$

Here $d_1=d_2=0$ unless the physical mass of θ^{in} is zero.[10] When the constants c_1, c_2, d_1, and d_2 are specified, we write down the expansion of ψ and θ in terms of physical fields and then follow the steps of the self-consistent procedure. Any specific choice of the constants is justified only when it leads to a sensible self-consistent answer. It can even happen that θ^{in} and N^{in} do not transform ($c_1=c_2=d_1=d_2=0$). Still there may appear other composite fields whose transformations still induce the Heisenberg field transformations in (9).

III. CONCLUSION

In Sec. II we illustrated the main steps of the self-consistent procedure by means of a solvable model. In this model it so happened that the existence of a composite particle was required not by the basic field equations, but by the condition of microcausality. The equal-time commutation relations for the Heisenberg fields were not assumed, but calculated; and the result of this model turned out to be the canonical commutators. The method gives rise to an interesting question: How general are the canonical commutators? It would be an interesting problem to apply the self-consistent procedure to quantum electrodynamics without the Gupta-Bleuler photons of negative probability.

APPENDIX

We shall now show how the coefficients of the higher-order normal-product terms in the dynamical map (2) can be determined.

The expansion for θ is

$$\begin{aligned}\theta(x)=\theta^{\mathrm{in}}(x)&+\int d^3pd^3qd^3r\, N_{\mathbf{q}}^{\mathrm{in}\dagger}N_{\mathbf{p+q-r}}^{\mathrm{in}}\theta_{\mathbf{r}}^{\mathrm{in}}e^{i\mathbf{p}\cdot\mathbf{x}}c_{\mathbf{p}}(\mathbf{q},\mathbf{r})e^{-iw_{\mathbf{r}}t}+\int d^3pd^3l\, g_{\mathbf{p}}(\mathbf{l})N_{\mathbf{l}}^{\mathrm{in}\dagger}B_{\mathbf{p+l}}^{\mathrm{in}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{i(m-M)t}\\ &+\int d^3pd^3qd^3rd^3sd^3u\, Y_{\mathbf{p}}(\mathbf{q},\mathbf{r},\mathbf{s},\mathbf{u})N_{\mathbf{q}}^{\mathrm{in}\dagger}\theta_{\mathbf{r}}^{\mathrm{in}\dagger}\theta_{\mathbf{s}}^{\mathrm{in}}\theta_{\mathbf{u}}^{\mathrm{in}}N_{\mathbf{p+q+r-s-u}}^{\mathrm{in}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{i(w_{\mathbf{r}}-w_{\mathbf{s}}-w_{\mathbf{u}})}\\ &+\int d^3pd^3qd^3rd^3sd^3u\, N_{\mathbf{q}}^{\mathrm{in}\dagger}N_{\mathbf{r}}^{\mathrm{in}\dagger}N_{\mathbf{s}}^{\mathrm{in}}N_{\mathbf{p+q+r-s-u}}^{\mathrm{in}}\theta_{\mathbf{u}}^{\mathrm{in}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{-iw_{\mathbf{u}}t}Z_{\mathbf{p}}(\mathbf{q},\mathbf{r},\mathbf{s},\mathbf{u})\\ &+\int d^3pd^3qd^3rd^3s\, N_{\mathbf{q}}^{\mathrm{in}\dagger}\theta_{\mathbf{r}}^{\mathrm{in}\dagger}B_{\mathbf{p+q+r-s}}^{\mathrm{in}}\theta_{\mathbf{s}}^{\mathrm{in}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i(w_{\mathbf{s}}-w_{\mathbf{r}})t}e^{-i(M-m)t}G_{\mathbf{p}}(\mathbf{q},\mathbf{r},\mathbf{s})\end{aligned}$$

[10] When $d_1\neq 0$ or $d_2\neq 0$ there occurs the spontaneous breakdown of symmetry.

