Crossing-Symmetric Expansions of Scattering Amplitudes, Threshold Behavior, and Asymptotics*

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A two-variable, explicitly crossing-symmetric expansion of the scattering amplitude is discussed for the two-body scattering of spinless particles (with arbitrary masses). It converges simultaneously in the physical regions of at least two channels, has the correct threshold behavior, and allows for amplitudes growing asymptotically as arbitrary powers of s and t. The expansion is based on the representation theory of the group $O(2,1)$ in a basis not corresponding to any subgroup reduction and making use of Lamé functions.

I. INTRODUCTION

THE purpose of this paper is to discuss a new expansion formula representing a general class of scattering amplitudes and having the following properties.

(i) It is a two-variable expansion, simultaneous in terms of both the Mandelstam variables s and t.

(ii) The expansion is explicitly crossing-symmetric term by term in two channels and converges simultaneously for amplitudes defined in the physical regions of both channels. Both the direct and inverse expansion formulas involve only amplitudes defined in physical regions.

(iii) It automatically incorporates the correct threshold behavior.

(iv) The simplest assumptions about the analytic continuation of the expansion leads to amplitudes growing asymptotically as arbitrary powers of s and t.

(v) The expansion is based on the representation theory of the group $O(2,1)$, using a basis that does not correspond to the reduction of $O(2,1)$ to any subgroups. The basis functions turn out to be Lamé functions.

Our approach is part of a general program,[1–4] the aim of which is to develop a scattering theory based on two-variable (or more generally, multivariable) expansions of scattering amplitudes. The dependence on all kinematic parameters (energies, angles, etc.) is made explicit, the dynamics are contained in the expansion coefficients, so that a greater separation of "kinematics" and "dynamics" is achieved, thus giving a tool for implementing general principles, making dynamical assumptions, and performing phenomenological fits to larger bodies of data.

The method of this approach (so far for two-body scattering) is to consider the scattering amplitude as a function of the momenta p_i (or rather the relativistic velocities $v_i=p_i/m_i$) of the particles involved ($i=1, \ldots, 4$) and then show that, making use of the conservation laws and relativistic invariance, it is possible to express the components of three of the momenta in terms of the fourth. The scattering amplitude can thus be considered to be the function of one four-momentum only, i.e., the function of a point on a three-dimensional hyperboloid $v^2=v_0^2-v_1^2-v_2^2-v_3^2=1$ (or on the mass shell of one of the particles). The group of motions of this space is the homogeneous Lorentz group $O(3,1)$ and the natural thing to do is to expand the amplitude $f(v)$ in terms of the basis functions of this group.

Crucial in the obtaining of such expansions are three related features—the choice of a Lorentz frame of reference, choice of coordinates on the hyperboloid, and choice of a basis for group representations.

In this paper our principal aim is to discuss crossing-symmetric expansions, so we must choose such a frame of reference and such coordinates that we obtain a symmetric mapping of the Mandelstam variables s and t onto some curvilinear coordinates α, β on the hyperboloid. In a previous publication,[5] we have considered this problem for the two-body scattering of spinless particles with equal masses. Here we shall consider a more general case.

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[1] N. Vilenkin and J. Smorodinsky, Zh. Eksperim. i Teor. Fiz. **46**, 1793 (1964) [Soviet Phys. JETP **19**, 1209 (1964)].

[2] P. Winternitz, J. Smorodinsky, and M. Sheftel, Yadern. Fiz. **8**, 833 (1968) [Soviet J. Nucl. Phys. **8**, 489 (1969)].

[3] P. Pajas and P. Winternitz, Phys. Rev. D **1**, 1105 (1970).

[4] P. Winternitz, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Gordon and Breach, New York, 1971), Vol. XIII. This paper and Ref. 3 contain numerous references to previous work.

[5] N. W. Macfadyen and P. Winternitz, J. Math. Phys. **12**, 281 (1971), hereafter referred to as I.

II. MAPPING OF MANDELSTAM VARIABLES ONTO VELOCITY SPACE

As in I, we simplify the mathematics by restricting ourselves to spinless particles and choosing the scattering plane to be such that the third space components of all the momenta vanish. This makes it possible to consider the momenta as three-dimensional objects and to consider an $O(2,1)$ hyperboloid, instead of an $O(3,1)$ one.

