scattering. We have shown how the approach, making use of the geometry of velocity space, leads to crossing-symmetric expansions (for particles with arbitrary masses) having reasonable threshold and asymptotic behavior.

We plan to continue our investigation, in particular to consider the problem of analytic continuation into nonphysical regions and between physical regions, and to establish the exact connection with Regge-pole theory, i.e., to show how moving poles in the complex angular momentum plane manifest themselves in the properties of our expansion coefficients. Further problems under consideration are the relation of the obtained expansions to various dual-resonance models, the calculation of expansion coefficients in various models, and (one may hope) applications to the description of specific scattering or decay processes. The mathematical problems which arise are related both to the development of group representation theory in bases not corresponding to the reduction of a group to any subgroup, and to expansions of non-square-integrable functions in terms of nonunitary representations of noncompact groups.

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Some Features of Chiral Symmetry Breaking

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Some rather paradoxical features of chiral symmetry breaking are shown to be directly related to the fact that the vacuum is supposed to be degenerate in the limit of exact symmetry. The importance of picking the correct one of the many symmetry-limit vacua is stressed. A peculiar phenomena of spontaneous CP violation appearing alongside $SU(3)\otimes SU(3)$ breaking is observed. (In the model investigated here, the effect is much too large to be at all related to that seen in weak interactions.) For the $(3,\bar{3})\oplus(\bar{3},3)$ model of symmetry breaking, it is shown that the rate for $K_S^0 \to 2\pi$ should be suppressed by a factor of $[SU(3)\otimes SU(3)\text{ breaking}]^4$. This may or may not be a difficulty. A number of other topics in chiral symmetry breaking are discussed briefly.

I. INTRODUCTION

IT appears to be reasonable to adopt a picture of the strong interactions in which the strong Hamiltonian H can be meaningfully written as

$$H = H_0 + \epsilon H', \tag{1}$$

where H_0 is $SU(3)\otimes SU(3)$ symmetric and H' contains the (small) departures from exact symmetry.[1,2] We suppose that H' takes care of not only the corrections to chiral symmetry, e.g., corrections to partial conservation of axial-vector current (PCAC), but also the corrections to $SU(3)$ itself, e.g., mass splitting. An advantage of this way of looking at the strong interactions is that one can hopefully relate these two types of symmetry breaking.

In the limit $\epsilon=0$ the eight pseudoscalar mesons are supposed to be Goldstone bosons and, hence, massless. To first order in ϵ, their masses squared are given by the elementary formula

$$m_\alpha^2 = \langle \mathbf{P}_\alpha | \epsilon \mathcal{H}'(0) | \mathbf{P}_\alpha \rangle + O(\epsilon^2), \tag{2}$$

where $\alpha = 1, \ldots, 8$ labels the mesons; $\mathcal{H}'$ is the density of H', i.e.,

$$H' = \int d^3x\, \mathcal{H}(\mathbf{x},0); \tag{3}$$

and $\langle \mathbf{P}|$ is a covariantly normalized state, i.e., $\langle \mathbf{P}|\mathbf{P}'\rangle = 2P_0\delta^3(\mathbf{P}-\mathbf{P}')$. Taken at face value, Eq. (2) has a peculiar property. The left-hand side is necessarily positive, but there does not appear to be any reason why the matrix elements on the right-hand side cannot be negative. One is thus led to ask the question of whether positivity of the squares of the meson masses places some restriction on $\mathcal{H}'$.

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[1] The original suggestion that PCAC is related to chiral symmetry was due to Nambu and collaborators. Later Weinberg pointed out that current-algebra results could be interpreted as the consequences of an approximate chiral symmetry. The present paper is based on ideas expressed in Ref. 2. More recent papers on chiral symmetry are too numerous to list in any reasonably fair manner.

[2] R. Dashen, Phys. Rev. 183, 1245 (1969).

The question just stated turns out to be closely related to an apparent paradox which has been discussed by Kuo[3] and by Okubo and Mathur.[4] Kuo points out that a specific symmetry-breaking term $\mathcal{H}'$ can be subjected to $SU(3)\otimes SU(3)$ transformations and that the new $\mathcal{H}'$ thus obtained looks as though it describes a world which is different, in a physically significant way, from that described by the old one. This is to be contrasted with $SU(3)$ alone. Making an $SU(3)$ rotation on $\mathcal{H}'$ does not introduce new physics. It appears, according to Kuo's work, there might be an inherent ambiguity in the definition of $\mathcal{H}'$.

The answer to both questions raised above is buried in the fact that the vacuum is not supposed to be an $SU(3)\otimes SU(3)$ singlet even in the limit $\epsilon\to 0$. In fact, there is a whole continuum of vacua which are all equivalent as far as H_0 is concerned. On the other hand, the complete Hamiltonian $H=H_0+\epsilon H'$ is supposed to have a unique vacuum. As ϵ goes to zero, this unique physical vacuum goes over to one of the many vacua which are degenerate in the symmetry limit. Therefore, $\mathcal{H}'$ not only determines how the symmetry is broken but also which of the many vacua of H_0 is relevant for physics. This has the following consequence. The physical meaning of $SU(3)$ is that subgroup of $SU(3)\otimes SU(3)$ under which the vacuum is a singlet in the limit $\epsilon\to 0$, and hence can be used to classify particle states. But since $\mathcal{H}'$ determines which of the vacua of H_0 is the relevant one, it follows that the physical $SU(3)$ group is not independent of $\mathcal{H}'$. Therefore, if we choose a coordinate system in $SU(3)\otimes SU(3)$ space in which physical $SU(3)$ plays a special role, then $\mathcal{H}'$ cannot have arbitrary properties in that coordinate system. This, it turns out, resolves the paradoxes of possibily negative squares of meson masses and Kuo's transformations.

A large part of this paper is devoted to a thorough explanation of the points mentioned above. The behavior of a ferromagnet in an external field is used to bring out the physics of the problem—the problem evidently being one of, given $\mathcal{H}'$, how to find the correct vacuum of H_0. The machinery to do this work is developed for an arbitrary group rather than just $SU(3)\otimes SU(3)$. There is a reason for this. The group $SU(3)\otimes SU(3)$ has some special properties which tend to obscure the basic physics. Later, specializing to $SU(3)\otimes SU(3)$, several examples are worked out in detail. Also, it is shown that if one does everything correctly, the squares of the meson masses come out positive for any $\mathcal{H}'$.

Most of the above is done in Sec. II. The discussion may seem rather long and formal to some readers. It is the author's feeling, however, that the determination of what it is that breaks $SU(3)\otimes SU(3)$ and $SU(3)$ is one of the most important problems in particle theory. To do this in the case of $SU(3)\otimes SU(3)$, one needs a real understanding of how theories with degenerate vacua work. Some surprising things can happen. In the middle of Sec. III a phenomenon of spontaneous CP violation is illustrated. Here one starts with a CP-invariant H_0 and an $\mathcal{H}'$ that appears to preserve CP, but the resulting theory shows CP violation. What happens is that $\mathcal{H}'$ picks out not one but two vacua which are CP images of each other. The CP violation found in a model in Sec. III is much too large to be in any way related to the effects observed in weak interactions. It does suggest, however, that further research along this line might be interesting. The reasons for taking the vacuum of H_0 to be $SU(3)$ symmetric, not just $SU(2)$ symmetric, and what this implies about the hypothetical κ meson are also reviewed in Sec. III. Section III ends with a discussion of the "allowed domains" for symmetry-breaking parameters discovered by Mathur and Okubo.[5] It is found that their disconnected domains are actually interconnected by theories exhibiting spontaneous CP violation.

Section IV contains some points of more immediate relevance. After deriving a general formula for the trilinear coupling of Goldstone bosons, it is shown that the decay amplitude for $K\to 2\pi$ is of order $G\epsilon^2$ in the $(3,\bar{3})\oplus(\bar{3},\bar{3})$ model,[6,7] where G is the Fermi constant. This is to be contrasted with a decay like $\Lambda\to\pi+N$, whose amplitude is simply of order G. The rate for $K_s^0\to 2\pi$ does not appear to be depressed by the indicated factor of ϵ^4. This may or may not prove to be a problem for the $(3,\bar{3})\oplus(\bar{3},3)$ model. In connection with this, the rationale for $\mathcal{H}'$ belonging to $(3,\bar{3})\oplus(\bar{3},3)$ is reviewed. It is pointed out that the meson masses can be fitted with $\mathcal{H}'$, a linear combination of $SU(3)$ singlet and octet from any representation $(\bar{X},X)\oplus(X,\bar{X})$ of $SU(3)\otimes SU(3)$. A consequence of this is that the smallness of $m_\pi{}^2$ does not necessarily imply that $SU(2)\otimes SU(2)$ is a better symmetry than $SU(3)$ or $SU(3)\otimes SU(3)$, contrary to what seems to be commonly believed.

The various sections and subsections are written so that to some extent they can be read independently. Notational conventions relating to $SU(3)\otimes SU(3)$ are listed in the beginning of Sec. III.