$$+\int d^3pd^3qd^3rd^3s\, N_{\mathbf{q}}^{\text{in}\dagger}N_{\mathbf{r}}^{\text{in}\dagger}B_{\mathbf{p+q+r-s}}^{\text{in}}N_{\mathbf{s}}^{\text{in}}X_{\mathbf{p}}(\mathbf{q,r,s})e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i(M-m)t}$$

$$+\int d^3pd^3qd^3rd^3s\, B_{\mathbf{q+r+s-p}}^{\text{in}\dagger}N_{\mathbf{q}}^{\text{in}}\theta_{\mathbf{r}}^{\text{in}}\theta_{\mathbf{s}}^{\text{in}}e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i(M-m-w_{\mathbf{s}}-w_{\mathbf{r}})t}H_{\mathbf{p}}(\mathbf{q,r,s})$$

$$+\int d^3pd^3qd^3r\, B_{\mathbf{q}}^{\text{in}\dagger}B_{\mathbf{p+q-r}}^{\text{in}\dagger}\theta_{\mathbf{r}}^{\text{in}}R_{\mathbf{p}}(\mathbf{q,r})e^{i\mathbf{p}\cdot\mathbf{x}}e^{-iw_{\mathbf{r}}t}+\cdots.$$

The expansion for $\psi(x)$ is

$$\psi(x)=\psi^{\text{in}}(x)+\int d^3pd^3qd^3re^{i\mathbf{p}\cdot\mathbf{x}}e^{-i(w_{\mathbf{r}}-w_{\mathbf{q}})t}d_{\mathbf{p}}(\mathbf{q,r})\theta_{\mathbf{q}}^{\text{in}\dagger}N_{\mathbf{p+q-r}}^{\text{in}}\theta_{\mathbf{r}}^{\text{in}}+\int d^3pd^3l\, h_{\mathbf{p}}(\mathbf{l})\theta_{\mathbf{l}}^{\text{in}\dagger}B_{\mathbf{p+l}}^{\text{in}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{+i(w_{\mathbf{l}}-M)t}$$

$$+\int d^3pd^3qd^3rd^3sd^3u\, N_{\mathbf{q}}^{\text{in}\dagger}\theta_{\mathbf{r}}^{\text{in}\dagger}\theta_{\mathbf{s}}^{\text{in}}N_{\mathbf{u}}^{\text{in}}N_{\mathbf{p+q+r-s-u}}^{\text{in}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{i(w_{\mathbf{r}}-w_{\mathbf{s}}-M)t}D_{\mathbf{p}}(\mathbf{q,r,s,u})$$

$$+\int d^3pd^3qd^3rd^3s\, N_{\mathbf{q}}^{\text{in}\dagger}\theta_{\mathbf{r}}^{\text{in}\dagger}B_{\mathbf{p+q+r-s}}^{\text{in}}N_{\mathbf{s}}^{\text{in}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{+i(w_{\mathbf{s}}-M)t}A_{\mathbf{p}}(\mathbf{q,r,s})$$

$$+\int d^3pd^3qd^3rd^3s\, B_{\mathbf{p+q+r+s}}^{\text{in}\dagger}N_{\mathbf{q}}^{\text{in}}N_{\mathbf{r}}^{\text{in}}\theta_{\mathbf{s}}^{\text{in}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{i(M-2m-w_{\mathbf{s}})t}B_{\mathbf{p}}(\mathbf{q,r,s})$$

$$+\int d^3pd^3qd^3rd^3sd^3u\, \theta_{\mathbf{q}}^{\text{in}\dagger}\theta_{\mathbf{r}}^{\text{in}\dagger}\theta_{\mathbf{s}}^{\text{in}}\theta_{\mathbf{u}}^{\text{in}}N_{\mathbf{p+q+r-s-u}}^{\text{in}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{-imt}c_{\mathbf{p}}(\mathbf{q,r,s,u})$$

$$+\int d^3pd^3qd^3rd^3s\, \theta_{\mathbf{q}}^{\text{in}\dagger}\theta_{\mathbf{r}}^{\text{in}\dagger}\theta_{\mathbf{s}}^{\text{in}}B_{\mathbf{p+q+r-s}}^{\text{in}}K_{\mathbf{p}}(\mathbf{q,r,s})e^{i\mathbf{p}\cdot\mathbf{x}}e^{i(w_{\mathbf{q}}+w_{\mathbf{r}}-w_{\mathbf{s}}-M)t}$$

$$+\int d^3pd^3qd^3r\, B_{\mathbf{q}}^{\text{in}\dagger}B_{\mathbf{r}}^{\text{in}}N_{\mathbf{p+q-r}}^{\text{in}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{-imt}+\cdots.$$

Here the three dots stand for higher-order normal-product terms.