To make the choice of the curvilinear coordinates unique, we demand that the Laplace operator on the hyperboloid allow the separation of variables in these coordinates and that the corresponding eigenfunctions —the basis functions for our expansions—can be written as the product of two identical functions. An inspection of separable coordinates[6,7] leads us directly to elliptic coordinates (see I), in which the nonzero components of the momenta are

$$p_r=m_r(-\mathrm{cn}\alpha_r\,\mathrm{cn}\beta_r,\, i\,\mathrm{sn}\alpha_r\,\mathrm{dn}\beta_r,\, i\,\mathrm{dn}\alpha_r\,\mathrm{sn}\beta_r,\, 0)\,,\quad r=1,\ldots,4\,. \tag{1}$$

Here cn$z$, sn$z$, and dn$z$ are the Jacobian elliptic functions[8] of modulus $k=1/\sqrt{2}$, with arguments in the regions $\alpha_r\in(iK,iK+2K)$, $\beta_r\in(iK,iK+2K)$, where $K=[\Gamma(\frac{1}{4})]^2/4\sqrt{\pi}$ is the quarter-period of the elliptic functions.

To eliminate the redundant variables we must choose a convenient frame of reference. We shall use the coordinates of momentum p_2 as our basic variables. In order to obtain simple properties under the crossing transformation, we let $\alpha_2\equiv\alpha$, $\beta_2\equiv\beta$ and demand that the transformation $\alpha\to\beta$, $\beta\to 2K-\alpha$, $m_2\to m_3$, $m_3\to m_2$, $m_1\to m_1$, $m_4\to m_4$ should correspond to $p_2\to -p_3$, $p_3\to -p_2$, $p_1\to p_1$, $p_4\to p_4$. Further we specify our frame of reference (a generalization of the brick-wall system) by putting $\mathbf{p}_1\|\mathbf{p}_4$ (see Fig. 1). It is easy to check that in such a frame of reference we have

$$\begin{aligned}
p_1&=(-m_1\,\mathrm{cn}\alpha_1\,\mathrm{cn}\beta_1,\, -i(a+1)A,\, -i(a+1)B,\, 0)\,,\\
p_2&=m_2(-\mathrm{cn}\alpha\,\mathrm{cn}\beta,\, i\,\mathrm{sn}\alpha\,\mathrm{dn}\beta,\, i\,\mathrm{dn}\alpha\,\mathrm{sn}\beta,\, 0)\,,\\
p_3&=m_3(-\mathrm{cn}\alpha\,\mathrm{cn}\beta,\, -i\,\mathrm{dn}\alpha\,\mathrm{sn}\beta,\, -i\,\mathrm{sn}\alpha\,\mathrm{dn}\beta,\, 0)\,,\\
p_4&=(-m_1\,\mathrm{cn}\alpha_1\,\mathrm{cn}\beta_1+(m_3-m_2)\,\mathrm{cn}\alpha\,\mathrm{cn}\beta,\\
&\qquad -iaA,\, -iaB,\, 0)\,,
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
A&=m_2\,\mathrm{sn}\alpha\,\mathrm{dn}\beta+m_3\,\mathrm{dn}\alpha\,\mathrm{sn}\beta\,,\\
B&=m_2\,\mathrm{dn}\alpha\,\mathrm{sn}\beta+m_3\,\mathrm{sn}\alpha\,\mathrm{dn}\beta\,,
\end{aligned} \tag{3}$$

$$\left\{\begin{matrix}\mathrm{cn}^2\alpha_1\\ \mathrm{cn}^2\beta_1\end{matrix}\right\}=\frac{\mp(a+1)^2(A^2-B^2)-\{[2m_1^2-(a+1)^2(A+B)^2][2m_1^2-(a+1)^2(A-B)^2]\}^{1/2}}{2m_1^2}\,, \tag{4}$$

and

$$a=[-Y+(Y^2-4XZ)^{1/2}]/2X\,, \tag{5}$$

with

$$\begin{aligned}
X&=4(A^2+B^2)(m_2+m_3)^2[2m_2m_3(A+B)^2+(m_2^2-m_3^2)^2]\,,\\
Y&=4(A^2+B^2)(m_2+m_3)^2\{2m_2m_3(A+B)^2\\
&\qquad+(m_2+m_3)^2[(m_2-m_3)^2-m_1^2+m_4^2]\}\,,\\
Z&=\{2m_2m_3(A+B)^2+(m_2+m_3)^2[(m_2-m_3)^2-m_1^2\\
&\qquad-m_4^2]\}^2+4m_4^2(m_2+m_3)^4(A^2+B^2-m_1^2)\,.
\end{aligned} \tag{6}$$

Such a frame of reference exists if a is real, the condition for which is

$$(m_2-m_3)^2\leq(m_1-m_4)^2\,. \tag{7}$$

In the chosen frame of reference, all momenta obviously depend on α and β only, which in turn can be related to the Mandelstam variables s and t, so that the scattering amplitude $f(s,t)\equiv f(\alpha,\beta)$ is given as a function of a point on the hyperboloid $p_2^2=m_2^2$.