II. GENERAL PICTURE

A. Apparent Paradox

Suppose for the sake of argument that $\epsilon\mathcal{H}'$ is $-\hat{u}_8$, an $I=0$, $Y=0$, even-parity member of $(3,\bar{3})\oplus(\bar{3},3)$. (We write $-\hat{u}_8$ rather than just u_8 for reasons which will become apparent later.) Then according to Eq. (2),

$$m_\alpha{}^2=-\langle\mathbf{P}_\alpha|\hat{u}_8(0)|\mathbf{P}_\alpha\rangle+O(\epsilon^2)$$
$$=(\text{const})d_{\alpha\alpha 8}+O(\epsilon^2)\,, \qquad (4)$$

[3] T. Kuo, Phys. Rev. D 2, 342 (1970).
[4] S. Okubo and V. Mathur, Phys. Rev. D 2, 394 (1970).
[5] V. Mathur and S. Okubo, Phys. Rev. D 1, 2046 (1970).
[6] M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1969).
[7] S. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968).

in which we have *naively* used $SU(3)$, which is supposed to be exact in the limit $\epsilon \to 0$, to evaluate the matrix element. The trouble with Eq. (4) is that $d_{\alpha\alpha 8}$ changes sign as α varies so that at least one m_α^2 must be negative. Where is the mistake? We have tacitly assumed in Eq. (4) that the $SU(3)$ group under which π, K, and η form an exact octet when $\epsilon=0$ is the same $SU(3)$ under which $-\hat{u}_8$ is the eighth component of an octet. This is not necessarily the case. The two may differ by an $SU(3) \otimes SU(3)$ rotation.

Some insight into the above problem can be gained by looking at the behavior of a ferromagnet in an external magnetic field. This will also give us an opportunity to state some basic facts about theories with degenerate vacua.

In the absence of an external field, a ferromagnet has an infinity of ground states $|\mathbf{M}\rangle$ which are labeled by a direction of magnetization $\mathbf{M}$. These states lie in different orthogonal Hilbert spaces. The Hilbert space built on any given ground state $|\mathbf{M}\rangle$ contains enough states to make a complete theory of the ferromagnet. Strictly speaking, then, we ought to think of $\mathbf{M}$ as defining a particular theory rather than a particular state. We will not always make this distinction, but the reader should take care to remember that since $|\mathbf{M}\rangle$'s lie in different Hilbert spaces, operations such as taking linear combinations of these ground states are not allowed. Also, it is intuitively clear that general angular momenta $\mathbf{J}$ and rotations $e^{i\omega\cdot\mathbf{J}}$ cannot exist in any one of the Hilbert spaces. Clearly, only rotations around an axis parallel to $\mathbf{M}$ can be defined in the particular Hilbert space built on $|\mathbf{M}\rangle$. A local spin density $\mathbf{S}(\mathbf{x})$ does exist and one can write a formal expression $\mathbf{J}=\int \mathbf{S}(\mathbf{x})d^3x$ for the angular momentum, but it is not really well defined because the integral will in general diverge at spatial infinity. The mathematical structure of more complicated theories with degenerate vacua or ground states is similar.

If the magnet is placed in an external field, the ground state is unique. Obviously, the state of lowest energy is attained when the magnetization is parallel to the external field. We can say, then, that a perturbing external field picks out one of the many Hilbert spaces which are equivalent as far as H_0 is concerned. If the perturbation is turned off, the magnet remains in this particular Hilbert space.

The Goldstone bosons in a magnet are spin waves and there is a formula similar to Eq. (2) which gives their "mass" (energy at rest) to first order in an external field. Now suppose that all our lives we had been talking about systems magnetized along the z axis. Someone might then calculate the first-order mass of spin waves for a magnet polarized along the z axis but with a perturbing field pointing along the x axis. Upon doing so he would find complex masses, just as Eq. (4) appears to give complex masses. In this case the interpretation is quite clear. A magnet polarized along the z axis is unstable when placed in an external field along the x axis. The complex-mass spin waves are simply an indication of this instability. The real masses of spin waves are, of course, to be computed by perturbing a magnet with a field pointing along its direction of magnetization.

The way to resolve the paradox of Eq. (4) is now clear. If $SU(3)\otimes SU(3)$ were as simple as the rotation group, things would be as easy and intuitive as in the ferromagnetic case. It is not, however; therefore we have to develop some machinery.

B. Two Theorems

Below we state and prove two theorems which are not quite trivial generalizations of the elementary variational principle for the lowest-energy level of a Hamiltonian.

Let $j_a{}^\mu(\mathbf{x},t)$, $a=1, 2, \ldots, n$, be currents satisfying

$$[j_a{}^0(\mathbf{x},t),j_b{}^0(\mathbf{y},t)]=iC_{abc}\delta^3(\mathbf{x}-\mathbf{y})j_c{}^0(\mathbf{x},t)\,, \qquad (5)$$

where C_{abc} are the structure constants of a compact Lie group. Furthermore, assume that the Hamiltonian H and its density $\mathcal{H}$ can be written in the form

$$H=H_0+\epsilon H'\,,$$
$$H=\int \mathcal{H}d^3x=\int \mathcal{H}_0 d^3x+\epsilon\int \mathcal{H}' d^3x\,, \qquad (6)$$

where the term $\epsilon\mathcal{H}'$ has the property that

$$i\epsilon[\mathcal{H}'(\mathbf{x},t),j_a{}^0(\mathbf{y},t)]=\delta^3(\mathbf{x}-\mathbf{y})\partial_\mu j_a{}^\mu(\mathbf{x},t)\,. \qquad (7)$$

Now let us define *formal* charges Q_a by

$$Q_a=\int j^0(\mathbf{x},0)d^3x\,, \qquad (8)$$

which may not exist as operators because of troubles related to the integration over infinite space. Nevertheless, by virtue of the δ function in (7), the commutator of Q_a with $\mathcal{H}'$ exists and is

$$-i\epsilon[Q_a,\mathcal{H}'(0)]=\partial_\mu j_a{}^\mu(0)\,. \qquad (9)$$

Assuming further local commutators of the general form

$$[j_a{}^0(\mathbf{x},t),\partial_\mu j_b{}^\mu(\mathbf{y},t)]=i\delta^3(\mathbf{x}-\mathbf{y})\sigma_{ab}(\mathbf{x})\,, \qquad (10)$$

where σ_{ab} is some local operator, allows one to define the double commutator of the Q's with $\mathcal{H}'$, i.e.,

$$-\epsilon[Q_a,[Q_b,\mathcal{H}'(0)]]=\sigma_{ab}(0)\,. \qquad (11)$$

Continuing in this way, one can clearly define a finite transformation of $\mathcal{H}'(0)$ through the formula

$$e^{i\omega\cdot\mathbf{Q}}\mathcal{H}'(0)e^{-i\omega\cdot\mathbf{Q}}=\mathcal{H}'(0)+i[\omega\cdot\mathbf{Q},\mathcal{H}'(0)]+\cdots\,, \quad (12)$$

where $\omega=(\omega_1,\ldots,\omega_n)$ are the group parameters. The point here is that $e^{i\omega\cdot\mathbf{Q}}\mathcal{H}'(0)e^{-i\omega\cdot\mathbf{Q}}$ exists as a local operator even when $\mathbf{Q}$ itself fails to exist because of the spatial integration in (8). Note also that terms like $\mathbf{V}_x\delta(\mathbf{x}-\mathbf{y})$ on the right-hand side of Eq. (10) can be allowed and will have no effect on $e^{i\omega\cdot\mathbf{Q}}\mathcal{H}'(0)e^{-i\omega\cdot\mathbf{Q}}$. We now state and prove a general theorem.

Theorem 1. Let $|\text{vac}\rangle$ be the vacuum state for $H=H_0+\epsilon H'$; then with the above notation,

$$\langle\text{vac}|e^{i\omega\cdot\mathbf{Q}}\epsilon\mathcal{H}'(0)e^{-i\omega\cdot\mathbf{Q}}|\text{vac}\rangle,$$

the vacuum expectation value considered, is a function of $\boldsymbol{\omega}$ and has a (at least local) minimum at $\boldsymbol{\omega}=0$.