By taking the following matrix element $\langle B_{\mathbf{l}}^{\text{in}}|\text{Eq. (7)}|B_{\mathbf{m}}^{\text{in}}\theta_{\mathbf{n}}^{\text{in}}\rangle$, we obtain

$$R_{\mathbf{n+m-l}}(\mathbf{l,n})=\frac{\lambda(2\pi)^{3/2}}{w_{\mathbf{n}}-w_{\mathbf{n+m-l}}}\int d^3r\, h_{\mathbf{l-n}}^*(\mathbf{n})g_{\mathbf{r}}(\mathbf{m-r})\frac{\alpha(w_{\mathbf{r}})\alpha(w_{\mathbf{l-n-m}})}{(4w_{\mathbf{r}}w_{\mathbf{l-n-m}})^{1/2}}+\frac{\lambda\sqrt{(2\pi)^{3/2}}}{w_{\mathbf{n}}-w_{\mathbf{n+m-l}}}\int d^3r\, h_{\mathbf{l-r}}^*(\mathbf{r})h_{\mathbf{m-r}}(\mathbf{r})\frac{\alpha(w_{\mathbf{n}})\alpha(w_{\mathbf{l-m-n}})}{(4w_{\mathbf{n}}w_{\mathbf{l-m-n}})^{1/2}}.$$

Similarly, by taking the matrix element

$$\langle N_{\mathbf{u}}^{\text{in}}N_{\mathbf{v}}^{\text{in}}|\text{Eq. (7)}|N_{\mathbf{w}}^{\text{in}}B_{\mathbf{m}}^{\text{in}}\rangle,$$

we obtain

$$X_{\mathbf{m+w-v-u}}(\mathbf{u,w,v})-X_{\mathbf{m+w-v-u}}(\mathbf{v,w,u})=\frac{\lambda}{m-M}\frac{\alpha(w_{\mathbf{v+u-m-w}})}{(2w_{\mathbf{v+u-m-w}})^{1/2}}\left\{\frac{\alpha(w_{\mathbf{m-u}})}{(2w_{\mathbf{m-u}})^{1/2}}g_{\mathbf{m-u}}(\mathbf{u})-\frac{\alpha(w_{\mathbf{m-v}})}{(2w_{\mathbf{m-v}})^{1/2}}g_{\mathbf{m-v}}(\mathbf{v})\right\}$$

$$+\frac{\lambda}{m-M}\int d^3l\frac{\alpha(w_{\mathbf{m+w-l-u}})\alpha(w_{\mathbf{v+u-m-w}})}{(4w_{\mathbf{m+w-l-u}}w_{\mathbf{v+u-m-w}})^{1/2}}[X_{\mathbf{w+m-l-u}}(\mathbf{l,w,u})-X_{\mathbf{w+m-l-u}}(\mathbf{u,w,l})]$$

$$+\frac{\lambda}{m-M}\int d^3l\frac{\alpha(w_{\mathbf{m+w-l-v}})\alpha(w_{\mathbf{v+u-m-w}})}{(4w_{\mathbf{v+u-m-w}}w_{\mathbf{m+w-l-v}})^{1/2}}[X_{\mathbf{w+m-l-v}}(\mathbf{l,w,v})-X_{\mathbf{w+m-l-v}}(\mathbf{v,w,l})].$$

Similarly the other coefficients may be computed by taking the appropriate matrix elements. We may find that these coefficients do not admit a consistent solution. In that event we must modify the dynamical map (2) by adding new physical fields.