The properties under the crossing transformation are of interest mainly when the two exchanged particles are identical. Let us further assume that $m_2=m_3$. In such a case the formulas (2)–(6) simplify considerably.

Indeed for $m_2=m_3$ we can give the relation between s, t, u and α, β explicitly:

$$\begin{aligned}
\left\{\begin{matrix}s\\ t\end{matrix}\right\}&=m_1^2+m_2^2+2\{\mp m_1m_2y\\
&\quad\times[1-(1/8m_1^2m_2^2x^2)(2m_2^2x^2+m_1^2-m_4^2)^2]^{1/2}\\
&\quad-\tfrac{1}{4}(2m_2^2x^2+m_1^2-m_4^2)\}\,,\\
u&=2m_2^2x^2\,,
\end{aligned} \tag{8}$$

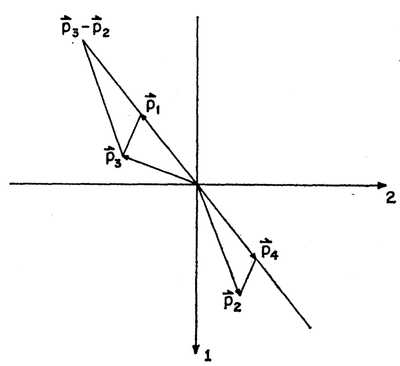

FIG. 1. Symmetric frame of reference.

[6] M. N. Olevsky, Mat. Sbornik **27** (69), 379 (1950).

[7] P. Winternitz, I. Lukač, and J. Smorodinsky, Yadern. Fiz. **7**, 192 (1968) [Soviet J. Nucl. Phys. **7**, 139 (1968)].

[8] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi *et al.* (McGraw-Hill, New York, 1953), Vol. II, Chap. 13.

where

$$x = \mathrm{sn}\alpha\, \mathrm{dn}\beta + \mathrm{dn}\alpha\, \mathrm{sn}\beta, \qquad y = \mathrm{cn}\alpha\, \mathrm{cn}\beta. \tag{9}$$

Inverting these formulas, we obtain

$$\begin{Bmatrix}\mathrm{cn}^4\alpha\\ \mathrm{cn}^4\beta\end{Bmatrix} = \frac{u}{8m_2^4[(m_1-m_4)^2-u][(m_1+m_4)^2-u]} \times\{(u-4m_2^2)[(m_1-m_4)^2-u][(m_1+m_2)^2-u] -(u-2m_2^2)(s-t)^2 \mp 4[(4m_2^2-u)(m_2^2(s+t)-st +m_1^2m_4^2-m_2^4)\phi(s,t,u)]^{1/2}\}, \tag{10}$$

where

$$\phi(s,t,u) = stu - \Sigma^2(as+bt+cu) \tag{11}$$

is the Kibble function[9] determining the boundaries of the physical regions as solutions of the equation $\phi(s,t,u)=0$. In (11) we have (for $m_2=m_3$)

$$\begin{aligned}\Sigma &= m_1^2+2m_2^2+m_4^2,\\ \Sigma^3 a &= \Sigma^3 b = m_2^2(m_1^2-m_4^2)^2,\\ \Sigma^3 c &= (m_1^2m_4^2-m_2^4)(m_1^2+m_4^2-2m_2^2).\end{aligned} \tag{12}$$

In the equal-mass case $m_1=m_2=m_3=m_4=1$ formulas (8) and (10) simplify to those, given in I, namely,

$$\begin{Bmatrix}s\\ t\end{Bmatrix} = 2-x^2 \mp 2y(1-\tfrac{1}{2}x^2)^{1/2}, \quad u=2x^2 \tag{13}$$

$$\mathrm{cn}^4\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} = \frac{(s+t)^2+2st(2-s-t)}{4(s+t)} \pm \frac{1}{2}\left[\frac{stu(s+t-st)}{s+t}\right]^{1/2}. \tag{14}$$

We now notice that $\alpha\in(iK, iK+2K)$, $\beta\in(iK, iK+2K)$ corresponds to the physical region of the s channel; $\alpha\in(iK, iK+2K)$, $\beta\in(-iK, -iK+2K)$ to the t channel; and $\alpha\in(iK, iK+2K)$, $\beta\in(0, 2iK)$ to the u channel. Comparing with (1) we see that the physical regions of the s and t channels get mapped onto the entire upper and lower sheets of the hyperboloid $v^2=v_0^2-v_1^2-v_2^2=1$, respectively.