Proof. Let $F(\boldsymbol{\omega})$ equal $\langle\text{vac}|e^{i\omega\cdot\mathbf{Q}}\epsilon\mathcal{H}'(0)e^{-i\omega\cdot\mathbf{Q}}|\text{vac}\rangle$. First, we prove that $\boldsymbol{\omega}=0$ is a stationary point of $F(\boldsymbol{\omega})$. This is almost trivial since $\partial F(0)/\partial\omega_a$ is equal to $\langle\text{vac}|[Q_a,\mathcal{H}'(0)]|\text{vac}\rangle=\langle\text{vac}|\partial_\mu j_a{}^\mu(0)|\text{vac}\rangle=0$ by Eq. (10) and translational invariance. To show that $\boldsymbol{\omega}=0$ is actually a local minimum, we need the matrix of second derivatives

$$\begin{aligned}N_{ab}&\equiv\frac{\partial^2F(0)}{\partial\omega_a\partial\omega_b}\\&=\langle\text{vac}|[Q_a,[Q_b,\epsilon\mathcal{H}'(0)]]|\text{vac}\rangle\\&=\langle\text{vac}|[Q_b,[Q_a,\epsilon\mathcal{H}'(0)]]|\text{vac}\rangle\\&=\langle\text{vac}|\sigma_{ab}(0)|\text{vac}\rangle=\langle\text{vac}|\sigma_{ba}(0)|\text{vac}\rangle,\end{aligned}\tag{13}$$

where the symmetry of the right-hand side (under $a\leftrightarrow b$) is a consequence of the Jacobi identity [Eq. (5)] and the vanishing expectation value of $[Q_c,\mathcal{H}'(0)]$. Since N_{ab} is a real symmetric matrix, it can be diagonalized by an orthogonal transformation. Let us suppose that this has been done and call the transformed matrix $\tilde{N}_{ab}=\delta_{ab}\chi_a$, where χ_a are the eigenvalues. Applying the same numerical index transformation to currents, divergences, and so on, we obtain new operators $\tilde{j}_a{}^\mu$, $\partial_\mu\tilde{j}_a{}^\mu$, $\tilde{\sigma}_{ab}$, etc. We are, of course, looking for a positivity condition on the eigenvalues χ_a, To this end, we note that using Eq. (10) and standard current-algebra techniques, one has

$$\begin{aligned}\langle\text{vac}|\tilde{\sigma}_{aa}(0)|\text{vac}\rangle&=\chi_a\\&=i\int d^4x\langle\text{vac}|T^*(\partial_\mu\tilde{j}_a{}^\mu(x),\partial_\mu\tilde{j}_a{}^\mu(0))|\text{vac}\rangle,\end{aligned}\tag{14}$$

where T^* is the covariant time-ordering symbol, which we assume takes care of any Schwinger terms which should be added to Eq. (10). The time-ordered product on the right-hand side of (14) is one of those which is well known to be non-negative. More precisely, for each index a, either the integrated time-ordered product is positive definite, or else the operator $\partial_\mu\tilde{j}_a{}^\mu(0)$ annihilates the vacuum and hence vanishes identically. Thus each eigenvalue χ_a is strictly positive or else the corresponding current $\tilde{j}_a{}^\mu$ is conserved and Q_a commutes with $\mathcal{H}'$. From this, it clearly follows that $\boldsymbol{\omega}=0$ is an (at least local) minimum of $F(\omega)$.

Question. Is $\boldsymbol{\omega}=0$ always a global minimum of $F(\boldsymbol{\omega})$? We conjecture that it is, on the basis that if the operators $e^{-i\omega\cdot\mathbf{Q}}$ exist and can be applied to $|\text{vac}\rangle$, then Theorem 1 is a direct consequence of the elementary variational principle for the lowest-energy state of H. The minimum in the elementary variational principle is a global one. We pause only to note that $F(\boldsymbol{\omega})$ does have a global minimum somewhere. We may assume that F is continuous. Furthermore, the parameter space of a compact group is compact. By the usual theorems then, $F(\boldsymbol{\omega})$ does have a global minimum.

Now let us assume that the parameter ϵ in $H=H_0+\epsilon H'$ is small. If we expand in powers of ϵ, an interesting result emerges.

Theorem 2. Let $|0\rangle$ be the vacuum of H_0, $H_0|0\rangle=0$, which is the limit of the vacuum of $H_0+\epsilon H'$ as $\epsilon\to0$; i.e., $|0\rangle=\lim_{\epsilon\to0}|\text{vac}\rangle$, where $(H_0+\epsilon H')|\text{vac}\rangle=0$. Then $\langle0|e^{i\omega\cdot\mathbf{Q}}\epsilon\mathcal{H}'(0)e^{-i\omega\cdot\mathbf{Q}}|0\rangle$ considered as a function of $\boldsymbol{\omega}$ has a minimum at $\boldsymbol{\omega}=0$.

This result follows immediately from Theorem 1.

We can use Theorem 2 to answer the following kind of question. Suppose we have some symmetrical Hamiltonian H_0 which has a degenerate vacuum. We might chose one particular vacuum $|0\rangle$ and ask what perturbations $\epsilon H'$ have the property that the vacuum for $H_0+\epsilon H'$ goes over to $|0\rangle$ as $\epsilon\to0$. The answer is, of course, just those perturbations whose densities satisfy Theorem 2. For example, let H_0 be the Hamiltonian for a ferromagnet in the absence of external fields, and let $\epsilon\mathcal{H}'$ be $-\mu\mathbf{S}\cdot\mathbf{B}$, where $\mathbf{S}$ is the spin density, $\mathbf{B}$ is the external field, and μ is a number. The Q's are now angular momenta $\mathbf{J}$, and $|0\rangle$ becomes some particular ground state $|\mathbf{M}\rangle$. The matrix element in Theorem 2 is then

$$\begin{aligned}-\mu\mathbf{B}\cdot\langle\mathbf{M}|e^{i\omega\cdot\mathbf{J}}\mathbf{S}(0)e^{i\omega\cdot\mathbf{J}}|\mathbf{M}\rangle&\\&=-\mu\sum_{ij}B_iR_{ij}(\boldsymbol{\omega})\langle\mathbf{M}|S_j|\mathbf{M}\rangle\\&=-\sum_{ij}B_iR_{ij}(\boldsymbol{\omega})M_j,\end{aligned}\tag{15}$$

since $\mu<\mathbf{M}|\mathbf{S}|\mathbf{M}>=\mathbf{M}$, where R_{ij} is the obvious rotation matrix. Clearly, $-\sum_{ij}B_iR_{ij}(\boldsymbol{\omega})M_j$ is a minimum at $\boldsymbol{\omega}=0$ if and only if $\mathbf{B}$ and $\mathbf{M}$ are parallel. Thus the state $|\mathbf{M}\rangle$ is obtained as the $\epsilon\to0$ limit of the perturbed ground state only if $\mathbf{B}$ is parallel to $\mathbf{M}$. This is the expected result.

C. Symmetry Breaking in Theories with Goldstone Bosons

Let the group generated by the formal charges $Q_1, Q_2,\ldots, Q_n$ be called $\mathcal{G}$. Evidently, $\mathcal{G}$ is the symmetry group of H_0 with $\epsilon H'$ breaking the symmetry. As before, we denote the $\epsilon=0$ limit of the vacuum of H by $|0\rangle$. In general, $|0\rangle$ will be invariant only under some subgroup $\mathcal{G}'\subset\mathcal{G}$. Let us label the generators Q_a so that $Q_1, Q_2,\ldots, Q_m$ $(m\leq n)$ are the generators of $\mathcal{G}'$.

They have property that

$$Q_a|0\rangle=0, \quad a=1,2,\ldots,m. \tag{16}$$

The remaining currents j_a^μ, $a=m+1, m+2,\ldots, n$, must make Goldstone bosons out of the vacuum. That is, there must be $n-m$ spinless bosons ξ_a whose mass vanishes in the limit $\epsilon=0$ and which have matrix elements

$$\langle\xi_a(q)|j_a^\mu(0)|\mathrm{vac}\rangle=-iq^\mu(2f_a)^{-1}, \quad a=m+1, m+2,\ldots,n \tag{17}$$

satisfying

$$\lim_{\epsilon\to 0}(2f_a)^{-1}\neq 0, \quad a=m+1, m+2,\ldots,n. \tag{18}$$

Let us recall the physical meaning[2] of $\mathcal{G}$ and $\mathcal{G}'$. It is that for small enough ϵ, the particle states of $H=H_0+\epsilon H'$ will fall into (slightly split) multiplets corresponding to irreducible representations of $\mathcal{G}'$. The full group $\mathcal{G}$ does not produce multiplets. The rest of the approximate symmetry is accounted for by the (now slightly massive) bosons ξ_a. They dominate the divergences $\partial_\mu j_a^\mu$ for $a>m$ in a PCAC-like way and thus satisfy low-energy theorems, etc., in the usual way.

Note that we have not yet said how $\mathcal{G}'$ is determined. The first question is to what extent is $\mathcal{G}'$ determined by H_0 alone. To answer this, let us note that since H_0 is invariant under $\mathcal{G}$, then if one $\mathcal{G}'$ is a possible candidate so must be $U\mathcal{G}'U^{-1}$ where U is any transformation in $\mathcal{G}$. (The notation $U\mathcal{G}'U^{-1}$ means take all transformations V in $\mathcal{G}'$ and replace them with UVU^{-1}.) Moreover, we can assume that the transformations $\mathcal{G}'\to U\mathcal{G}'U^{-1}$ are all the freedom that H_0 allows. Further freedom would imply additional symmetry in H_0. Thus, looking at the symmetric Hamiltonian alone tells us that $\mathcal{G}'$ belongs to a class of (isomorphic) subgroups of $\mathcal{G}$. The perturbation $\epsilon H'$ then has to select a particular member of this class. Note that if the vacuum of $H_0+\epsilon H'$ is unique then, because of the condition $|0\rangle=\lim_{\epsilon\to 0}\times|\mathrm{vac}\rangle$, $\mathcal{G}'$ must be unique.

We leave to the reader the instructive exercise of translating the ferromagnetic example into the above abstract language.

It should be recognized at this point that for many purposes, one does not really need to pin down $\mathcal{G}'$. After all, $\mathcal{G}'\to U\mathcal{G}'U^{-1}$ is just a transformation of coordinates in group space. Many things in our approximately symmetrical world are independent of such a change of coordinates. Examples are the numbers of particles in $\mathcal{G}'$ multiplets and the number of (almost) Goldstone bosons. In fact, it is rather easy to see that pinning down $\mathcal{G}'$ is necessary only if we wish to look at the details of symmetry breaking. In the latter case, however, it is mandatory. At the risk of repeating the obvious, let us return to the paradox of Eq. (4) for an illustration of this point. There H_0 is supposed to be $SU(3)\otimes SU(3)$ symmetric. The perturbation $-\epsilon\hat{u}_8$ is to be thought of as being the eighth component of an octet with respect to some standard $SU(3)$ subgroup of $SU(3)\otimes SU(3)$, call it $S\hat{U}(3)$. In Eq. (4) we assumed that $\mathcal{G}'=S\hat{U}(3)$, but it really is $U^{-1}S\hat{U}(3)U$, where U is some $SU(3)\otimes SU(3)$ transformation. With respect to $U^{-1}S\hat{U}(3)U$, which is the ordinary $SU(3)$ one uses to classify states, $-\hat{u}_8$ is *not* just the eighth component of an octet. Hence Eq. (4) is wrong. Below we show how to compute U explicitly.

D. Resolving the Paradox

We can use Theorem 2 to compute the U which relates $\mathcal{G}'$ and $S\hat{U}(3)$ in the example mentioned above. The full group $\mathcal{G}$ is now $SU(3)\otimes SU(3)$ and $\mathcal{G}'$ is ordinary $SU(3)$.

If u_8 is the member of $(3,\bar{3})\oplus(\bar{3},3)$ which is the eighth component of an octet with respect to ordinary $SU(3)$ [not $S\hat{U}(3)$] then, by definition,

$$\epsilon\mathcal{H}'=-\hat{u}_8=-Uu_8U^{-1}. \tag{19}$$

This form for $\epsilon\mathcal{H}'$ is convenient because Theorem 2 then tells us that the vacuum expectation value

$$-\langle 0|VUu_8U^{-1}V^{-1}|0\rangle$$

considered as a function of the arbitrary $SU(3)\otimes SU(3)$ transformation V is a minimum when V is the identity transformation. Combining V and U into one transformation W, this is equivalent to saying the $-\langle 0|Wu_8W^{-1}|0\rangle$ will have a minimum when $W=U$. To parametrize W it is easier not to use the exponential form $e^{i\omega\cdot Q}$, but rather to label W by the two three-by-three unitary matrices with determinant 1, and call them α and β, which appear in the defining representation of $SU(3)\otimes SU(3)$. Thus, we write $W(\alpha,\beta)$ where α and β are suitable matrices. The special case $W(\alpha,\alpha^{-1})$ is a pure $SU(3)$ rotation.

Now since u_8 is a member of $(3,\bar{3})\oplus(\bar{3},3)$, one knows that

$$W(\alpha,\beta)u_8W^{-1}(\alpha,\beta)=d_{80}(\alpha,\beta)u_0 +[\text{other members of } (3,\bar{3})\oplus(\bar{3},3)], \tag{20}$$

with

$$d_{80}(\alpha,\beta)=\mathrm{Re}(\tfrac{1}{2}\sqrt{\tfrac{2}{3}})\,\mathrm{Tr}(\alpha\lambda_8\beta)=\mathrm{Re}(\tfrac{1}{2}\sqrt{\tfrac{2}{3}})\,\mathrm{Tr}(\beta\alpha\lambda_8), \tag{21}$$

where u_0 is the even-parity $SU(3)$ singlet in $(\bar{3},3)\oplus(\bar{3},3)$, and λ_8 is the usual $SU(3)$ matrix

$$\lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}. \tag{22}$$

Equation (20) is useful because u_0 is the only member of $(3,\bar{3})\oplus(\bar{3},3)$ which can have an expectation value in the vacuum $|0\rangle$; thus

$$-\langle 0|W(\alpha,\beta)u_8W^{-1}(\alpha,\beta)|0\rangle =-\mathrm{Re}(\tfrac{1}{2}\sqrt{\tfrac{2}{3}})\,\mathrm{Tr}(\beta\alpha\lambda_8)\langle 0|u_0|0\rangle. \tag{23}$$

It is obvious now, that once the sign of $\langle 0|u_0|0\rangle$ is known, the problem has to be reduced to maximizing or minimizing $\mathrm{Re}\,\mathrm{tr}(\beta\alpha\lambda_8)$. We shall assume for this example that $\langle 0|u_0|0\rangle$ is negative. The minimum of $\mathrm{Re}\,\mathrm{tr}(\beta\alpha\lambda_8)$ is then required. To find it, we define the matrix

$$\varphi=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{24}$$

and write

$$\begin{aligned}\mathrm{Re}\,\mathrm{tr}(\alpha\beta\lambda_8)&=\mathrm{Re}\,\mathrm{tr}(\alpha\beta\varphi^{-1}\varphi\lambda_8)\\ &=-\mathrm{Re}\{(1/\sqrt{3})[(\alpha\beta\varphi^{-1})_{11}+(\alpha\beta\varphi^{-1})_{22}+2(\alpha\beta\varphi^{-1})_{33}]\}\\ &\geq-(1/\sqrt{3})(4)\end{aligned} \tag{25}$$

in an obvious notation. The inequality in the last line of Eq. (25) follows from the fact that $\alpha\beta\varphi^{-1}$ is unitarity and the two sides are equal if and only if $\alpha\beta\varphi^{-1}$ is the unit matrix. Thus, the W which minimizes

$$-\langle 0|Wu_8W^{-1}|0\rangle$$

is one for which $\alpha\beta=\varphi$. This determines W up to a pure $SU(3)$ rotation which can clearly be ignored. Without any loss of generality then, we can take $W=W(0,\varphi)$. Working out Eq. (19) one then finds

$$\epsilon\mathcal{H}'=\tfrac{1}{3}(2\sqrt{2}u_0-u_8)\,. \tag{26}$$

It will be shown below that this $\epsilon\mathcal{H}'$ gives positive squares for the meson masses.

E. Positivity of the Squares of Meson Masses

We will now show directly that the squares of the masses of Goldstone bosons come out positive in order ϵ. First, we do the general case and then return to the example.

It was shown in Ref. 2 that in order ϵ the mass-squared matrix for the Goldstone particles $\mu^2{}_{ab}$ is given by

$$\mu^2{}_{ab}=4f_af_b\langle 0|\sigma_{ab}(0)|0\rangle+O(\epsilon^2)\,,\quad a,b=m+1,m+2,\ldots,n\,, \tag{27}$$

where σ_{ab} and the f's are defined by Eqs. (10) and (17). The matrix $\mu^2{}_{ab}$ clearly has non-negative eigenvalues if and only if the matrix $\langle 0|\sigma_{ab}(0)|0\rangle$ does. But as one can see from the proof of Theorem 1, $\langle 0|\sigma_{ab}(0)|0\rangle$ has non-negative eigenvalues provided only that $|0\rangle=\lim_{\epsilon\to 0}\times|\mathrm{vac}\rangle$. Therefore, if one does things correctly, the masses squared always come out positive.

For the $\epsilon\mathcal{H}'$ given in Eq. (26), $\langle 0|\sigma_{ab}|0\rangle$ is easily evaluated; one finds

$$\langle 0|\sigma_{ab}|0\rangle=\tfrac{1}{2}(\sqrt{\tfrac{2}{3}})\,\mathrm{Re}\,\mathrm{tr}[(\lambda_a\lambda_b\varphi\lambda_8)]\langle 0|u_0|0\rangle\,,\quad a,b=1,2,\ldots,8\,. \tag{28}$$

The eigenvalues are $-(\sqrt{3})\langle 0|u_0|0\rangle$, $-(5\sqrt{2}/9)\langle 0|u_0|0\rangle$, and $(-\sqrt{2}/2)\langle 0|u_0|0\rangle$ which are all positive by virtue of our assumption that $\langle 0|u_0|0\rangle<0$. From Eq. (27) and the fact that $f_\pi=f_\eta=f_K$ to zeroth order in ϵ, one can compute the masses for π, K, and η. Their squares come out positive but wrong. The $\mathcal{H}'$ of Eq. (26) is simply an example, not a correct theory of $SU(3)\otimes SU(3)$ breaking.