Thus, we have the physical scattering amplitudes as functions on homogeneous manifolds and we can proceed to expand them in terms of the basis functions of irreducible representations of the group acting on these spaces, namely, the group $O(2,1)$.

III. CROSSING-SYMMETRIC EXPANSIONS

As was shown in I, the basis functions for irreducible representations of $O(2,1)$, separable in the elliptic coordinates (1), are eigenfunctions of a complete set of commuting operators,

$$\begin{aligned}\Delta f_{lh}^{pq}(\alpha,\beta) &= (L_3^2-K_1^2-K_2^2)f_{lh}^{pq}(\alpha,\beta)\\ &= -l(l+1)f_{lh}^{pq}(\alpha,\beta),\\ Hf_{lh}^{pq}(\alpha,\beta) &= (K_1^2-\tfrac{1}{2}L_3^2)f_{lh}^{pq}(\alpha,\beta) = hf_{lh}^{pq}(\alpha,\beta),\\ Xf_{lh}^{pq}(\alpha,\beta) &= qf_{lh}^{pq}(\alpha,\beta), \quad Yf_{lh}^{pq}(\alpha,\beta) = pf_{lh}^{pq}(\alpha,\beta).\end{aligned} \tag{15}$$

Here L_3 is the generator of a rotation about axis 3, K_1 and K_2 are Lorentz boosts, and X and Y are reflection operators, changing the sign of v_1 and v_2, respectively.

The solutions of (15) can be written as

$$f_{lh}^{pq}(\alpha,\beta) = \Lambda_{lh}^p(\alpha)\Lambda_{lh'}^q(\beta), \tag{16}$$

where $h+h'=l(l+1)$, $p,q=\pm1$, and $\Lambda_{lh}^p(z)$ satisfy the Lamé equation[10]

$$\frac{d^2\Lambda_{lh}^p(z)}{dz^2} + [h-\tfrac{1}{2}l(l+1)\,\mathrm{sn}^2 z]\Lambda_{lh}^p(z) = 0. \tag{17}$$

To fix the Lamé functions completely, we standardize them by imposing

$$\begin{aligned}\Lambda_{lh}^+(iK+K) &= 1, \quad \frac{d}{dz}\Lambda_{lh}^+(z)\,|_{iK+K} = 0,\\ \Lambda_{lh}^-(iK+K) &= 0, \quad \frac{d}{dz}\Lambda_{lh}^-(z)\,|_{iK+K} = -1.\end{aligned} \tag{18}$$

[Note that $\Lambda_{lh}^+(z)$ and $\Lambda_{lh}^-(z)$ are symmetric and antisymmetric about the center point $z=K+iK$, respectively.]

It is now possible to obtain an expansion formula for a function $f(\alpha,\beta)$ defined over the hyperboloid, in terms of the basis functions (16). Such an expansion was derived in I using the Gelfand-Graev method of horospheres[11] for functions square integrable with respect to the invariant measure $d^2p/p_0 = -\frac{1}{2}(\mathrm{cn}^2\alpha + \mathrm{cn}^2\beta)d\alpha d\beta$. Here we only give the resulting expansion:

$$f(\alpha,\beta) = \frac{1}{8\pi^2 i}\int_{\kappa-i\infty}^{\kappa+i\infty} dl(2l+1) \times \cot\pi l \sum_h \sum_{p=\pm} \sum_{q=\pm p} N_{lh}^p |\lambda_{pq}(l,h)|^2 \times[A^{pq}(l,h)\Lambda_{lh}^p(\alpha)\Lambda_{lh'}^q(\beta) + A^{qp}(l,h')\Lambda_{lh'}^q(\alpha)\Lambda_{lh}^p(\beta)], \tag{19}$$

$$A^{pq}(l,h) = -\frac{1}{2}\int_I d\alpha \int_I d\beta\,(\mathrm{cn}^2\alpha+\mathrm{cn}^2\beta) \times f(\alpha,\beta)\Lambda_{lh}^p(\alpha)\Lambda_{lh'}^q(\beta), \tag{20}$$

where $\kappa=-\frac{1}{2}$, I is the interval $(iK, iK+2K)$, and the

[9] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

[10] Bateman Manuscript Project, *Higher Transdendental Functions*, edited by A. Erdélyi *et al.* (McGraw-Hill, New York, 1955), Vol. III, Chap. 15.