F. Kuo's Transformations

Kuo[3] has pointed out that transformations of the form $-u_8\to\frac{1}{3}(2\sqrt{2}u_0-u_8)$, which carries Eq. (4) into Eq. (26), appear to make $SU(3)\otimes SU(3)$ breaking ambiguous. This is not the case, however. The reason that it *looks* as though there might be an ambiguity is that operators connected by Kuo's transformations *appear* to have different $SU(3)$ properties. One just has to remember that physically $SU(3)$ is that subgroup of $SU(3)\otimes SU(3)$ that leaves $|0\rangle$ invariant. Therefore what one means by $SU(3)$ depends on $\epsilon\mathcal{H}'$. When everything is worked out, one will find that the *physical* $SU(3)$ properties of $\epsilon\mathcal{H}'$ are unchanged by Kuo's transformations.

III. SOME DETAILS ABOUT $SU(3)\otimes SU(3)$

A. Assumptions and Conventions

The Q's of the previous section become the vector and axial-vector charges defined by

$$F_a=\int V_a{}^0(x,0)d^3x\,,\qquad F_a{}^5=\int A_a{}^0(\mathbf{x},0)d^3x\,,\quad a=1,2,\ldots,8 \tag{29}$$

where $V_a{}^\mu$ and $A_a{}^\mu$ are the vector and axial-vector currents whose time components are supposed to satisfy Gell-Mann's well-known $SU(3)\otimes SU(3)$ commutation relations.

Breaking the strong Hamiltonian up into $H_0+\epsilon H'$ as in Eq. (1), *it is assumed that $\epsilon H'$ is small enough so that the $SU(3)\otimes SU(3)$-symmetric world described by H_0 alone bears some resemblance to the real strong interactions.* Allowing this assumption, we need to ask what the symmetric world looks like. Defining as usual $|0\rangle$ as the limit of the vacuum of $H_0+\epsilon H'$ as $\epsilon\to 0$, *it is further assumed that the group which leaves $|0\rangle$ invariant is some $SU(3)$ subgroup of $SU(3)\otimes SU(3)$.* Just which $SU(3)$ subgroup this is depends, as we saw in Sec. II, on the properties of $\epsilon H'$. As has been discussed previously, the above assumptions lead to a world with approximate $SU(3)$ multiplets of particles and approximate PCAC for π, K, and η.

By a suitable choice of coordinates in $SU(3)\otimes SU(3)$ space, we can always take the $SU(3)$ group leaving $|0\rangle$ invariant to be that generated by the vector charges. Therefore, we write

$$F_a|0\rangle=0\,,\quad a=1,2,\ldots,8\quad\text{(convention)}. \tag{30}$$

Having made this convention, $\epsilon H'$ can no longer have arbitrary properties with respect to the $SU(3)$ group generated by $F_1, F_2, \ldots, F_8$. We saw this in detail in Sec. II and there is no need to go through the arguments again. From now on $SU(3)$ will be assumed to have the usual meaning, i.e., the group generated by the vector charges.

The eight pseudoscalar mesons π, K, and η become massless Goldstone bosons in the limit $\epsilon=0$. They have matrix elements

$$\begin{aligned}\langle\pi(q)|A_\pi{}^\mu(0)|0\rangle&=-q^\mu/2f_\pi\,,\\ \langle K(q)|A_K{}^\mu(0)|0\rangle&=-q^\mu/2f_K\,,\\ \langle\eta(q)|A_\eta{}^\mu(0)|0\rangle&=-q^\mu/2f_\eta\end{aligned}\tag{31}$$

in an obvious notation. As ϵ goes to zero, f_π^{-1}, f_K^{-1}, and f_η^{-1} all approach a common finite number f^{-1}.

B. Digression on κ Mesons

There seems to be a (perhaps natural) desire to have a strange scalar "κ" meson whose role would be to dominate the divergence of the strange vector currents. From a theoretical point of view the κ, if it does turn out to exist, would be a peculiar object. It sounds as though "κ dominance" would be similar to "pion dominance" or ordinary PCAC. This is not the case, however.

As explained in Ref. 2, the dominance of divergences of axial-vector currents by the pseudoscalar-meson poles is a consequence of the special role of π, K, and η in the limit $\epsilon=0$. If we want κ dominance to come about for the same reason, then the symmetry group of $|0\rangle$ should be $SU(2)$ not $SU(3)$. This would make κ a Goldstone boson like π, K, and η and would ensure κ dominance up to terms of order ϵ. The trouble with this idea is that if the symmetry group of $|0\rangle$ is only $SU(2)$, then there is no reason whatsoever to have $SU(3)$ multiplets of particles. For example, one can write an $SU(3)$-symmetric nonlinear Lagrangian in which the only particles are a massless κ and, say, a nucleon. This is not what we think of as $SU(3)$ symmetry. Of course, one can conceive of a theory where the κ is a Goldstone boson and $SU(3)$ groupings of states still exist. While such a theory is rather arbitrary and contrived, it is a possibility. Suppose such a picture were correct. Then a typical κ dominance relation would be

$$(M_N-M_\Lambda)g_{N\Lambda}{}^V=G_{\Lambda N\kappa}/2f_\kappa+O(\epsilon)\,,\tag{32}$$

where $g_{N\Lambda}{}^V$ is the matrix element of the vector charge between nucleon and Λ, and the error term $+O(\epsilon)$ has been included to remind the reader that our hypothetical κ-dominance contains errors of order ϵ. In the limit $\epsilon\to 0$, Eq. (32) becomes exact and nontrivial since M_N and M_Λ are no longer equal in the symmetry limit. For the real case with $\epsilon\neq 0$, Eq. (32) will be useful only if the error term is considerably smaller than $(M_N-M_\Lambda)g_{N\Lambda}{}^V$, or in other words, if the explicit $SU(3)$ breaking contained in $\epsilon\mathcal{H}'$ is smaller than the spontaneous breaking contained in the asymmetrical vacuum of H_0. This is unlikely to be the situation since, if it were, the mass squared of the κ should be small compared to a typical $SU(3)$ breaking like $m_{K^*}{}^2-m_\rho{}^2$.

We see then that there is no way to get a sensible, clean prediction of either the existence of a κ meson or meaningful κ dominance out of $SU(3)\otimes SU(3)$ symmetry. Furthermore, there is no experimental evidence for a strange scalar meson at a low enough mass to be interesting. It is for this reason that we take the vacuum of H_0 to be fully $SU(3)$, not just $SU(2)$, symmetric.

If the reader is still bothered by the apparent lack of symmetry between the vector and axial-vector divergences, the following point should be understood. It is the structure of H_0 which determines (up to a group rotation) the invariance group of $|0\rangle$ and hence the number of Goldstone bosons. The number of currents which have divergences, on the other hand, is determined by $\epsilon H'$. In general, there is no connection between the number of Goldstone bosons and the number of divergences.

These remarks should not be interpreted as meaning that scalar mesons cannot, for some unknown dynamical reason, still play an important role in symmetry breaking. What we have in mind here is something like the tadpole model of Coleman and Glashow. There is, of course, nothing wrong with using an "effective" κ pole to parametrize the strange vector divergences.

C. $(3,\bar{3})\oplus(\bar{3},3)$ Symmetry Breaking

It has been suggested by Gell-Mann, Oakes, and Renner[6] and by Glashow and Weinberg[7] that $\mathcal{H}'$ belongs to the $(3,\bar{3})\otimes(\bar{3},3)$ representation of $SU(3)\otimes SU(3)$. In the following, we shall assume that

(i) $\mathcal{H}'$ belongs to $(3,\bar{3})\otimes(\bar{3},3)$,
(ii) $\mathcal{H}'$ is invariant under some $SU(2)$ subgroup of $SU(3)\otimes SU(3)$ which can become isospin,
(iii) $\mathcal{H}'$ is invariant under CP *or* some $SU(3)\otimes SU(3)$ transformations of the usual CP operation.

Then it is easy to show that $\epsilon\mathcal{H}'$ can always be written in one of the two forms

$$\epsilon\mathcal{H}'=U(u_0-\sqrt{2}u_8+2\delta u_0+\sqrt{2}\delta u_8)U^{-1},\quad 0\leq\delta\tag{33}$$

or

$$\epsilon\mathcal{H}'=-U(u_0-\sqrt{2}u_8+2\delta u_0+\sqrt{2}\delta u_8)U^{-1},\quad 0\leq\delta\tag{34}$$

where U is some $SU(3)\otimes SU(3)$ transformation. We must distinguish between Eqs. (33) and (34), which differ only by a sign, because there is no $SU(3)\otimes SU(3)$ transformation that takes one into the other. By considering both cases, we are free to make the *convention*

$$\langle 0|u_0|0\rangle<0\,.\tag{35}$$

The parameter δ can be restricted to positive values, since there is an $SU(3)\otimes SU(3)$ transformation which has the effect of taking δ to $-\delta$.