[11] I. M. Gelfand, M. I. Graev, and N. J. Vilenkin, *Generalized Functions* (Academic, New York, 1966), Vol. 5.

h summation in (19) is over a set of real discrete values, determined by choosing one arbitrary value h_0 and then selecting all values of h such that

$$\int_I \Lambda_{lh}{}^p(z)\Lambda_{lh_0}{}^p(z)\frac{dz}{\sqrt{2}}=0. \tag{21}$$

$N_{lh}{}^p$ and $\lambda_{pq}(l,h)$ are normalization constants:

$$(N_{lh}{}^p)^{-1}=\int_I [\Lambda_{lh}{}^p(x)]^2\frac{dx}{\sqrt{2}} \tag{22}$$

and

$$\lambda_{++}(l,h)=\frac{1}{\sqrt{2}}\int_I \Lambda_{lh}{}^+(x)(i\,\mathrm{cn}x)^{-l-1}dx,$$

$$\lambda_{+-}(l,h)=\frac{l+1}{2}\int_I \Lambda_{lh}{}^+(x)(i\,\mathrm{cn}x)^{-l-2}\,\mathrm{sn}x\,dx,$$

$$\lambda_{-+}(l,h)=\frac{l+1}{\sqrt{2}}\int_I \Lambda_{lh}{}^-(x)(i\,\mathrm{cn}x)^{-l-2}(i\,\mathrm{dn}x)dx, \tag{23}$$

$$\lambda_{--}(l,h)=\frac{(l+1)(l+2)}{2}\int_I \Lambda_{lh}{}^-(x)(i\,\mathrm{cn}x)^{-l-3}\times(i\,\mathrm{dn}x\,\mathrm{sn}x)dx.$$

Formula (19) is the crossing-symmetric expansion for which we have been searching. This expansion, together with the inverse formula (20), was derived rigorously only for square integrable functions, corresponding to cross sections falling off to zero for $s\to\infty$. To describe physical amplitudes, corresponding to a more general asymptotic behavior, we must generalize expansion (19), making use of nonunitary infinite-dimensional representations. The simplest possibility, suggested by analogy with Regge-pole theory, is to retain formula (19), but integrate over a shifted path (with $\kappa\geq-\frac{1}{2}$). Formula (20) has to be modified in this case and is in general not unique.

Let us now discuss the sense in which our expansions are crossing symmetric. As in I, we only consider amplitudes in physical regions, postponing a discussion of the analytic continuation from one region to another to a future publication. Contrary to I, we consider the general expansion (19), without imposing additional symmetries on the coefficients.

Consider simultaneously the s-channel reaction $1+2\to3+4$ and the t-channel reaction $1+\bar{3}\to\bar{2}+4$. In the physical region of the s channel we have $\alpha\in(iK,\,iK+2K)$, $\beta\in(iK,\,iK+2K)$ and we write expansion (19) for the scattering amplitude $f^s(\alpha,\beta)$. In the physical region of the t channel we again have $\alpha\in(iK,\,iK+2K)$; however $\tilde{\beta}\in(-iK,\,-iK+2K)$. Making use of the symmetries of the Lamé equation, it is easy to obtain the corresponding expansion

$$f^t(\alpha,\tilde{\beta})=\frac{1}{8\pi^2 i}\int_{\kappa-i\infty}^{\kappa+i\infty}dl\,(2l+1)\times\cot\pi l\sum_h\sum_{p=\pm}\sum_{q=\pm p}N_{lh}{}^p|\lambda_{pq}(l,h)|^2\times\{qB^{pq}(l,h)\Lambda_{lh}{}^p(\alpha)\Lambda_{lh'}{}^q(2K-\tilde{\beta})+pB^{qp}(l,h')\Lambda_{lh'}{}^q(\alpha)\Lambda_{lh}{}^p(2K-\tilde{\beta})\}, \tag{24}$$

$$B^{pq}(l,h)=-\tfrac{1}{2}q\int_I d\alpha\int_I d\beta\,(\mathrm{cn}^2\alpha+\mathrm{cn}^2\beta)\times f^t(\alpha,\,2K-\beta)\Lambda_{lh}{}^p(\alpha)\Lambda_{lh'}{}^q(\beta). \tag{25}$$

Notice that the variables $2K-\tilde{\beta}$ in (24) are in the same region as β in (19) and that the two expansions converge simultaneously.