We now want to determine the U's in Eqs. (33) and (34). To do this, we require that the symmetry group of $|0\rangle$ be ordinary $SU(3)$ and use Theorem 2. The calculation is almost identical to the example worked out in Sec. II. Using the notation defined in Sec. II, we write

$$\langle 0|W(\alpha,\beta)(u_0-\sqrt{2}u_8+2\delta u_0+\sqrt{2}\delta u_8)W^{-1}(\alpha,\beta)|0\rangle = \mathrm{Re}\,\mathrm{tr}(\alpha\beta M)\langle 0|u_0|0\rangle, \quad (36)$$

where the matrix M is

$$M=\begin{bmatrix}\delta & 0 & 0\\ 0 & \delta & 0\\ 0 & 0 & 1\end{bmatrix}. \quad (37)$$

If $\epsilon\mathcal{H}'$ is given by Eq. (33) then, taking account of Eq. (35), U is equal to that $W(\alpha,\beta)$ for which $\mathrm{Re}\,\mathrm{tr}(\alpha\beta M)$ is maximized. On the other hand, if $\epsilon\mathcal{H}'$ is given by Eq. (34) we evidently want the minimum of $\mathrm{Re}\,\mathrm{tr}(\alpha\beta M)$. It is easy to find the maximum of $\mathrm{Re}\,\mathrm{tr}(\alpha\beta M)$. We state without proof that it comes when $\alpha\beta$ is the unit matrix. Then without loss of generality we can take U to be the identity transformation. Hence,

$$\epsilon\mathcal{H}'=u_0-\sqrt{2}u_8+2\delta u_0+\sqrt{2}\delta u_8, \quad 0\leq\delta \quad (33')$$

for the case when $\epsilon\mathcal{H}'$ is given by Eq. (33). Finding the minimum is more difficult, but again we only state the answer. If $\delta\leq 2$, the minimum comes when

$$\alpha\beta=\begin{bmatrix}-\frac{1}{2}\delta\pm i(1-\frac{1}{4}\delta^2)^{1/2} & 0 & 0\\ 0 & -\frac{1}{2}\delta\pm i(1-\frac{1}{4}\delta^2)^{1/2} & 0\\ 0 & 0 & -1+\frac{1}{2}\delta^2\pm i\delta(1-\frac{1}{4}\delta^2)^{1/2}\end{bmatrix}, \quad (38)$$

and if $2\leq\delta$ it comes when

$$\alpha\beta=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}. \quad (39)$$

Up to an $SU(3)$ rotation, the Hamiltonians for Eq. (34) are

$$\epsilon\mathcal{H}'=(1+\tfrac{1}{2}\delta^2)u_0-\sqrt{2}(1-\delta^2)u_8\pm 3\delta(1-\tfrac{1}{4}\delta^2)^{1/2}v_0, \quad 0\leq\delta\leq 2 \quad (34')$$

and

$$\epsilon\mathcal{H}'=-u_0+\sqrt{2}u_8+2\delta u_0+\sqrt{2}\delta u_8, \quad 2\leq\delta \quad (34'')$$

where v_0 is the odd parity $SU(3)$ singlet in $(3,\bar{3})\oplus(\bar{3},3)$. With assumptions (i)–(iii) above and with the convention $\langle 0|u_0|0\rangle<0$, Eqs. (33′), (34′), and (34″) list all possible symmetry breaking Hamiltonians for which $|0\rangle$ is invariant under ordinary $SU(3)$. Note that the $\mathcal{H}'$ of Eq. (34′) violates CP. We will return to this point later.

Kuo's transformations take the Hamiltonians of Eqs. (33′), (34′), and (34″) into objects not in this list. Thus applying one of Kuo's transformations to $\epsilon\mathcal{H}'$ takes one to a theory where the symmetry group of $|0\rangle$ is not ordinary $SU(3)$ but some transformed $SU(3)$.

If we assume that the true $\epsilon\mathcal{H}'$ belongs to $(3,\bar{3})\oplus(\bar{3},3)$, it is not hard to see which member of the above list is relevant. The Hamiltonian in Eq. (34′) violates CP and is out. The one in Eq. (34″) gives, by way of Eq. (27), $m_\eta^2\leq m_\pi^2$ and is also out. This leaves Eq. (33′) which, with small δ to get a small m_π^2, is the Gell-Mann–Oakes–Renner model.

D. Spontaneous *CP* Violation

As pointed out above, the Hamiltonian in Eq. (34′) violates CP. This may seem surprising since assumption (iii) above seems to preclude CP violation. The point here is that Eq. (34′) does not conserve the normal CP operator but is invariant under $UCPU^{-1}$, where U is the $SU(3)\otimes SU(3)$ transformation in Eq. (34). However, $UCPU^{-1}$ does not take $|0\rangle$ into itself and therefore the theory defined by Eq. (34′) violates CP. This is a real CP violation which cannot be "rotated away." To verify this, we will show in Sec. III E that for the $\epsilon\mathcal{H}'$ in Eq. (34′) the decay amplitude for $\eta\to 2\pi$ is nonvanishing in order ϵ.

The $\pm$ sign multiplying v_0 in Eq. (34′) indicates that there are two solutions for $\epsilon\mathcal{H}'$ in this case. These correspond to two separate worlds which are CP images of each other.

It would be nice to think that the CP violation observed in weak interactions comes about through a phenomenon like that described above. Clearly, Eq. (34′) has nothing to do with the observed CP violation. Equation (34′) gives CP violation whose strength is on the order of $SU(3)\otimes SU(3)$ breaking which is orders of magnitude too strong. The interesting thing about Eq. (34′) is that it shows that spontaneous CP violation can really occur and that, in certain circumstances, one can actually predict when it will occur. One simply has to take an otherwise harmless looking $\epsilon\mathcal{H}'$ of the class given by Eq. (34), choose $\delta<2$, and out comes CP violation. It is a very interesting question whether some more complex or simply more clever theory could give spontaneous CP-violating effects of the magnitude observed in weak interactions.

E. Domains of Mathur and Okubo

Mathur and Okubo[5] have pointed out that the parameters $\langle\mathrm{vac}|u_0|\mathrm{vac}\rangle$, $\langle\mathrm{vac}|u_8|\mathrm{vac}\rangle$, and $\sqrt{2}a$ = ratio of u_8 to u_0 in $\epsilon\mathcal{H}'$ cannot have arbitrary values, but are restricted to certain domains. Apart from the fact that they use $U(3)\otimes U(3)$ rather than $SU(3)\otimes SU(3)$, we can think of their results as just the requirement that Theorem 1 be satisfied. If $\epsilon\mathcal{H}'$ belongs to $(3,\bar{3})\oplus(\bar{3},3)$, then $\langle\text{vac}|W(\alpha\beta)\epsilon\mathcal{H}'W^{-1}(\alpha,\beta)|\text{vac}\rangle$ can be expressed in terms of $\langle\text{vac}|u_0|\text{vac}\rangle$, $\langle\text{vac}|u_8|\text{vac}\rangle$, and their parameter a. Requiring this quantity to be a (local) minimum when W is the identity gives the $SU(3)\otimes SU(3)$ version of Mathur and Okubo's domains. We shall not discuss the general case here but rather assume that ϵ is small so that $\langle\text{vac}|u_8|\text{vac}\rangle$ may be neglected and Theorem 1 becomes Theorem 2. Neglecting $\langle\text{vac}|u_8|\text{vac}\rangle$ in the formulas of Mathur and Okubo, being careful to use only $SU(3)\otimes SU(3)$ [not $U(3)\otimes U(3)$] results and remembering our convention $\langle 0|u_0|0\rangle<0$, one finds that

$$-1\leq a\leq 1 \tag{40}$$

is the allowed domain for a.

The work of Sec. III C is very illuminating with with regard to Eq. (40). For Eq. (33′) one has $a=(\delta-1)\times(1+2\delta)^{-1}$ with $0\leq\delta$, while for Eq. (34″) a is $(1+\delta)(-1+2\delta)^{-1}$ with $2\leq\delta$. One easily sees that these two cases cover the allowed range of a. Equations (33′) and (34″) join smoothly at $a=\frac{1}{2}$ or $\delta=\infty$. Nothing in particular happens at this point, because $\delta=\infty$ is only a singularity in our parametrization of $\epsilon\mathcal{H}'$ and not a true singularity in any physical sense. What happens if we try to push a out of the allowed range? Consider the point $a=-1$. We can get to the left of this by letting δ become negative in Eq. (33′), but since the resulting $\epsilon\mathcal{H}'$ is not in our list we have to make an $SU(3)\otimes SU(3)$ transformation to satisfy Theorem 2. The result is that one ends up with the CP-violating $\mathcal{H}'$ in Eq. (34′) for some small positive value of δ. Note that Eqs. (33′) and (34′) join continuously at $\delta=0$. Mathur and Okubo did not, of course, allow CP violation. The other end point $a=+1$ can be crossed by formally letting δ become less than 2 in Eq. (34″). Again an $SU(3)\otimes SU(3)$ rotation is required and again one ends up with Eq. (34′) but this time with δ near 2. As before, we note that Eqs. (34′) and (34″) join continuously at $\delta=2$ or $a=1$.