Now let us assume that the particles satisfy $2=\bar{3}$, i.e., particle 3 is the antiparticle of 2, so that the reactions in the two channels are identical. Independently of any analyticity problems, the amplitudes must satisfy $f^s(s,t,u)=\pm f^t(t,s,u)$, i.e.,

$$f^s(\alpha,\beta)=\pm f^t(\beta,\,2K-\alpha). \tag{26}$$

Comparing (19) and (24), we see that crossing symmetry (26) is satisfied trivially by putting

$$qB^{pq}(l,h)=\pm A^{qp}(l,h'),\quad pB^{qp}(l,h')=\pm A^{pq}(l,h). \tag{27}$$

In other words, we write the same expansion in both channels and crossing symmetry is imposed trivially, term by term, by means of Eq. (27).

IV. THRESHOLD AND ASYMPTOTIC BEHAVIOR

A. Threshold Behavior

Let us first consider the behavior of expansion (19) near to the threshold, for simplicity considering the case when all four masses are equal, so that α,β and s,t are related by (13) and (14). Let us approach the threshold from the s-channel physical region along any line, corresponding to a fixed angle in the c.m. system. Thus, putting $s=2t(\cos\theta-1)^{-1}+4$ we have

$$\mathrm{cn}^4\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}\xrightarrow[s\to4;\ \theta=\mathrm{const}]{}1+\tfrac{1}{8}[1+\cos\theta\mp4\sin\theta](s-4), \tag{28}$$

so that the threshold corresponds to the center point $\alpha=\beta=iK+K$. Putting $\alpha=iK+K+\epsilon_\alpha$, $\beta=iK+K+\epsilon_\beta$, and expanding $\mathrm{cn}^4\alpha(\mathrm{cn}^4\beta)$ about this point, we find

$$\begin{Bmatrix}\epsilon_\alpha\\ \epsilon_\beta\end{Bmatrix}=\tfrac{1}{4}(1+\cos\theta\mp4\sin\theta)^{1/2}(s-4)^{1/2}. \tag{29}$$

Further, let us make the usual assumption[12] of Hermitian analyticity $f^*(s+i\epsilon,\,t)=f(s-i\epsilon,\,t)$ along

[12] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge U. P., New York, 1966).

the cuts in the s plane, so that

$$f^*(\alpha,\beta)=f(2K+2iK-\alpha,\ 2K+2iK-\beta)\,. \quad (30)$$

We split the scattering amplitude into a symmetric and an antisymmetric part, putting

$$\begin{aligned} f(\alpha,\beta)&=\tfrac{1}{2}[f(\alpha,\beta)+f(2K+2iK-\alpha,\ 2K+2iK-\beta)] \\ &\quad+\tfrac{1}{2}[f(\alpha,\beta)-f(2K+2iK-\alpha,\ 2K+2iK-\beta)] \\ &=f_s(\alpha,\beta)+f_a(\alpha,\beta) \end{aligned} \quad (31)$$

and notice that (30) implies

$$\begin{aligned} f_s(\alpha,\beta)&=f_s^*(\alpha,\beta)=\mathrm{Re}f(\alpha,\beta)\,, \\ f_a(\alpha,\beta)&=-f_a^*(\alpha,\beta)=i\,\mathrm{Im}f(\alpha,\beta)\,. \end{aligned} \quad (32)$$

Near the center point the function $\Lambda_{lh}^+(iK+K+\epsilon)$ and $\Lambda_{lh}^-(iK+K+\epsilon)$ can be expanded in even and odd powers of ϵ, respectively. Obviously only terms with $p=q$ in (19) will contribute to $f_s(\alpha,\beta)$ and those with $p=-q$ to $f_a(\alpha,\beta)$. Thus we find that each term in the expansion of $\mathrm{Re}f(\alpha,\beta)$ approaches a constant as s goes to the threshold, whereas each term in the expansion of $\mathrm{Im}f(\alpha,\beta)$ goes to zero having a square-root singularity at the threshold. Assuming uniform convergence of the expansion, we immediately obtain the correct threshold behaviors:

$$\mathrm{Re}f(\alpha,\beta)\underset{s\to4}{\sim}\mathrm{const}\,,\quad \mathrm{Im}f(\alpha,\beta)\underset{s\to4}{\sim}(s-4)^{1/2}. \quad (33)$$