Evidently, the following picture has emerged. We may think of the parameter a as having any value between $-\infty$ and $+\infty$. If $|a|$ is greater than 1, however, an $SU(3)\otimes SU(3)$ rotation must be made and the result is a CP-violating Hamiltonian of the form in Eq. (34′). Since a is not an $SU(3)\otimes SU(3)$ invariant, this parameter loses its original significance after transforming to Eq. (34′).

Physically, the way one would know that $a=\pm1$ are peculiar points is that m_π^2 as computed from Eq. (27) vanishes at $a=-1$, while m_η^2 vanishes at $a=+1$. The vanishing of m_π^2 at $a=-1$ is a consequence of the fact that $\partial_\mu A_a{}^\mu$ for $a=1$, 2, 3 vanishes there and $SU(2)\otimes SU(2)$ is exact. On the other hand, at $a=-1$ the vanishing of m_η^2 to first order in ϵ is an accident; the divergence $\partial_\mu A_8{}^\mu$ is nonzero at $a=+1$. (Presumably m_η^2 is nonvanishing in order ϵ^2 at $a=+1$.) If one tries to go out of $-1\leq a\leq 1$ without rotating to Eq. (34′), then either m_π^2 or m_η^2 will become negative.

IV. MORE ABOUT $(3,\bar{3})\oplus(\bar{3},3)$

A. Formula

There is an amusing general formula for the trilinear couplings of Goldstone bosons. We will first derive it for the general case and then specialize to $SU(3)\otimes SU(3)$.

Using the notation of Sec. II, we define

$$F(q_a{}^2,q_b{}^2,q_c{}^2)=-i8f_af_bf_c\frac{m_a{}^2-q_a{}^2}{m_a{}^2}\frac{m_b{}^2-q_b{}^2}{m_b{}^2}\frac{m_c{}^2-q_c{}^2}{m_c{}^2}\times\int e^{iq_a\cdot x}e^{iq_b\cdot y}\langle 0|T^*(\partial_\mu j_a{}^\mu(x)\partial_\nu j_b{}^\nu(y)\partial_\lambda j_c{}^\lambda(0))|0\rangle\times d^4x d^4y\,, \tag{41}$$

where $q_c=-q_a-q_b$. By Eq. (17) one has

$$F(m_a{}^2,m_b{}^2,m_c{}^2)=-ig_{abc}\,, \tag{42}$$

where g_{abc} is the trilinear coupling of the Goldstone particles. Simple current-algebra calculations then give

$$F(0,m_b{}^2,m_c{}^2)=F(m_a{}^2,0,m_c{}^2)=F(m_a{}^2,m_b{}^2,0)=0 \tag{43}$$

and

$$F(0,0,0)=8f_af_bf_c[\langle 0|[Q_c,[Q_a,[Q_b,\epsilon\mathcal{H}'(0)]]]|0\rangle+\langle 0|[Q_b,[Q_a,[Q_c,\epsilon\mathcal{H}'(0)]]]|0\rangle]. \tag{44}$$

Now we expand F as

$$F(m_a{}^2,m_b{}^2,m_c{}^2)=F(0,0,0)+F_1(0,0,0)m_a{}^2+F_2(0,0,0)m_b{}^2+F_3(0,0,0)m_c{}^2+O(\epsilon^2)\,, \tag{45}$$

where $F_1(x,y,z)=(\partial/\partial x)F(x,y,z)$, etc. However, Eq. (43) then tells us that $F_1(0,0,0)m_a{}^2=F_2(0,0,0)m_b{}^2=F_3(0,0,0)\times m_c{}^2=-\frac{1}{2}F(0,0,0)+O(\epsilon^2)$. Therefore, we find that $F(m_a{}^2,m_b{}^2,m_c{}^2)=-\frac{1}{2}F(0,0,0)+O(\epsilon^2)$ and

$$g_{abc}=-i4f_af_bf_c[\langle 0|[Q_c,[Q_a,[Q_b,\epsilon\mathcal{H}'(0)]]]|0\rangle+\langle 0|[Q_b,[Q_a,[Q_c,\epsilon\mathcal{H}'(0)]]]|0\rangle]+O(\epsilon^2)\,, \tag{46}$$

which is the desired formula. The following remarks are in order.

(i) According to Eq. (46), g_{abc} always vanishes in the limit $\epsilon=0$. For $SU(3)\otimes SU(3)$ this is a trivial point since parity forbids a trilinear coupling of pseudoscalar mesons. This result will, however, be useful in discussing the weak decay $K\to 2\pi$.

(ii) Although Eq. (46) does not appear to be symmetric in a, b, and c, it actually is. This follows from the Jacobi identity and the fact that $\langle 0|[Q_a,\epsilon\mathcal{H}'(0)]|0\rangle=0+O(\epsilon^2)$.

The vacuum expectation values of $[Q_a,[Q_b,\epsilon\mathcal{H}']]$ and $[Q_a,[Q_b,[Q_c,\epsilon\mathcal{H}']]]$ give, respectively, the mass matrix and trilinear coupling of Goldstone bosons in order ϵ. What about fourth- and higher-order commutators? It is known[8] that $\langle 0|[Q_a,[Q_b,[Q_c,[Q_d,\epsilon\mathcal{H}']]]]|0\rangle$ is the order ϵ term in meson-meson scattering, i.e., it acts like a quadrilinear coupling. Presumably the pattern continues. Also, it is worth pointing out that $\langle 0|[Q_a,\epsilon\mathcal{H}'(0)]|0\rangle=0+O(\epsilon)$ is easily shown to imply that to order ϵ^2 the matrix element of $\epsilon\mathcal{H}'$ between Goldstone bosons and the vacuum vanishes. In other words, there are no tadpole diagrams. Thus, there seems to be a simple interpretation of the expectation values of all commutators of Q's with $\epsilon\mathcal{H}'$.

Equation (46) can be used to compute the $\eta\pi\pi$ coupling for the CP-violating interaction in Eq. (34′). A simple calculation shows that it is proportional to $\pm\delta(1-\frac{1}{4}\delta^2)^{1/2}\langle 0|u_0|0\rangle$ and is therefore not zero.

B. $K\rightarrow 2\pi$

Let us now apply Eq. (46) to the decay $K\rightarrow 2\pi$. Writing

$$H=H_0+GH_W+\epsilon H', \qquad (47)$$

where GH_W is the nonleptonic part of the weak interaction, we observe that if H_W is built of the $V+A$ currents then

$$[F_a^-, H_0+GH_W]=0, \quad a=1,2,\ldots,8 \qquad (48)$$

where $F_a^-=F_a-F_a^5$. We shall take Eq. (48) as our theory of nonleptonic weak interactions. It follows from Eq. (47) that H_0+GH_W describes a world with eight Goldstone bosons. That is, adding GH_W to H_0 leaves π, K, and η massless exact Goldstone particles. According to the above discussion then, the $K\pi\pi$ vertex vanishes[9] when $\epsilon=0$ and the term of order ϵ is given by Eq. (46). We will now show that for $\mathcal{H}'$ belonging to $(3,\bar{3})\oplus(\bar{3},3)$, the order-$\epsilon$ term also vanishes so that $K\rightarrow 2\pi$ occurs only in order $G\epsilon^2$. The proof is quite simple. For $\epsilon\mathcal{H}'$ belonging to $(3,\bar{3})\oplus(\bar{3}.3)$, one can easily show that

$$\langle 0|[F_c^-,[F_a^-,[F_b^-,\epsilon\mathcal{H}']]]|0\rangle + \langle 0|[F_b^-,[F_a^-,[F_c^-,\epsilon\mathcal{H}']]]|0\rangle = \sum_d C_{abcd}\langle 0|[F_d^-,\epsilon\mathcal{H}']|0\rangle + d_{abc}\langle 0|\epsilon\mathcal{H}''|0\rangle, \qquad (49)$$

where C_{abcd} and d_{abc} are numbers, $\mathcal{H}''$ is equal to $\mathcal{H}'$ with u_0 replaced by v_0 and u_8 replaced by v_8 and $\langle 0|$ is now the vacuum for H_0+GH_W. But $\langle 0|[F_d^-,\epsilon\mathcal{H}']|0\rangle$ vanishes by Theorem 2 and $\langle 0|\mathcal{H}''|0\rangle$ vanishes by CP. (We neglect CP violation in the weak interactions.)