B. Asymptotic Behavior

We postulate that expansion (19) converges for some positive value of κ, determining the integration path in the complex l plane, and study what sort of asymptotic behavior is compatible with this expansion [let us again stress that the inverse formula (20) must be modified for non-square-integrable functions, the rigorous mathematical treatment of which is in itself a difficult problem]. Applying the Fuchsian theory[13] of ordinary differential equations to the Lamé equation (17), we find that at the singular points $z_0=iK$ or $iK+2K$ the Lamé functions behave as

$$\Lambda_{lh}^p(z)=\phi_1^p(z)(z-z_0)^{l+1}+\phi_2^p(z)(z-z_0)^{-l}\,, \quad (34)$$

where $\phi_1^p(z)$ and $\phi_2^p(z)$ are regular at $z=z_0$.

We can use this asymptotic behavior to study the behavior of amplitudes for $s\to\infty$ (or $t\to\infty$) in any direction in the Mandelstam plane, as long as the variables α and β approach their end points.

For simplicity, let us again restrict ourselves to the case of four equal masses and consider three different asymptotic limits. Expanding about the end point z_0, we have $\mathrm{cn}^4(z_0+\epsilon)\sim4\epsilon^{-4}(1-\frac{1}{4}\epsilon^4+\cdots)$ for $\epsilon\to0$.

Consider first $s\to\infty$, $t=t_0<0$ fixed. It follows from the mapping (13) and (14) that both α and β approach the same end point. Putting $\alpha=z_0+\epsilon_\alpha$, $\beta=z_0+\epsilon_\beta$ ($z_0=iK$ or $iK+2K$), we find

$$\left\{\begin{matrix}\epsilon_\alpha\\ \epsilon_\beta\end{matrix}\right\}=8^{1/4}s^{-1/4}\{\tfrac{1}{2}(1-2t)\pm[-t(1-t)]^{1/2}\}^{1/2}. \quad (35)$$

Using (34), we see that the postulated convergence of our expansions is compatible with

$$f(\alpha,\beta)\underset{s\to\infty;\ t=t_0<0}{\sim}\epsilon_\alpha^{-\kappa}\epsilon_\beta^{-\kappa}\sim s^{\kappa/2}, \quad (36)$$

i.e., the amplitudes can grow as arbitrary polynomials in s.

If $s\to\infty$ for fixed $u=u_0<0$, we find that α and β approach opposite end points with

$$\left\{\begin{matrix}\epsilon_\alpha\\ \epsilon_\beta\end{matrix}\right\}=\sqrt{2}s^{-1/2}\left[\frac{2-u}{2(4-u)}\pm\frac{1}{2}\left(\frac{-u}{4-u}\right)^{1/2}\right]^{-1/4}, \quad (37)$$

so that we can have

$$f(\alpha,\beta)\underset{s\to\infty;\ u=u_0<\infty}{\sim}\epsilon_\alpha^{-\kappa}\epsilon_\beta^{-\kappa}\sim s^{\kappa}. \quad (38)$$

Finally, if $s\to\infty$ for fixed c.m. scattering angle ($\theta_{\mathrm{c.m.}}=\mathrm{const}\neq0$ or π) α and β approach the same end point, with

$$\begin{aligned} \epsilon_\alpha&=8^{1/4}(1-\cos\theta)^{-1/4}s^{-1/2}\,, \\ \epsilon_\beta&=8^{1/4}(1+\cos\theta)^{1/4}(1-\cos\theta)^{-1/2}s^{-1/4}\,, \end{aligned} \quad (39)$$

so that the amplitude can behave as

$$f(\alpha,\beta)\sim\epsilon_\alpha^{-\kappa}\epsilon_\beta^{-\kappa}\sim s^{3\kappa/4}\,. \quad (40)$$

The actual high-energy behavior naturally depends on the properties of the amplitudes $A^{pq}(l,h)$ in the expansion. A simple approach, analogous to Regge-pole theory, would be to assume that we can shift the integration path in (19) to the left in the l plane and that the coefficient $A^{pq}(l,h)$ has a pole in the l plane for $l=l_0$, $-\frac{1}{2}<\mathrm{Re}l_0<\kappa$, dominating the asymptotics. We then find that

$$f(s,t)=f(\alpha,\beta)\underset{s\to\infty}{\sim}(\epsilon_\alpha\epsilon_\beta)^{-l_0}, \quad (41)$$

which for $t=\mathrm{const}$, $u=\mathrm{const}$, or $\theta_{\mathrm{c.m.}}=\mathrm{const}$, corresponds to $s^{l_0/2}$, s^{l_0}, or $s^{3l_0/4}$, respectively.