The above result that $K\rightarrow 2\pi$ occurs only in order $G\epsilon^2$ is peculiar to $(3,\bar{3})\oplus(\bar{3},3)$ breaking. For $\mathcal{H}'$ in, say, $(8,8)$ the process takes place in order $G\epsilon$. Is this bad for the $(3,\bar{3})\oplus(\bar{3},3)$ model? It is hard to say. In order to decide, one has to look at some rates. For a two-body decay, the rate is given by

$$\Gamma(A\rightarrow B+C)=[g(ABC)]^2q, \qquad (50)$$

where q is the momentum of the decay products in the rest frame of A and g is a dimensionless number. For $(3,\bar{3})\oplus(\bar{3},3)$ breaking, one would expect $g(K_s^0\pi\pi)$ to be of order ϵ^2G while $g(\Lambda\pi N)$ or $g(\Sigma\pi N)$ should be of order G. (One can easily see from the old PCAC–current-algebra calculations of $Y\rightarrow\pi+N$ that hyperon decays are allowed when $H=H_0+GH_W$.) The momenta q in $K_s^0\rightarrow 2\pi$ and in $\Lambda\rightarrow\pi+N$, $\Sigma\rightarrow\pi+N$, and $\Xi\rightarrow\pi+\Lambda$ are all similar, differing by at most a factor of 2. Furthermore the mean lifetimes for $\Lambda\rightarrow\pi+N$, $\Sigma\rightarrow\pi+N$, and $\Xi\rightarrow\pi+N$ are all within a factor of 2×10^{-10} sec. Thus, in the $(3,\bar{3})\oplus(\bar{3},3)$ model, one would expect the lifetime for $K_s^0\rightarrow 2\pi$ to be roughly $2\epsilon^{-4}\times10^{-10}$ sec. If ϵ is, say, $\frac{1}{3}$ to $\frac{1}{5}$, this would give a lifetime of about 2×10^{-8} to 10^{-7} sec. Experimentally the lifetime for $K_s^0\rightarrow 2\pi$ is 0.8×10^{-10} sec. In other words, the ϵ^4 suppression factor in $\Gamma(K_0^s\rightarrow 2\pi)$ does not seem to be there in the real world. Admittedly, it is dangerous to compare decay of light bosons and heavy fermions. Perhaps the dimensionless g defined by Eq. (50) is not the relevant object, but it is not clear that any other quantity is more relevant. Also, it is quite possible to imagine a dynamical enhancement in the amplitude for $K_s^0\rightarrow 2\pi$ or a suppression in $Y\rightarrow\pi+N$ which could account for a factor of 10 in the amplitude or 10^2 in the lifetime. Note that any $\mathcal{H}'$ will give a factor of ϵ^2 in $\Gamma(K_s^0\rightarrow 2\pi)$; $(3,\bar{3})\oplus(\bar{3},3)$ simply makes things worse by adding another factor of ϵ^2.

We leave it to the reader to decide whether the above situation shows a real difficulty with either the $(3.\bar{3})\oplus(\bar{3},3)$ model for $\epsilon\mathcal{H}'$ or a weak interaction which satisfies Eq. (48). Below we will examine the $(3.\bar{3})\oplus(\bar{3},3)$ model from a different angle. Also it should be noted that the trouble with $K\rightarrow 2\pi$ could be due to slow convergence of the expansion in powers of ϵ. Such a possibility (which does *not* mean that the net effect of $\epsilon\mathcal{H}'$ is large) was discussed in Ref. 2.

C. Why $(3,\bar{3})\oplus(\bar{3},3)$? Or Is $SU(2)\otimes SU(2)$ Really Better Than $SU(3)$?

In view of the above discussion, it seems worthwhile to review the rationale for $(3,\bar{3})\oplus(\bar{3},3)$ symmetry breaking. Let us begin with Eq. (27) for Goldstone boson masses. Specializing to $SU(3)\otimes SU(3)$ and noting the definition σ_{ab} in Eq. (11), we have

$$m^2{}_{ab}=-4f^2\langle 0|[F_a^5,[F_b^5,\epsilon\mathcal{H}'(0)]]|0\rangle+O(\epsilon^2), \quad a,b=1,2,\ldots,8 \qquad (51)$$

with $m^2{}_{ab}$ being the mass-squared matrix for the pseudoscalar mesons. Now suppose that $\epsilon\mathcal{H}'$ belongs to the representation $(X,\bar{X})\oplus(\bar{X},X)$ of $SU(3)\oplus SU(3)$, where X is any $SU(3)$ representation. Let the (unique) even-parity $SU(3)$ singlet in $(\bar{X},X)\oplus(X,\bar{X})$ be called

[8] R. Dashen and M. Weinstein, Phys. Rev. **183**, 1261 (1969).

[9] Actually this is a consequence of $SU(3)$ and CP alone. It is well known that the amplitude for $K\rightarrow 2\pi$ must be of order G times $SU(3)$ violation.

O_0 and the (unique) even-parity eighth component of an octet be called O_8. Then we may suppose that

$$\epsilon\mathcal{H}'=C_0O_0+C_8O_8\,, \tag{52}$$

where C_0 and C_8 are numbers. Defining Clebsch-Gordan coefficients A_x and B_x by

$$[F_a{}^5,[F_b{}^5,O_0]]=A_x\delta_{ab}O_0 +[\text{other members of } (\bar{X},X)\oplus(X,\bar{X})] \tag{53}$$

and

$$[F_a{}^5,[F_b{}^5,O_8]]=B_xd_{ab8}O_0 +[\text{other members of } (\bar{X},X)\oplus(X,\bar{X})]\,, \tag{54}$$

where d_{abc} is the symmetric $SU(3)$ symbol, Eq. (51) becomes

$$m^2{}_{ab}=(C_0A_x\delta_{ab}+C_8B_xd_{ab8})\langle 0|O_0|0\rangle+O(\epsilon^2)\,. \tag{55}$$

Given the fact that the pseudoscalar masses squared satisfy the Gell-Mann–Okubo mass formula, it follows from Eq. (55.) *that by suitably choosing C_0 and C_8 one can fit the pseudoscalar masses with $\epsilon\mathcal{H}'$ belonging to any representation* $(\bar{X},X)\oplus(X,\bar{X})$. We note also that terms in $\epsilon\mathcal{H}'$ belonging to $(X,\bar{Y})\oplus(\bar{X},Y)$ for $Y\neq X$ do not contain an $SU(3)$ singlet, and therefore do not contribute to the pseudosclaar masses in lowest order.

Since the pseudoscalar masses are essentially the only violation of $SU(3)\otimes SU(3)$ [as opposed to $SU(3)$] about which we have much accurate experimental information, one might ask what is so special about $(3,\bar{3})\oplus(\bar{3},3)$? The answer, paraphrasing the arguments of Gell-Mann, Oakes, and Renner, runs as follows, The pion mass is very small compared to the K and η masses. Let us see what happens if we choose the ratio of C_8 to C_0 in Eq. (52) so that Eq. (55) gives $m_\pi{}^2=0$. The remarkable fact pointed out by Gell-Mann, Oakes, and Renner is that only for $(3,\bar{3})\oplus(\bar{3},3)$ does this produce an $\epsilon\mathcal{H}'$ which commutes with the axial-vector charges $F_a{}^5$ for $a=1$, 2, 3. In other words, for $\epsilon\mathcal{H}'$ of the form in Eq. (52), only $(3,\bar{3})\oplus(\bar{3},3)$ has the property that $SU(2)\otimes SU(2)$ becomes exact when $m_\pi{}^2$ goes to zero (keeping $m_K{}^2$ and $m_\eta{}^2$ finite). For other representations $(\bar{X},X)\oplus(X,\bar{X})$ with $X\neq 3$, the smallness of $m_\pi{}^2$ is accidental and $SU(2)\otimes SU(2)$ is no better than $SU(3)\otimes SU(3)$ in general.

The question then arises of whether $SU(2)\otimes SU(2)$ is really a much better symmetry than $SU(3)\otimes SU(3)$ or $SU(3)$ itself or have we been misled by an accidentally small $m_\pi{}^2$? The author's feeling is that $SU(2)\otimes SU(2)$ really is a better symmetry. One should, however, let experiments decide. The experimental situation is not that clear. A manifestation of the breaking of $SU(2)\otimes SU(2)$ is the error in the ordinary Goldberger-Treiman relation. It is in error by about 15%, which is not much different than some $SU(3)$ violations like $(M_\Lambda-M_N)/M_N\sim 0.2$. One's general feeling is that PCAC for K's and the η is much worse than PCAC for pions, but we really do not *know* that this is true. It could be that the larger masses of K and η just makes PCAC for these particles more difficult to handle but not intrinsically less accurate. There are not yet any really clean tests of K or η PCAC. Experimentally, one cannot argue with a statement that pion PCAC $[SU(2)\otimes SU(2)]$, K and η PCAC $[SU(3)\otimes SU(3)]$, and $SU(3)$ are all equally good, to about 20% on the average. Of course everyone's feeling, including the author's, runs to the contrary. Neveretheless, it is clearly important to obtain some hard facts. Weinstein and the author[10] have suggested some explicit tests of the $(3,\bar{3})\oplus(\bar{3},3)$ model. We may hope that some new imformation will be available in a few years.

[10] R. Dashen and M. Weinstein, Phys. Rev. **188**, 2330 (1969).