Comparing these results with those of Regge-pole theory, we find that the assumption that the crossing-symmetric expansion (19) is dominated by a pole for $s\to\infty$ corresponds to the dominance of a fixed pole in the complex angular momentum plane.

Note that the rigorously derived expansion (for $\kappa=-\frac{1}{2}$) corresponds to amplitudes behaving asymptotically like $s^{-1/4}$, $s^{-1/2}$, or $s^{-3/8}$ for t, u, or θ fixed.

V. CONCLUSION

As was stressed in the Introduction, this investigation is a step in a certain general formulation of two-body

[13] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).

scattering. We have shown how the approach, making use of the geometry of velocity space, leads to crossing-symmetric expansions (for particles with arbitrary masses) having reasonable threshold and asymptotic behavior.

We plan to continue our investigation, in particular to consider the problem of analytic continuation into nonphysical regions and between physical regions, and to establish the exact connection with Regge-pole theory, i.e., to show how moving poles in the complex angular momentum plane manifest themselves in the properties of our expansion coefficients. Further problems under consideration are the relation of the obtained expansions to various dual-resonance models, the calculation of expansion coefficients in various models, and (one may hope) applications to the description of specific scattering or decay processes. The mathematical problems which arise are related both to the development of group representation theory in bases not corresponding to the reduction of a group to any subgroup, and to expansions of non-square-integrable functions in terms of nonunitary representations of noncompact groups.

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Some Features of Chiral Symmetry Breaking

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Some rather paradoxical features of chiral symmetry breaking are shown to be directly related to the fact that the vacuum is supposed to be degenerate in the limit of exact symmetry. The importance of picking the correct one of the many symmetry-limit vacua is stressed. A peculiar phenomena of spontaneous CP violation appearing alongside $SU(3)\otimes SU(3)$ breaking is observed. (In the model investigated here, the effect is much too large to be at all related to that seen in weak interactions.) For the $(3,\bar{3})\oplus(\bar{3},3)$ model of symmetry breaking, it is shown that the rate for $K_S{}^0\to 2\pi$ should be suppressed by a factor of $[SU(3)\otimes SU(3)\text{ breaking}]^4$. This may or may not be a difficulty. A number of other topics in chiral symmetry breaking are discussed briefly.

I. INTRODUCTION

It appears to be reasonable to adopt a picture of the strong interactions in which the strong Hamiltonian H can be meaningfully written as

$$H=H_0+\epsilon H', \tag{1}$$

where H_0 is $SU(3)\otimes SU(3)$ symmetric and H' contains the (small) departures from exact symmetry.[1,2] We suppose that H' takes care of not only the corrections to chiral symmetry, e.g., corrections to partial conservation of axial-vector current (PCAC), but also the corrections to $SU(3)$ itself, e.g., mass splitting. An advantage of this way of looking at the strong interactions is that one can hopefully relate these two types of symmetry breaking.

In the limit $\epsilon=0$ the eight pseudoscalar mesons are supposed to be Goldstone bosons and, hence, massless. To first order in ϵ, their masses squared are given by the elementary formula

$$m_\alpha{}^2=\langle \mathbf{P}_\alpha|\epsilon\mathcal{H}'(0)|\mathbf{P}_\alpha\rangle+O(\epsilon^2), \tag{2}$$

where $\alpha=1,\ldots,8$ labels the mesons; $\mathcal{H}'$ is the density of H', i.e.,

$$H'=\int d^3x\,\mathcal{H}(\mathbf{x},0); \tag{3}$$

and $\langle\mathbf{P}|$ is a covariantly normalized state, i.e., $\langle\mathbf{P}|\mathbf{P}'\rangle=2P_0\delta^3(\mathbf{P}-\mathbf{P}')$. Taken at face value, Eq. (2) has a peculiar property. The left-hand side is necessarily positive, but there does not appear to be any reason why the matrix elements on the right-hand side cannot be negative. One is thus led to ask the question of whether positivity of the squares of the meson masses places some restriction on $\mathcal{H}'$.

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[1] The original suggestion that PCAC is related to chiral symmetry was due to Nambu and collaborators. Later Weinberg pointed out that current-algebra results could be interpreted as the consequences of an approximate chiral symmetry. The present paper is based on ideas expressed in Ref. 2. More recent papers on chiral symmetry are too numerous to list in any reasonably fair manner.

[2] R. Dashen, Phys. Rev. 183, 1245 (1969).